Lorentz transformations via Pauli matrices

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Abstract.- We exhibit expressions, in terms of Pauli matrices, which directly generate Lorentz transformations in Minkowski space.

Key words.- Pauli matrices, Lorentz transformations, Infeld-van der Waerden symbols

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In space time an event is represented by $\left(x^j\right)=(ct,x,y,z)$, $j=0,\ldots,3$, with the metric $\left(g_{jr}\right)=\mathrm{Diag}\,(1,-1,-1,-1)$. If it is necessary to employ another frame of reference, then the new coordinates $\tilde{x}^r$ are connected with x^j via the linear transformation:

$$\tilde{x}^j = L^j{}_r x^r \quad , \tag{1}$$

where the Lorentz matrix $\underset{\sim}{L}$ verifies the restriction :

$$L^j{}_a \; g_{rj} L^r{}_b = g_{ab} \quad , \tag{2}$$

because the Minkowskian line element must remain invariant under $\underset{\sim}{L}$, that is, $\tilde{x}^r\tilde{x}_r = x^r x_r$.

From (2) we see that $\underset{\sim}{L}$ has six degrees of freedom, which permits to work with four complex numbers $\alpha,\beta,\gamma,\delta$ such that $\alpha\delta-\beta\gamma=1$, then the components of homogeneous Lorentz transformation $\underset{\sim}{L}$ can be written in the form [1-4]:

$$\begin{aligned}
L^0{}_0 &= \frac{1}{2}\left(\alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + \delta\delta^*\right) , & L^0{}_1 &= \frac{1}{2}\left(\alpha^*\beta + \gamma^*\delta\right) + \text{c.c.} , \\
L^0{}_2 &= \frac{i}{2}\left(\alpha^*\beta + \gamma^*\delta\right) + \text{c.c.} , & L^0{}_3 &= \frac{1}{2}\left(\alpha\alpha^* - \beta\beta^* + \gamma\gamma^* - \delta\delta^*\right) , \\
L^1{}_0 &= \frac{1}{2}\left(\alpha^*\gamma + \beta^*\delta\right) + \text{c.c.} , & L^1{}_1 &= \frac{1}{2}\left(\alpha^*\delta + \beta\gamma^*\right) + \text{c.c.} , \\
L^1{}_2 &= \frac{i}{2}\left(\alpha^*\delta + \beta\gamma^*\right) + \text{c.c.} , & L^1{}_3 &= \frac{1}{2}\left(\alpha^*\gamma - \beta^*\delta\right) + \text{c.c.} , \\
L^2{}_0 &= \frac{i}{2}\left(\alpha\gamma^* - \beta^*\delta\right) + \text{c.c.} , & L^2{}_1 &= \frac{i}{2}\left(\alpha\delta^* + \beta\gamma^*\right) + \text{c.c.} , \\
L^2{}_2 &= \frac{1}{2}\left(\alpha^*\delta - \beta^*\gamma\right) + \text{c.c.} , & L^2{}_3 &= \frac{i}{2}\left(\alpha\gamma^* + \beta^*\delta\right) + \text{c.c.} , \\
L^3{}_0 &= \frac{1}{2}\left(\alpha\alpha^* + \beta\beta^* - \gamma\gamma^* - \delta\delta^*\right) , & L^3{}_1 &= \frac{1}{2}\left(\alpha^*\beta - \gamma^*\delta\right) + \text{c.c.} , \\
L^3{}_2 &= \frac{i}{2}\left(\alpha^*\beta - \gamma^*\delta\right) + \text{c.c.} , & L^3{}_3 &= \frac{1}{2}\left(\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \delta\delta^*\right) ,
\end{aligned} \tag{3}$$

where c.c. means the complex conjugate of all the previous terms. Therefore, any complex 2x2 matrix [4-7]:

$$\underset{\sim}{U} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} , \quad \text{Det}\;\underset{\sim}{U} = \alpha\delta - \beta\gamma = 1 \quad , \tag{4}$$

generates a Lorentz matrix via (3).

The following relations, which are not explicitly in the literature, give us directly all the components (3):

$$L^{\mu}{}_{\upsilon} = -\frac{1}{2} \mathrm{U}^{ar}\, \sigma^{\mu}{}_{aj}\sigma_{\upsilon br}\, \mathrm{U}^{\dagger bj} \quad , \quad \mu,\upsilon = 1,2,3$$

$$L^{\mu}{}_{0} = \frac{1}{2}\sigma^{\mu}{}_{jr}\mathrm{Q}^{jr} \quad , \quad \mu = 0,\ldots,3 \, , \qquad L^{0}{}_{\upsilon} = -\frac{1}{2}\sigma_{\upsilon jk}\mathrm{R}^{jk} \quad , \quad \upsilon = 1,2,3 \tag{5}$$

such that:

$$\underset{\sim}{\mathrm{U}}^{\dagger} = \begin{pmatrix} \alpha^{*} & \gamma^{*} \\ \beta^{*} & \delta^{*} \end{pmatrix} \quad , \quad \underset{\sim}{\mathrm{Q}} = \underset{\sim}{\mathrm{U}}\,\underset{\sim}{\mathrm{U}}^{\dagger} \quad , \quad \underset{\sim}{\mathrm{R}} = \underset{\sim}{\mathrm{U}}^{\dagger}\underset{\sim}{\mathrm{U}} \quad , \tag{6}$$

with the Infeld-van der Waerden symbols [8-11]:

$$\left(\sigma^{0}{}_{ab}\right) = \left(\sigma_{0ab}\right) = \underset{\sim}{\mathrm{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \left(\sigma^{1}{}_{ab}\right) = \left(\text{-}\sigma_{1ab}\right) = \sigma_{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\left(\sigma^{2}{}_{ab}\right) = \left(\text{-}\sigma_{2ab}\right) = \text{-}\sigma_{y} = \begin{pmatrix} 0 & \mathrm{i} \\ \text{-i} & 0 \end{pmatrix} \quad , \quad \left(\sigma^{3}{}_{ab}\right) = \left(\text{-}\sigma_{3ab}\right) = \sigma_{z} = \begin{pmatrix} 1 & 0 \\ 0 & \text{-}1 \end{pmatrix} \tag{7}$$

where $\sigma_j, \; j = x, y, z$ are the known Cayley-Sylvester-Pauli matrices [4, 6, 12-14].

The expressions (5) show explicitly a direct relationship between $\underset{\sim}{L}$ and $\underset{\sim}{U}$, which may be useful in applications of spinorial calculus [11] in Minkowski spacetime.

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