ISSN 2321-1091

ELEMENTARY FUNCTIONS OF A QUATERNION VARIABLE AND SOME APPLICATIONS

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ABSTRACT

In this paper, an effective formula for the calculation of the elementary functions of a quaternion variable obtained using the methods of differential equations. Also the elementary functions are obtained from the quaternion matrices.

Indexing terms/Keywords

Quaternion algebra, an analogue of the Euler formula, matrix exponential.

Academic Discipline And Sub-Disciplines

Mathematics

SUBJECT CLASSIFICATION

Mathematics Subject Classification

TYPE (METHOD/APPROACH)

Mathematical logic

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Social Sciences Research
Vol .7, No.2
jssreditor.cir@gmail.com
www.jssronline.com

1. QUATERNION ALGEBRA

According to Hamilton, quaternion is a mathematical object which we can write in the form [1]

$$q \equiv q_0 + iq_1 + jq_2 + kq_3, \tag{1}$$

where q_0, q_1, q_2, q_3 are real numbers, q is called quaternion's components, the basis element «1» is the identity element of q_0, and $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are three imaginary units. Quaternion product is denoted by «○» sign; and defined by the following rules for quaternion units multiplication given by Hamilton's definition:

$$i \circ i = j \circ j = k \circ k = -1,\ i \circ j = k,\ j \circ k = i,\ k \circ i = j. \tag{2}$$

We may also recognize i, j, k as unit vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ of Cartesian basis, and then by analogy with complex numbers, quaternion $q \in H$ can be represented as a formal sum of the scalar part q_0 and the vector part $\vec{q}$:

$$q = q_0 + q_1\overrightarrow{e_1} + q_2\overrightarrow{e_2} + q_3\overrightarrow{e_3} = q_0 + \vec{q}, \tag{3}$$

and the multiplication rules of basis vectors (2) can be expressed in the form of scalar and vector products:

$$\vec{e_i} \circ \vec{e_j} = -\left(\vec{e_i} \cdot \vec{e_j}\right) + \vec{e_i} \times \vec{e_j} \text{ or}$$

$$\vec{e_i} \circ \vec{e_j} = -\left(\vec{e_i} \cdot \vec{e_j}\right) + \varepsilon_{ijk}\,\vec{e_k},$$

where ε_{ijk} – Levi-Civita symbols, and $i, j, k = 1,2,3$.

These relationships allow us to interpret quaternions multiplication $\Lambda = \lambda_0 + \vec{\lambda}$, $M = \mu_0 + \vec{\mu}$ via scalar and vector products

$$\Lambda \circ M = \lambda_0\mu_0 - \left(\vec{\lambda} \cdot \vec{\mu}\right) + \lambda_0\vec{\mu} + \mu_0\vec{\lambda} + \vec{\lambda} \times \vec{\mu}. \tag{4}$$

It follows from multiplication rules for quaternion imaginary units that q multiplication is non-commutative

$$q_1 \circ q_2 \neq q_2 \circ q_1,$$

so, there is a concept of the left and right multiplication although q multiplication is still associative

$$(q_1 \circ q_2)q_3 = q_1(q_2 \circ q_3).$$

Following the procedure of obtaining conjunction we introduce the operation of quaternion conjunction

$$q^* \equiv q_0 - iq_1 - jq_2 - kq_3$$

and define the modulus of q - number

$$|q| = \sqrt{q \circ q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

2. ANALOGUE OF EULER'S FORMULA

We present an analogue of Euler's formula as an example of quaternion's operations. For this, let us consider quaternion's exponential function in the form:

$$e^{q\varphi} = e^{(q_0 + iq_1 + jq_2 + kq_3)\varphi}, \tag{5}$$

where φ is a variable of the quaternion's variable.

By virtue of the fact that q_0 is a real number, it commutes with its basis unit as well as with other imaginary units:

$$e^{(q_0 + iq_1 + jq_2 + kq_3)\varphi} = e^{q_0\varphi} \cdot e^{(iq_1 + jq_2 + kq_3)}. \tag{6}$$

Assume that the exponential function is given in the next form:

$$e^{(iq_1 + jq_2 + kq_3)\varphi} = A_0(\varphi) + iA_1(\varphi) + jA_2(\varphi) + kA_3(\varphi).$$

Taking derivative in (6) with respect to φ we obtain the following equation:

$$(iq_1 + jq_2 + kq_3)e^{(iq_1 + jq_2 + kq_3)\varphi} = A_0'(\varphi) + iA_1'(\varphi) + jA_2'(\varphi) + kA_3'(\varphi) \text{ or}$$

ISSN 2321-1091

$$(iq_1 + jq_2 + kq_3)[A_0(\varphi) + iA_1(\varphi) + jA_2(\varphi) + kA_3(\varphi)] = A_0^{'}(\varphi) + iA_1^{'}(\varphi) + jA_2^{'}(\varphi) + kA_3^{'}(\varphi).$$

Hence it follows from quaternion's equality that

$$\begin{aligned} A_0^{'}(\varphi) &= -a_1A_1(\varphi) - a_2A_2(\varphi) - a_3A_3(\varphi), \\ A_1^{'}(\varphi) &= -a_1A_0(\varphi) - a_3A_2(\varphi) + a_2A_3(\varphi), \\ A_2^{'}(\varphi) &= a_2A_0(\varphi) + a_3A_1(\varphi) - a_1A_3(\varphi), \\ A_3^{'}(\varphi) &= a_3A_0(\varphi) + a_1A_2(\varphi) - a_2A_1(\varphi). \end{aligned} \tag{7}$$

Having differentiated the first equation with respect to φ variable we get

$$A_0^{''}(\varphi) = -a_1A_1^{'}(\varphi) - a_2A_2^{'}(\varphi) - a_3A_3^{'}(\varphi) = -a_1[-a_1A_0(\varphi) - a_3A_2(\varphi) + a_2A_3(\varphi)] - a_2[a_2A_0(\varphi) + a_3A_1(\varphi) -$$

a1A3φ−a3a3A0φ+a1A2φ−a2A1φ=−a12−a22−a32A0φ, i.e.

$$A_0^{''}(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_0(\varphi) = 0. \tag{8}$$

We apply initial conditions for equation (8), i.e. at $\varphi = 0$:

$$A_0(\varphi) = 1, A_0^{'}(\varphi) = 0.$$

After differentiating second, third and forth equations in (7) with respect to φ variable we obtain similar equations of the type (8) for $A_1(\varphi)$, $A_2(\varphi)$ and $A_3(\varphi)$ under appropriate initial conditions:

$A_1^{''}(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_1(\varphi) = 0$, here at $\varphi = 0$, $A_1(\varphi) = 0$, $A_1^{'}(\varphi) = q_1$;

$A_2^{''}(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_2(\varphi) = 0$, here at $\varphi = 0$, $A_2(\varphi) = 0$, $A_2^{'}(\varphi) = q_2$;

$A_3^{''}(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_3(\varphi) = 0$, here at $\varphi = 0$, $A_3(\varphi) = 0$, $A_3^{'}(\varphi) = q_3$.

Solving these equations under corresponding initial conditions we have:

$$A_0(\varphi) = \cos\sqrt{q_1^2 + q_2^2 + q_3^2}\,\varphi,$$

$$A_1(\varphi) = \frac{q_1}{\sqrt{q_1^2+q_2^2+q_3^2}}\sin\sqrt{q_1^2 + q_2^2 + q_3^2}\varphi,$$

$$A_2(\varphi) = \frac{q_2}{\sqrt{q_1^2+q_2^2+q_3^2}}\sin\sqrt{q_1^2 + q_2^2 + q_3^2}\varphi,$$

$$A_3(\varphi) = \frac{q_3}{\sqrt{q_1^2+q_2^2+q_3^2}}\sin\sqrt{q_1^2 + q_2^2 + q_3^2}\varphi.$$

In turn, an analogue of Euler's formula for quaternion can be written as

$$e^{(iq_1+jq_2+kq_3)\varphi} = \cos\sqrt{q_1^2 + q_2^2 + q_3^2}\,\varphi + \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2+q_2^2+q_3^2}}\sin\sqrt{q_1^2 + q_2^2 + q_3^2}\varphi. \tag{9}$$

Hence if the quaternion is used as an argument of elementary function, it can be represented as a conditional complex number with a conditional imaginary unit:

$$q = q_0 + \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2+q_2^2+q_3^2}}\sqrt{q_1^2 + q_2^2 + q_3^2},$$

$q = Q_0 + IQ_1$, где $Q_0 = q_0$,

$$I = \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2+q_2^2+q_3^2}},\ Q_1 = \sqrt{q_1^2 + q_2^2 + q_3^2}.$$

Here the conditioned imaginary unit has a vector meaning in which it is a unit vector directed along $\vec{I} = \mathrm{Im}\,q$ vector. In such notation the quaternion retains complex number's properties:

$$q^2 = Q_0^2 + Q_1^2 = q_0 + q_1^2 + q_2^2 + q_3^2.$$

$$q^2 = q \circ q = q_0^2 - (q_1^2 + q_2^2 + q_3^2) + 2q_0(iq_1 + jq_2 + kq_3) = Q_0^2 - Q_1^2 + 2IQ_0Q_1,\ I^2 = -1.$$

Using these properties, we can find the elementary functions of a quaternion variable. For this, we

1) replace the quaternion with a conditional complex number $q \Longrightarrow Q_0 + IQ_1$;
2) expand an elementary function as a function of a complex variable $Q_0 + IQ_1$;
3) and after that proceed to a converse replacement $Q_0 \Longrightarrow q_0$,

$$I \Longrightarrow \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2+q_2^2+q_3^2}},$$

$$Q_1 \Longrightarrow \sqrt{q_1^2+q_2^2+q_3^2}.$$

Some elementary functions of quaternion are written below for illustrative purposes.

$$\sin q = \sin(Q_0 + IQ_1) = \sin Q_0 \cos IQ_1 + \cos Q_0 \sin IQ_1 = \sin Q_0 \operatorname{ch} Q_1 + I \cos Q_0 \operatorname{sh} Q_1 = \sin q_0 \operatorname{ch}\sqrt{q_1^2+q_2^2+q_3^2} + \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2+q_2^2+q_3^2}} \cos q_0 \operatorname{sh}\sqrt{q_1^2+q_2^2+q_3^2},$$

$$\cos q = \cos(Q_0 + IQ_1) = \cos Q_0 \cos IQ_1 - \sin Q_0 \sin IQ_1 = \cos Q_0 \operatorname{ch} Q_1 - I \sin Q_0 \sin Q_1 = \cos q_0 \operatorname{ch}\sqrt{q_1^2+q_2^2+q_3^2} - \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2+q_2^2+q_3^2}} \sin q_0 \operatorname{sh}\sqrt{q_1^2+q_2^2+q_3^2}.$$

3. MATRIX FORM OF QUATERNION ALGEBRA

It is interesting to present quaternion multiplication in matrix form [2]. Let

$$A = a_0 + ia_1 + ja_2 + ka_3,$$

$$B = b_0 + ib_1 + jb_2 + kb_3.$$

Then the product of two quaternions gives third quaternion $C = A \circ B$ and the resulting quaternion components are defined by the formula (4):

$$C_0 = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3,$$

$$C_1 = a_1b_0 + a_0b_1 - a_3b_2 + a_2b_3,$$

$$C_2 = a_2b_0 + a_2b_1 + a_0b_2 - a_1b_3,$$

$$C_3 = a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3.$$

Next we assign four-dimensional $\overrightarrow{V_A} = (a_0, a_1, a_2, a_3)^{\mathrm{T}}$ vector to A quaternion and four-dimensional $\overrightarrow{V_B} = (b_0, b_1, b_2, b_3)^{\mathrm{T}}$ vector to B quaternion, respectively.

Then C quaternion can be associated with its own four-dimensional vector defined by as follows:

$$\overrightarrow{V_C} = \overrightarrow{V_{A\circ B}} = G_1(A) \cdot \overrightarrow{V_B} = G_2(B) \cdot \overrightarrow{V_A}. \quad (10)$$

Matrix $G_1(A)$ and matrix $G_2(B)$ in the expression (10) equal to

$$G_1(A) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}, G_2(B) = \begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix}.$$

For an arbitrary quaternion Q matrices $G_1(A)$ and $G_2(B)$ can be represented as

$$G_1(Q) = \begin{pmatrix} q_0 & -\vec{q}^{\mathrm{T}} \\ \vec{q} & q_0E_3 + K(\vec{q}) \end{pmatrix}, \quad G_2(Q) = \begin{pmatrix} q_0 & -\vec{q}^{\mathrm{T}} \\ \vec{q} & q_0E_3 - K(\vec{q}) \end{pmatrix}$$

where E_3 is 3-by-3 unity matrix,

$$K(\vec{q}) = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}.$$

Some properties of $K(\vec{a})$ matrix:

$$K(\vec{a}) \cdot \vec{r} = \vec{a} \times \vec{r},$$

$$K(\vec{a}) \cdot \vec{a} = \vec{a} \times \vec{a} = 0,$$

$$K^{\mathrm{T}}(\vec{a}) = -K(\vec{a}),$$

$$K(\vec{a})K(\vec{b}) = \vec{b} \times \vec{a} - \vec{a}^{\mathrm{T}} \cdot \vec{b} \cdot E_3.$$

Some properties of G_1 and G_2 matrices:

$$G_m(A^*) = G_m^{\mathrm{T}}(A), \qquad G_m(A+B) = G_m(A) + G_m(B), \quad m = 1, 2;$$
$$G_1(A \circ B) = G_1(A)G_2(B), \qquad G_2(A \circ B) = G_2(B)G_1(A),$$
$$G_1(A)G_2(B) = G_2(B)G_1(A), \qquad \det G_m(A) = \|A\|^4, \qquad m = 1, 2.$$

Lemma. For any $A \neq 0$ quaternion the following equations are valid: $G_1(A^{-1}) = G_1^{-1}(A)$, $G_2(A^{-1}) = G_2^{-1}(A)$.

Using G_1 and G_2 matrices we can easily replace equations written in quaternions to equations in matrices. In particular, for $A = B \circ C \circ D$ quaternion the matrix form will be as follows:

$$A = G_1(B) \cdot G_2(D) \cdot \vec{C},$$

where $\vec{C} = (c_0, c_1, c_2, c_3)^{\mathrm{T}}$ is a four-dimensional vector related to C quaternion.

4. MATRIX EXPONENTIAL

We apply spectral decomposition of matrix function so that to find matrix exponential.

There is an isomorphism between quaternions $q = q_0 + iq_1 + jq_2 + kq_3$ and special form 4-by-4 matrix in terms of quaternion and matrix operations.

$$Q = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}.$$

Let find characteristic polynomial of Q quaternion matrix

$$\det(Q - \lambda E) = [(q_0 - \lambda)^2 + q_1^2 + q_2^2 + q_3^2]^2.$$

Then the complex values $\lambda = q_0 + i\sqrt{q_1^2 + q_2^2 + q_3^2}$, $\bar{\lambda} = q_0 - i\sqrt{q_1^2 + q_2^2 + q_3^2}$ are the eigenvalues of the quaternion $q = q_0 + iq_1 + jq_2 + kq_3$.

The minimal polynomial of Q quaternion matrix is given by

$$\mu_Q(\lambda) = \frac{(-1)^n \det(Q - \lambda E)}{D_{n-1}(\lambda)},$$

where $D_{n-1}(\lambda)$ is $(Q - \lambda E)$ characteristic matrix's the greatest common divisor of subdeterminant of order $(n-1)$.

In our case $n = 4$, $D_{n-1} = (q_0 - \lambda)^2 + q_1^2 + q_2^2 + q_3^2$. Then

$$\mu_Q(\lambda) = (q_0 - \lambda)^2 + q_1^2 + q_2^2 + q_3^2 = (\lambda - q_0 - i\langle q \rangle) \cdot (\lambda - q_0 + i\langle q \rangle),$$

where $\langle q \rangle = \sqrt{q_1^2 + q_2^2 + q_3^2}$.

Or

$$\mu_Q(\lambda) = [\lambda - (q_0 + i|q - q_0|)] \cdot [\lambda - (q_0 - i|q - q_0|)],$$

where $\langle q \rangle = |q - q_0|$. Here $i^2 = -1$.

Then the basic formula for $f(Q)$ is as follows:

$$f(Q) = f(\lambda_1)z_{11} + f(\lambda_2)z_{21},$$

where z_{11}, z_{21} Q matrix components, and

$$\lambda_1 = q_0 + i|q - q_0|,$$

$$\lambda_2 = q_0 - i|q - q_0|.$$

Substituting $\lambda - \lambda_1$, $\lambda - \lambda_2$ consistently in place of $f(\lambda)$ we obtain

$$-2i|q - q_0|z_{21} = Q - \lambda_1 E,$$

$$2i|q - q_0|z_{11} = Q - \lambda_2 E,$$

where E is a unity matrix.

Hence

$$2i|q - q_0|f(Q) = (Q - \lambda_2 E)f(\lambda_1) - (Q - \lambda_1 E)f(\lambda_2). \quad (11)$$

Let us consider some applications of this formula.

If $f(\lambda) = \frac{1}{\lambda}$, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ numbers are formula's value on Q matrix spectrum. Therefore, thus function is defined on Q matrix spectrum.

For this reason the basic formula (11) can be used to find Q^{-1} inverse matrix.

Substituting the values of $f(\lambda_1) = \frac{1}{\lambda_1} = \frac{\lambda_2}{\Delta}$, $f(\lambda_2) = \frac{1}{\lambda_2} = \frac{\lambda_1}{\Delta}$ into the basic formula $f(Q)$ we have $Q^{-1} = \frac{1}{\Delta}(-Q + 2q_0 E)$. Here

$$\Delta = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Validity of the obtained formula can be verified by direct calculation.

Now we find the exponent of quaternion matrix. For this, consider $f(\lambda) = e^{\lambda}$ function, which is also defined Q matrix spectrum

$$\exp(Q) = e^{q_0}\left[\frac{\sin|q - q_0|}{|q - q_0|}Q + \left(\cos|q - q_0| - \frac{q_0 \sin|q - q_0|}{|q - q_0|}E\right)\right].$$

By proceeding this process we can get entire spectrum of elementary matrix of Q quaternion matrix.

REFERENCES

1. Hamilton, W. R., 1994. Reasoning about naming systems. Selecta: Optics. Dynamics. Quaternions. Moscow, Nauka. Russian edition.
2. Bellman, R. 1969. Introduction to matrix analysis. Moscow, Nauka. Russian edition.