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Transient solution of $M^{[X]}$/G/1 with Second Optional Service Subject to Reneging during Vacation and Breakdown time

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ABSTRACT

In this paper, we present a batch arrival non- Markovian queuing model with second optional service. Batches arrive in Poisson stream with mean arrival rate λ, such that all customers demand the first essential service, whereas only some of them demand the second 'optional' service. We consider reneging to occur when the server is unavailable during the system breakdown or vacations periods. The time-dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been derived explicitly. Also the mean queue length and the mean waiting time have been found explicitly.

Keywords: $M^{[X]}/G/1$ Queue First essential service; Second optional service; Breakdowns; reneging: Steady State Queue Size.

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1. INTRODUCTION

The research study on queuing systems with impatient customers has become an extensive and interesting area in queuing theory literature. In many real life situations, the arriving customers may be discouraged due to long queue, and decide not to join the queue and leave the system at once. This behavior of customers is referred as balking. Sometimes customers get impatient after joining the queue and leave the system without getting service. This behavior of customer recognizes as reneging. In the last few years, we see studies on queues with balking and reneging gaining significant importance. We see applications of queue with reneging in emergency services in hospitals dealing serious patients, communication systems, production and inventory system and many more. A queue with balking and reneging was initially studied by Haight (1957), Barrer (1957). Since then, extensive amount of work has been done on queuing systems related to impatient customers. Queues with balking and reneging have been studied by authors like Altman and Yechiali (2006), Ancker et al. (1963), Choudhury and Medhi (2011), in the last few years.

Vacation queuing models has been modeled effectively in various situations such as production, banking service, communication systems, and computer networks etc. Numerous authors are interested in studying queuing models with various vacation policies including single and multiple vacation policies. Batch arrival queue with server vacations was investigated by Yechiali (1975). An excellent comprehensive studies on vacation models can be found in Takagi (1991) and Doshi (1986) research papers. One of the classical vacation model in queuing literature is Bernoulli scheduled server vacation. Keilson and Servi(1987) introduced and studied vacation scheme with Bernoulli schedule discipline.

In this paper we consider queuing system such that the customers are arriving in batches according to Poisson stream. The server provides a first essential service to all incoming customers and a second optional service will be provided to only some of them those who demand it. We extend and develop this model by adding new assumptions reneging and system breakdowns.

Customers may renege (leave the queue after joining) during server breakdowns or during the time when the server takes vacation due to impatience. In real world, this is a very realistic assumption and often we come across such queuing situations.

MATHEMATICAL DESCRIPTION OF THE MODEL

The following assumptions are to be used describe the mathematical model of our study:

- Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided service one by one on a ‘first come first served’ basis. Let $\lambda c_k dt\ (k = 1, 2, 3, \ldots)$ be the first order probability that a batch of k customers arrives at the system during a short interval of time $(t, t+dt]$, $where\ 0 \le c_k \le 1\ and\ \sum_{k=1}^{\infty} c_k = 1\ and\ \lambda > 0$ is the mean arrival rate of batches.

- There is a single server which provides the first essential service to all arriving customers. Let $B_1(v)\ and\ b_1(v)$ respectively be the distribution function and the density function of the first service times respectively.

- As soon as the first service of a customer is completed, then he may demand for the second service with probability r, or else he may decide to leave the system with probability 1- r in which case another customer at the head of the queue (if any) is taken up for his first essential service.

- The second service times as assumed to be general with the distribution function $B_2(v)$ and the density function $b_2(v)$. Further, Let $\mu_i(x)dx$ be the conditional probability density function of i^{th} service completion during the interval (x, x +d x] given that the elapsed service time is x, so that

$$\mu_i = \frac{b_i(x)}{1 - B_i(x)}, i = 1,2$$

and therefore

$$b_i(v) = \mu_i(v) e^{\int_0^v \mu_i(x)dx}, i = 1,2$$

- As soon as the customer’s service is completed, the server may go for a vacation of random length V with probability $p(0 < p < 1)$or it may continue to serve the next customer with probability (1-p).

- On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.

- The vacation time of the server follows general (arbitrary) distribution with distribution function V(s) and the density function v(s). Let $\gamma(x)dx$ be the condition probability of a completion of a vacation during the interval (x, x +d x] given that the elapsed service time is x, so that

$$\eta(x) = \frac{W(x)}{1 - W(x)}$$

and thus

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$$w(s) = \gamma(s)e^{\int_0^s \eta(x)dx}$$

- The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate α > 0. Further the repair time follows general (arbitrary) distribution with distribution function R (x) and the density function $r(x)$. Let $\beta(x)\,dx$ be the condition probability of a completion of a repair process, so that

$$\beta(x) = \frac{R(x)}{1 - R(x)}$$

and thus

$$F(s) = \beta(s)e^{\int_0^t \gamma(x)dx}$$

- In addition, customers arriving for service may become impatient and renege (leave the queue) after joining during vacations and breakdown times. Reneging is assumed to follow exponential distribution with parameter γ. Thus $f(t) = \gamma e^{-\gamma t}dt, \gamma > 0$.Thus η dt is the probability that a customer can renege during a short interval of time (t, t +d t]
- Various stochastic processes involved in the system are independent of each others.

3. DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

$P_n^i(x,t) =$ probability that at time 't' the server is active providing ith service and there are 'n' customers in the queue including the one being served and the elapsed service time for this customer is x. Consequently $P_n^i(t) =$ denotes the probability that at time't' there are 'n' customers in the queue excluding the one customer in i^{th} service irrespective of the value of x.

- $V_n(x,t)$ = probability that at time 't' the server is on vacation with elapsed vacation time x, and there are 'n' customers in the waiting in the queue for service. Consequently $V_n(t)$ = probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.
- $R_n(x,t)$ = Probability that at time t, the server is inactive due to break down and the system is under repair while there are 'n' customers in the queue.
- $Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha)P_n^{(1)}(x,t) = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{(1)}(x,t), n \geq 1 \quad (3.1)$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + \left(\lambda + \mu_1(x) + \alpha\right)P_0^{(1)}(x,t) = 0 \quad (3.2)$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + (\lambda + \mu_2(x) + \alpha)P_n^{(2)}(x,t) = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{(2)}(x,t), \ n \geq 1 \quad (3.3)$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + \left(\lambda + \mu_2(x) + \alpha\right)P_0^{(2)}(x,t) = 0 \quad (3.4)$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + (\lambda + \eta(x) + \gamma)V_n(x,t) = \lambda \sum_{k=1}^{\infty} C_k V_{n-k}(t), \quad n \geq 1 \quad (3.5)$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + \left(\lambda + \eta(x)\right)V_0(x,t) = \gamma V_0(x,t) \quad (3.6)$$

$$\frac{\partial}{\partial x}R_0(x,t) + \frac{\partial}{\partial t}R_0(x,t) + (\lambda + \beta(x))\, R_0(x,t) = 0 \quad (3.7)$$

$$\frac{\partial}{\partial x}R_n(x,t) + \frac{\partial}{\partial t}R_n(x,t) + (\lambda + \beta(x) + \gamma)R_n(x,t) = \lambda \sum_{k=1}^{\infty} C_k R_{n-k}(x,t) + \gamma R_{n+1}, \ n \geq 1 \quad (3.8)$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t) + (1-r)(1-p)\int_0^{\infty} P_0^1(x,t)\mu_1(x)dx + (1-p)\int_0^{\infty} P_0^2(x,t)\mu_2(x)dx$$
$$+ \int_0^{\infty} V_0(x,t)\eta(x) + \int_0^{\infty} R_0(x,t)\beta(x)\, dx \quad (3.9)$$

Equations are to be solved subject to the following boundary conditions:

$$P_n^{(1)}(0,t) = \lambda c_n Q + \int_0^{\infty} V_n(x,t)\eta(x)dx + \int_0^{\infty} R_{n+1}(x,t)\beta(x)dx$$
$$+(1-p)(1-r)\int_0^{\infty} P_{n+1}^{(1)}(x,t)\mu_2(x)dx + (1-p)\int_0^{\infty} P_{n+1}^{(2)}(x,t)\mu_2(x)dx; n \geq 1 \quad (3.10)$$

$$P_n^{(2)}(0,t) = r\int_0^\infty P_n^{(1)}(x,t)\ \mu_1(x)dx\ , n \geq 0 \tag{3.11}$$

$$V_n(0,t) = p(1-r)\int_0^\infty P_n^{(1)}(x)\ \mu_1(x)dx\ + p\int_0^\infty P_n^{(2)}(x,t)\ \mu_2(x)dx, n \geq 0 \tag{3.12}$$

$$R_n(0,t) = \alpha\int_0^\infty P_{n-1}^{(1)}(x,t)dx + \alpha\int_0^\infty P_{n-1}^{(2)}(x,t)dx,\ \ n \geq 1 \tag{3.13}$$

$$R_0(0) = 0 \tag{3.14}$$

4. TIME DEPENDENT SOLUTION

Generating functions of the queue length

Now we define the probability generating function as follows

$$P^{(1)}(x,t) = \sum_0^\infty P_n^{(1)}(x,z,t)z^n;\quad P^{(1)}(z,t) = \sum_0^\infty P_n^{(1)}(t)z^n, |z| \leq 1, x > 0$$
$$P^{(2)}(x,z,t) = \sum_0^\infty P_n^{(2)}(x,t)z^n;\ P^{(2)}(z,t) = \sum_0^\infty P_n^{(2)}(t)z^n, |z| \leq 1, x > 0$$

$$V(z,t) = \sum_0^\infty z^n V_n(t)\quad ;\quad R(z,t) = \sum_0^\infty z^n R_n(t)\quad ;\quad C(z) = \sum_0^\infty C_n z^n,\ \ |z| \leq 1 \tag{4.1}$$

Taking Laplace transforms of equations (3.1) to (3.14)

$$\frac{\partial}{\partial x}\ \bar{P}_n^{(1)}(x,s) + \ (s + \lambda + \mu_1(x) + \ \alpha)\bar{P}_n^{(1)}(x,s) = \lambda\sum_{k=1}^\infty C_k\bar{P}_{n-k}^{(1)}(x,s),\quad n \geq 1 \tag{4.2}$$

$$\frac{\partial}{\partial x}\ \bar{P}_0^{(1)}(x,s) + \ (s + \lambda + \mu_1(x) + \ \alpha)\bar{P}_0^{(1)}(x,s) = 0 \tag{4.3}$$

$$\frac{\partial}{\partial x}\ \bar{P}_n^{(2)}(x,s) + \ (s + \lambda + \mu_2(x) + \ \alpha)\bar{P}_n^{(2)}(x,s) = \lambda\sum_{k=1}^\infty C_k\bar{P}_{n-k}^{(2)}(x,s)\ ,\quad n \geq 1 \tag{4.4}$$

$$\frac{\partial}{\partial x}\ \bar{P}_0^{(2)}(x,s) + \ \left(s + \lambda + \mu_2(x) + \ \alpha\right)\bar{P}_0^{(2)}(x,s) = 0 \tag{4.5}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + \ (s + \lambda + \eta(x) + \gamma)\bar{V}_n(x,s) = \lambda\sum_{k=1}^\infty C_k\bar{V}_{n-k}(x,s) + \gamma\bar{V}_{n+1}\ ,\quad n \geq 1 \tag{4.6}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + \ \left(s + \lambda + \eta(x)\right)\bar{V}_0(x,s) = \gamma\bar{V}_1(x,s) \tag{4.7}$$

$$\frac{\partial}{\partial x}\bar{R}_0(x,s) + (s + \lambda\ +\ \beta(x))\ \bar{R}_0(x,s) = \gamma\bar{R}_1(x,s) \tag{4.8}$$

$$\frac{\partial}{\partial x}\bar{R}_n(x,s) + \ (s + \lambda\ +\ \beta(x) + \gamma)\bar{R}_n(x,s)\ = \lambda\sum_{k=1}^\infty C_k\bar{R}_{n-k}(x,s) + \gamma\bar{R}_{n+1},\ \ n \geq 1 \tag{4.9}$$

$$\lambda\bar{Q}(s) = (1-r)(1-p)\int_0^\infty \bar{P}_0^1(x,s)\mu_1(x)dx + (1-p)\int_0^\infty \bar{P}_0^2(x,s)\mu_2(x)dx + \int_0^\infty \bar{V}_0(x,s)\eta(x)\ dx + \int_0^\infty R_0(x,s)\beta(x)dx \tag{4.10}$$

$$\bar{P}_n^{(1)}(0,s) = \lambda c_n\bar{Q}(s) + \int_0^\infty \bar{V}_n(x,s)\eta(x)dx + \int_0^\infty \bar{R}_{n+1}(x,s)\beta(x)dx$$
$$+(1-p)(1-r)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_2(x)dx + \ (1-p)\int_0^\infty \bar{P}_{n+1}^{(2)}s\mu_2(x)dx\ ; n \geq 1 \tag{4.11}$$

$$\bar{P}_n^{(2)}(0,s) = r\int_0^\infty \bar{P}_n^{(1)}(x,s)\ \ \mu_1(x)dx\ , n \geq 0 \tag{4.12}$$

$$\bar{V}_n(0,s) = p(1-r)\int_0^\infty \bar{P}_n^{(1)}(x,s)\ \ \mu_1(x)dx\ + p\int_0^\infty \bar{P}_n^{(2)}(x,s)\ \mu_2(x)dx, n \geq 0 \tag{4.13}$$

$$\bar{R}_n(0,s) = \alpha\int_0^\infty \bar{P}_{n-1}^{(1)}(x,s)dx + \alpha\int_0^\infty \bar{P}_{n-1}^{(2)}(x,s)dx,\ \ n \geq 1 \tag{4.14}$$

We multiply both sides of equations (4.2) and (4.3) by suitable powers of z, sum over n and use (4.1) and simplify. We thus have after algebraic simplifications

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s) + [s + \lambda - \lambda C(z) + \mu_1(x) + \alpha]\bar{P}^{(1)}(x,z,s)\ = 0 \tag{4.15}$$

Performing similar operations on equations (4.4) and (4.5) and using (4.1), We have

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s) + \left[s + \lambda - \lambda C(z) + \mu_2(x) + \alpha\right]\bar{P}^{(2)}(x,z,s) = 0 \tag{4.16}$$

Similar operations on equations (4.6),(4.7),(4.8) and (4.9) yields

$$\frac{\partial}{\partial x}\bar{V}(x,z,s) + \left[s + \lambda - \lambda C(z) + \eta(x) + \gamma - \frac{\gamma}{z}\right]\bar{V}(x,z,s) = 0 \tag{4.17}$$

$$\frac{\partial}{\partial x}\bar{R}(x,z,s) + \left[s+\lambda-\lambda C(z)+\eta(x)+\gamma-\frac{\gamma}{z}\right]\bar{R}(x,z,s) = 0 \qquad (4.18)$$

Now we multiply both sides of equation (4.11),(4.12),(4.13) and (4.14) by z^n, sum over n from 1to ∞ ,yields.

$$z\,\bar{P}^{(1)}(0,z,s) = \lambda[C(z)-1]\bar{Q}(s) + [1-s\bar{Q}(s)] + (1-r)(1-p)\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx + (1-p)\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx$$

$$+\int_0^\infty \bar{V}(x,z,s)\eta(x)\,dx + \int_0^\infty \bar{R}(x,z,s)\,\beta(x)dx - (1-r)\int_0^\infty \bar{P}_0^{(1)}(x,s)\mu_1(x)\,dx - \int_0^\infty \bar{P}_0^{(2)}(x,s)\mu_2(x)\,dx \qquad (4.19)$$

$$\bar{P}^{(2)}(0,z,s) = r\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx \qquad (4.20)$$

$$z\bar{V}(0,z,s) = (1-r)p\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx + p\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)\,dx \qquad (4.21)$$

$$\bar{R}(0,z,s) = z\alpha\int_0^\infty \bar{P}^{(1)}(x,z,s)dx + z\alpha\int_0^\infty \bar{P}^{(2)}(x,z,s)dx \qquad (4.22)$$

Integrating equations (4.15), (4.16) (4.17) and (4.18) between 0 and x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)\quad e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^\infty \mu_{1(t)}dt} \qquad (4.23)$$

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)\quad e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^\infty \mu_{2(t)}dt} \qquad (4.24)$$

$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)x-\int_0^\infty \eta(x)dx} \qquad (4.25)$$

$$\bar{R}(x,z,s) = \bar{R}(0,z,s)e^{-\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)x-\int_0^\infty \eta(x)dx} \qquad (4.26)$$

Again integrating equation (4.23) w.r.to x , we have

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\quad \left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \qquad (4.27)$$

$$where\ \bar{B}_1(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}d\bar{B}_1(x) \qquad (4.28)$$

is the Laplace transform of first essential service time.

Now multiplying both sides of equation (4.27) by $\mu_1(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)\,dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) \qquad (4.29)$$

Again integrating equation (4.24) w.r.to x, we have

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\quad \left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \qquad (4.30)$$

$$where\ \bar{B}_2(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}d\bar{B}_2(x) \qquad (4.31)$$

is the Laplace transform of second optional service time

Now multiplying both sides of equation (4.30) by $\mu_2(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)\,dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2(s+\lambda-\lambda C(z)+\alpha) \qquad (4.32)$$

Again integrating equation (4.25) w.r.to x , we have

$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{W}\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)}{\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)}\right] \qquad (4.33)$$

$$where\ \bar{W}\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right) = \int_0^\infty e^{-\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)x}d\bar{W}(x) \qquad (4.34)$$

is the Laplace transform of vacation time.

Now multiplying both sides of equation (4.22) by $\eta(x)$ and integrating over x, we get

$$\int_0^\infty \bar{V}(x,z,s)\eta(x)\,dx = \bar{V}(0,z,s)\,\bar{W}(s+\lambda-\lambda C(z)+\alpha) \qquad (4.35)$$

Again integrating equation (4.26) w.r.to x , we have

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$$\bar{R}(z,s) = \bar{R}(0,z,s)\left[\frac{1-\bar{F}\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)}{\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)}\right] \tag{4.36}$$

$$where\ \bar{W}\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right) = \int_0^\infty e^{-\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)x} d\bar{W}(x) \tag{4.37}$$

is the Laplace transform of vacation time.

Now multiplying both sides of equation (4.36) by $\beta(x)$ and integrating over x, we get

$$\int_0^\infty \bar{R}(x,z,s)\beta(x)dx = \bar{R}(0,z,s)\bar{F}(s+\lambda-\lambda C(z)+\alpha) \tag{4.38}$$

$$where\ \bar{F}\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right) = \int_0^\infty e^{-\left(s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}\right)x} d\bar{F}(x) \tag{4.39}$$

is the Laplace transform of repair time.

Now using equations (4.29) (4.32), (4.35) and (4.38) in equation (4.19) –(4.22) we get

$$\bar{P}^{(1)}(0,z,s) = \frac{f_1(z)[\lambda(C(z)-1)Q(s)+(1-sQ(s))\ ]}{DR} \tag{4.40}$$

where $DR = f_1(z)\{z-[(1-p)+p\bar{W}(f_2(z))][(1-r)\bar{B}_1(f_1(z))+r(1-p)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]\}$

$-\alpha z\bar{F}(f_2(z))(1-(1-r)\bar{B}_1(f_1(z))-r\,\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))$

$f_1(z) = s+\lambda-\lambda C(z)+\alpha$ and $f_2(z) = s+\lambda-\lambda C(z)+\gamma-\frac{\gamma}{z}$

$$\bar{P}^{(2)}(0,z,s) = \frac{r\,f_1(z)[\lambda(C(z)-1)Q(s)+(1-sQ(s))\ ]B_1(f_1(z))}{DR} \tag{4.41}$$

$$\bar{V}(0,z,s) = \frac{p\,f_1(z)[\lambda(C(z)-1)Q(s)+(1-sQ(s))\ ][(1-r)B_1(f_1(z))+rB_1(f_1(z))B_2(f_1(z))]}{DR} \tag{4.42}$$

Using (4.27)& (4.30) in (4.22) we get,

$$f_1(z)\bar{R}(0,z,s) = \alpha z\bar{P}^{(1)}(0,z,s)[1-\bar{B}_1(s+\lambda-\lambda C(z)+\alpha)\bar{B}_2(s+\lambda-\lambda C(z)+\alpha)] \tag{4.43}$$

Substituting the value from equation (4.40),(4.41) and (4.42) in equations (4.27), (4.30) ,(4.33)& (4.36) we get

$$\bar{P}^{(1)}(z,s) = \frac{f_2(z)[1-B_1(f_1(z))]}{DR}\left[\lambda(C(z)-1)\bar{Q}(s)+\left(1-s\bar{Q}(s)\right)\ \right] \tag{4.44}$$

$$\bar{P}^{(2)}(z,s) = \frac{rf_2(z)B_1(f_1(z))[1-B_2(f_1(z))]}{DR}\left[\lambda(C(z)-1)\bar{Q}(s)+\left(1-s\bar{Q}(s)\right)\ \right] \tag{4.45}$$

$$\bar{R}(z,s) = \frac{\alpha z[1-(1-r)\bar{B}_1(f_1(z))-r\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]}{DR}\left[\frac{1-\bar{F}(f_2(z))}{f_2(z)}\right] \tag{4.46}$$

$$\bar{V}(z,s) = \frac{pf_2(z)[(1-r)\bar{B}_1(f_1(z))+r\,\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]}{DR}\left[\frac{1-\bar{W}(f_2(z))}{f_2(z)}\ \right] \tag{4.47}$$

In this section we shall derive the steady state probability distribution for our Queuing model. To define the steady state probabilities, suppress the arguments where ever it appears in the time dependent analysis. By using well known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

$$P^{(1)}(z) = \frac{[1-B_1(f_1(z))]}{DR}\lambda(C(z)-1)Q \tag{4.48}$$

$$P^{(2)}(z) = \frac{rB_1(f_1(z))[1-B_2(f_1(z))]}{DR}\lambda(C(z)-1)Q \tag{4.49}$$

$$R(z) = \frac{p[1-(1-r)\bar{B}_1(f_1(z))-r\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))}{DR}\left[\frac{1-\bar{F}(f_2(z))}{f_2(z)}\right]Q \tag{4.50}$$

$$V(z) = \frac{\alpha z[(1-r)\bar{B}_1(f_1(z))+r\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))}{DR}\left[\frac{1-\bar{W}(f_2(z))}{f_2(z)}\right]Q \tag{4.51}$$

In order to determine $P^{(1)}(z), P^{(2)}(z), \mathrm{R}(z)$ completely, we have yet to determine the unknown $V(1)$ which appears in the numerator of the right sides of equations (4.34), (4.35) and (4.36). For that purpose, we shall use the normalizing condition.

$$P^{(1)}(1) + P^{(2)}(1) + V(1) + R(1) = 1 \quad (4.52)$$

$$P^{(1)}(1) = \frac{\lambda C'(1)(1-\bar{B}_1(\alpha))}{dr}Q \quad (4.53)$$

$$P^{(2)}(1) = \frac{\lambda r C'(1)\bar{B}_1(\alpha)(1-\bar{B}_2(\alpha))}{dr}Q \quad (4.54)$$

$$R(1) = \frac{\lambda\alpha C'(1)E(R)(1-(1-r)\bar{B}_1(\alpha)-r\bar{B}_1(\alpha)\bar{B}_2(\alpha))}{dr}Q \quad (4.55)$$

$$V(1) = \frac{\lambda p\alpha C'(1)E(V)((1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha))}{dr}Q \quad (4.56)$$

where $dr = -(\lambda C'(1) + \alpha(\lambda C'(1) - \gamma)E(R)) + [\alpha + \lambda C'(1) + \alpha(\lambda C'(1) - \gamma)E(R) - p\alpha(\lambda C'(1) - \gamma)E(V)]\bar{B}_1(\alpha)\bar{B}_2(\alpha)$

$P^{(1)}(1), P^{(2)}(1)$ *and* $R(1)$ denote the steady state probabilities that the server is providing first stage of service, second stage of service and server under repair without regard to the number of customers in the queue. Now using equations (4.53), (4.54), (4.55) and (4.56) into the normalizing condition (4.52) and simplifying, we obtain

$$Q = 1 - \frac{\lambda C'(1)[\{1+\alpha E(R)\} - \{1+\alpha E(R) - p\alpha E(V)\}((1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha))]}{\alpha\gamma E(R)(1-(1-r)\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)) + p\alpha[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} \quad (4.57)$$

and hence, the utilization factor ρ of the system is given by

$$\rho = \frac{\lambda C'(1)[\{1+\alpha E(R)\} - \{1+\alpha E(R) - p\alpha E(V)\}((1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha))]}{\alpha\gamma E(R)(1-(1-r)\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)) + p\alpha[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} \quad (4.58)$$

where $\rho < 1$ is the stability condition under which the steady states exits.

5. The Mean queue size and the mean system size

Let $P_q(z)$ denote the probability generating function of the queue size irrespective of the server state. Then adding equation (4.27), (4.28) and (4.29) we obtain

$$P_q(z) = P^{(1)}(z) + P^{(2)}(z) + R(z) + V(z)$$

$$P_q(z) = \frac{N(z)}{D(z)} \quad (5.1)$$

Let L_q denote the mean number of customers in the queue under the steady state. Then we have

$$L_q = \frac{d}{dz}[P_q(z)] \text{ at } z = 1$$

$$L_q = \lim_{z\to 1}\frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2} \quad (5.2)$$

where primes and double primes in (4.36) denote first and second derivative at z = 1, respectively. Carrying out the derivative at z = 1 we have

$$N'(1) = Q[\lambda E(I)\{1+\alpha E(R)\} - \lambda E(I)\{1+\alpha E(R) - p\alpha E(V)][(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] \quad (5.3)$$

$$N''(1) = Q\begin{bmatrix} \lambda E(I/I-1)\begin{Bmatrix} [1-(1-r)\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] \\ +\alpha E(R)[1-(1-r)\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] + p\alpha[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]E(V) \end{Bmatrix} \\ -\lambda E(I)\{1+\alpha E(R) - p\alpha E(V)\}[(1-r)\bar{B}'_1 + r(\bar{B}'_1(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}'_2(\alpha))] \\ \alpha(\lambda E(I) - \gamma)E(R^2)[1-(1-r)\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] \\ +(\lambda E(I) - \gamma)E(V^2)[1-(1-r)\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] \end{bmatrix} \quad (5.4)$$

$$D'(1) = -\{\lambda E(I) + \alpha(\lambda E(I) - \gamma)E(R)\} + \{\lambda E(I) + \alpha(\lambda E(I) - \gamma)E(R) - p\alpha(\lambda E(I) - \gamma)E(V) + \alpha\}$$
$$[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] \quad (5.5)$$

$$
\begin{aligned}
D''(1) = & -\lambda E(I/I-1)[(1+\alpha E(R)) - [(1-\mathrm{r})\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]\{1-\alpha E(R) + p\alpha E(V)\}] \\
& -2\alpha\gamma E(R)[1-(1-\mathrm{r})\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] + \alpha(\lambda E(I)-\gamma)^2 E(R^2)[1-(1-\mathrm{r})\bar{B}_1(\alpha) - r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] \\
& -p\alpha\{2\gamma E(V) - (\lambda E(I)-\gamma)^2 E(V^2)\}[(1-\mathrm{r})\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]
\end{aligned}
$$

$$
+\{\lambda E(I) + \alpha(\lambda E(I)-\gamma)E(R) - p\alpha(\lambda E(I)-\gamma)E(V)\}[(1-\mathrm{r})\bar{B}'_1 + r(\bar{B}'_1(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\,\bar{B}'_2(\alpha)] \quad (5.6)
$$

Then if we substitute the values from (5.3), (5.4), (5.5) and (5.6) into (5.2) we obtain L_q in the closed form. Further we find the mean system size L using Little's formula. Thus we have

$$L = Lq + \rho$$

where L_q has been found by equation (5.2) and ρ is obtained from equation (4.58).

6. Conclusion

In this paper we have studied an $M^{[X]}$/G/1 with Second Optional Service, reneging to occur when the server is unavailable during the system breakdown or vacations periods. The probability generating function of the number of customers in the queue is found using the supplementary variable technique. This model can be utilized in large scale manufacturing industries and communication networks.

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