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Four Bosons EM Gauge Invariance and EM Flux

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Abstract:

Electromagnetic phenomena is based on electric charge and spin. However, Maxwell equations are just macroscopic. There is a microscopic EM phenomena to be understood. A performance originated from electric charge microscopic behavior. Thus, keeping on mind the two basic EM postulates, which are light invariance and charge conservation, Maxwell is extended to a Four Bosons Electromagnetism. It says that, the macroscopic Maxwell equations does not describe all electromagnetism.

The electric charge physics has been studied microscopically through elementary particle physics. A new perception of EM phenomena emerges based on three interconnected charges $\{+, 0, -\}$ under four intermediated bosons $\{A_\mu, U_\mu, V_\mu^\pm\}$. From Maxwell photon, EM becomes a systemic cooperation between four fields. This quadruplet originates a new electric charge physics. New features for electric charge conservation, exchange, conduction, interaction are derived.

The research is to analyze the Four Bosons Electromagnetism gauge invariance and EM flux. The model is studied under U(1), $SO(2)_{global}$ and $U(1) \times SO(2)_{global}$ symmetries. Two approaches are considered. Based on fields strengths and on fields. A gauge invariant quadruplet physics is obtained under free coefficients conditions.

A nonlinear EM flux appears. Without requiring electric charge as source nonlinear fields work as own sources for flowing spin-1 and spin-0 waves and particles. It flows through a four potential field quadruplet, granular and collective fields strengths. A self contained EM is constituted providing a nonlinear physicality that precedes physical constants as electric charge and medium parameters.

The EM meaning is enlarged and we have to understand on the physical structures generated by this antireductionist nonlinear four bosons microscopic electromagnetism. Determine the corresponding fields blocks which are real and gauge invariants. They are identified as the electromagnetic domains. The Four Bosons EM develops interdependent EM domains. Interlaced physical sectors sharing a common EM energy context. Lagrangian, equations of motion, conservation laws are expressing such domains physics. They correspond to physical sectors where each one contains its own EM energy.

1 Introduction

Electromagnetism is the subject of electric charge physics. Nevertheless at the 20$^{\text{th}}$ century elementary particle physics has discovered a microscopic electric charge behavior beyond Maxwell [1]. QED developments by Dirac [2], and later by Tomonaga [3], Schwinger [4], Feynman [5], Dyson [6], Kӓllen [7], Landau [8] introduced the EM interaction as being intermediated by the photon. However, electric charge physics is not limited by quantum electrodynamics [9]. Scalar electromagnetism [10], vectorial electromagnetism [11], [12] are showing about a microscopic charge exchange beyond usual QED electromagnetism. A feature which incorporates charged particles with different flavors and spins.

A next performance leads to a microscopic electromagnetic model beyond photons. Consider that electric charge transmission requires another intermediate bosons. It is based on three interconnected charges $\{+, 0, -\}$ and four mediators [13]. It extends QED in the sense that electromagnetism is no more restrict to electric charge being intermediated by photons. An approach where weak interaction [14] should be considered as a sector of this enlarged electromagnetism [15].

The four bosons electromagnetic model understands the electric charge with a physicality beyond QED. It provides physical processes accomplished by charge exchange $\Delta Q = 0, \pm 1$. It adds to the photon three more mediators. It yields an electromagnetism intermediated by the fields set $A_{\mu I} = \{A_\mu, U_\mu, V_\mu^\pm\}$. A vectorial bosons quadruplet constituted by A_μ-usual photon, U_μ-massive photon and $V_\mu^\pm$-charged photons duplet. This performance introduces new features to EM energy. The electric charge meaning on conservation, exchange, conduction, interaction is extended.

Analyzing first on the electric charge conservation law, the simplest gauge model to get together these four fields should have an abelian nature. This fact, lead us to insight the following extended abelian symmetry beyond Maxwell,

$$
\begin{aligned}
{A_\mu}' &= A_\mu + k_1 \partial_\mu \alpha, \\
{U_\mu}' &= U_\mu + k_2 \partial_\mu \alpha, \\
V_\mu^{+\prime} &= e^{iq\alpha} V_\mu^+ + k_+ \partial_\mu \alpha, \\
V_\mu^{-\prime} &= e^{-iq\alpha} V_\mu^- + k_- \partial_\mu \alpha.
\end{aligned} \tag{1.1}
$$

Eq.(1.1) is proposing the $U(1) \times SO(2)_{\text{global}}$ symmetry for understanding on the microscopic behavior of the electric charge [13]. Its physical principle is that, due to the fact that microscopically electric charge appears interchanging the set $\{+, 0, -\}$, their physical processes should be in terms of four fields intermediations. Consequently, the corresponding charge conservation requires to associate the gauge parameter to all fields. It derives an electromagnetic model under a common abelian symmetry where the intermediate EM fields work out gathered through a whole .

Eq.(1.1) proposes an antireductionist physical set. A nonlinear abelian model is obtained for the electric charge microscopic physics. It produces a class of Born-Infeld electrodynamics [16], Euler-Heisenberg [17], and others nonlinear electrodynamics (NLED) [18]. However, diversely, it is not an effective theory. Eq.(1.1) means a fundamental nonlinear model. There are more nonlinear effects than to change the Coulomb potential [19]. Based just on the electric charge microscopic behavior it yields four new performances. They are a subatomic EM; EM energy originated from three interchanging charges $\{+, 0, -\}$; the number of EM mediators extended to four fields; and, an EM subdivided in physical regions producing their own EM flux.

Thus there is a new electromagnetic physics to be studied. Firstly, eq.(1.1) stipulates a systemic microscopic electromagnetism where electric charge physics acts through a set $\{+, 0, -\}$ accomplished with four bosons. A transmission based on transverse (spin-1) and longitudinal (spin-0) mediators. The second relevance is that the fields set $\{A_\mu, U_\mu, V_\mu^\pm\}$ identification is open. Actually particle physics contains nearly 250 particles exchanging electric charges [20] where the four bosons quadruplet can be chosen as much by fundamental or by composite particles. As fundamental, it propitiates for electroweak interaction a quadruplet $A_{\mu I} = \{A_\mu, Z_\mu^0, W_\mu^\pm\}$; while as composite particles, for instance, J/Ψ, be interpreted as a charged spin-1 photon, and pions as spin-0 mediators. For axionic electromagnetism, the axion be associate to the scalar sector instead of the usual relationship $\Theta F_{\mu\nu}\tilde{F}^{\mu\nu}$ [21]. By third, EM be reinterpreted as a phenomena with a generic four fields interlaced dynamics interchanging three kinds of charges (positive, negative, zero).

The fourth extension is that this systemic four bosons electromagnetism contains diverse physical regions. Sectors derived from eq.(1.1) which are separately gauge invariant and real. This is the investigation to be done here. Our purpose is to classify the corresponding electromagnetic domain physics. These domains are EM energies developing physical structures with a nonlinear performance which promote an EM flux working as own source. They flow through pieces at the Lagrangian, equations of motion and conservation laws. An energy flux build up by potential fields conglomerates, granular and collective field strengths which couplings precede the physical constants as dielectric constant, magnetic permeability, fine structure constant. Nonlinear domains provide an EM energy more fundamental than electric charge.

Our effort is to understand on the systemic four bosons EM. The paper is structured by grouping the four intermediate bosons through an abelian gauge symmetry. At section 2 and section 3, it studies the $U(1)$ and $SO(2)$ symmetries separately. It analyses and systematizes the physical entities developed by the model as the corresponding fields conglomerates, granular and collective fields strengths. The $U(1) \times SO(2)_{\text{global}}$ symmetry is studied at sections 4 to 7 by two gauge invariance methods. They are based on fields strengths and potential fields respectively. Two complementary approaches to understand on the four bosons EM gauge invariance. Their corresponding gauge invariant blocks are classified and physical sectors distinguished. The EM energies sustained by each domain are considered. At conclusion, the presence of nonlinear four EM fields is considered. An EM based on own fields is a new development for EM physics . It moves EM beyond to photon, electric charge and medium constants. A new frontier is crossed.

2 U(1) symmetry

Considering as origin the presence of Yang-Mills families at $\left(\frac{1}{2}, \frac{1}{2}\right)$ Lorentz group representation, one associates the corresponding fields through the following abelian gauge transformation

$$A_{\mu I} \longrightarrow A'_{\mu I} = A_{\mu I} + k_I \partial_\mu \alpha \tag{2.1}$$

Eq.(2.1) allows the existence of an abelian gauge model involving different potential fields where $A_{\mu I} \equiv \{A_{\mu 1}, ..., A_{\mu N}\}$ means N-potential fields transforming under a same U(1)-group. Given that such fields satisfy the Kamefuchi, O'Raifeartaigh, Salam condition [22] they can be redefined. Thus, in order to get a better transparency on symmetry, one first rewrites the model in terms of the so-called constructor basis $\{D_\mu, X_{\mu i}\}$ basis. It gives,

$$\begin{aligned} D_\mu &= \sum_{I=1}^{N} A_{\mu I} \\ X_{\mu i} &= A_{\mu\, i+1} - A_{\mu\, i}; i = 2...N \end{aligned} \tag{2.2}$$

which yields the following gauge transformations

$$\begin{aligned} D_\mu &\longrightarrow D'_\mu = D_\mu + N\partial_\mu \alpha \\ X_{\mu i} &\longrightarrow X'_{\mu i} = X_{\mu i} \end{aligned} \tag{2.3}$$

Nevertheless, $\{D, X_i\}$ basis is not the physical basis. The physical fields are those which diagonalize the equations of motion (physical masses being the poles of two-point Green functions). Thus, in order to diagonalize the transverse sector we have to introduce a matrix Ω [23]. Consequently, the physical basis $\{G_I\}$ is expressed as

$$D_\mu = \Omega_{1I} G^I_\mu, \qquad X_{\mu i} = \Omega_{iI} G^I_\mu \tag{2.4}$$

where the Ω matrix is a function of free coefficients associated to the Lagrangian. It contains the invertible property

$$\Omega_{IK} \Omega^{-1}_{KJ} = \delta_{IJ} \tag{2.5}$$

Writing down the gauge transformation in terms of physical fields, one gets

$$G_{\mu I}(x) \longrightarrow G'_{\mu I}(x) = G_{\mu I}(x) + \Omega^{-1}_{I1} \partial_\mu \alpha(x) \tag{2.6}$$

Thus, it is in terms of these $\{G_{\mu\, I}\}$ fields that the corresponding physical laws will be written. They will describe the electromagnetic fields to be measured. Differently from the usual gauge theory, the model introduces a variety of physical entities. It yields three entities to be understood at this physical basis. They are the potential fields conglomerates, granular and collective fields strengths. By first, it contains potential fields conglomerates with a physicity to be understood. There are (N-1) relationships $\Omega_{iJ} G^J_\mu$ derived from (2.4) and (2.6), and also, the $\gamma_{[IJ]} G^J_\mu$ and $\gamma_{(IJ)} G^J_\mu$ combinations.

Next, we should study on the possible gauge invariants fields strengths. Separating at antisymmetric and symmetric sectors, one gets by first the granular antisymmetric field strengths $G^I_{\mu\nu}$:

$$G^I_{\mu\nu} = \partial_\mu G^I_\nu - \partial_\nu G^I_\mu \tag{2.7}$$

Replacing eq.(2.6) in (2.7), it yields

$$G^I_{\mu\nu}{}' = G^I_{\mu\nu}$$

Follows on the collective antisymmetric field strengths, $z_{[\mu\nu]}$:

$$z'_{[\mu\nu]} = \gamma_{[IJ]} G^I_\mu{}' G^J_\nu{}' \tag{2.8}$$

From eq.(2.5) we have

$$z'_{[\mu\nu]} = \gamma_{[ij]} \Omega^i_I \Omega^j_J G^I_\mu G^J_\nu + \cancelto{0}{\gamma_{[ij]} \Omega^i_I \Omega^j_J \Omega^{-1J}_1} G^I_\mu \partial_\nu \alpha + \cancelto{0}{\gamma_{[ij]} \Omega^i_I \Omega^j_J \Omega^{-1I}_1} G^J_\mu \partial_\nu \alpha +$$
$$+ \cancelto{0}{\gamma_{[ij]} \Omega^i_I \Omega^j_J \Omega^{-1I}_1 \Omega^{-1J}_1} \partial_\mu \alpha \partial_\nu \alpha$$

which gives under (2.5)

$$z'_{[\mu\nu]} = z_{[\mu\nu]} \tag{2.9}$$

Others collective fields gauge invariants under eq(2.6) are

$$s_{\mu\nu} = s_{IJ} G_\mu^I G_\nu^J, \quad s_{IJ} = \sum_{i,j=2}^{N} \Omega_{iI} \Omega_{jJ}, \tag{2.10}$$

$$t_{\mu\nu} = t_{IJ} G_\mu^I G_\nu^J, \quad t_{IJ} = \sum_{K,L=1}^{N} \gamma_{KI} \gamma_{LJ}, \tag{2.11}$$

$$u_{\mu\nu} = u_{IJ} G_\mu^I G_\nu^J, \qquad u_{IJ} = \sum_{i,K} \Omega_{iI} \gamma_{KJ} \tag{2.12}$$

which yields the extended collective field

$$e_{\mu\nu} = z_{\mu\nu} + t_{\mu\nu} + s_{\mu\nu} + u_{\mu\nu} \tag{2.13}$$

where

$$e_{\mu\nu} = e_{IJ} G_\mu^I G_\nu^J \tag{2.14}$$

with

$$e_{IJ} = \gamma_{IJ} + s_{IJ} + t_{IJ} + u_{IJ} \tag{2.15}$$

where

$$e_{IJ} = \sum_{K,L=1}^{N} \sum_{\substack{i,j \\ k,l=2}}^{N} \left(1 + \gamma_{ij} + \gamma_{kj} \Omega_{kK}\right) \Omega_{iI} \Omega_{jJ} + \\ + \gamma_{ki} \gamma_{lj} \Omega_{kK} \Omega_{lL} \Omega_{iI} \Omega_{jJ} \tag{2.16}$$

Thus, the generic antisymmetric gauge invariant tensor becomes,

$$Z_{[\mu\nu]} = a_I G_{\mu\nu}^I + e_{[\mu\nu]} \tag{2.17}$$

where

$$a_I = d\Omega_{1I} + \alpha_i \Omega_{iI} \tag{2.18}$$

and

$$e_{[\mu\nu]} = z_{[\mu\nu]} + t_{[\mu\nu]} + s_{[\mu\nu]} + u_{[\mu\nu]} = e_{[IJ]} G_\mu^I G_\nu^J \tag{2.19}$$

with

$$e_{[IJ]} = \gamma_{[IJ]} + \sum_{K,L=1}^{N} \gamma_{K[I} \gamma_{LJ]} + \sum_{i,j=2}^{N} \Omega_{[iI} \Omega_{jJ]} + \sum_{I,K}^{N} \Omega_{i[I} \gamma_{KJ]} \tag{2.20}$$

Eq (2.17) provides $(N+4)$ gauge invariant fields strengths. Appendix A shows on coefficients $\alpha, \alpha_i, \beta_i, \rho_i$ and σ on origins.

For symmetric sector, we have a similar $Z_{(\mu\nu)}$:

$$Z_{(\mu\nu)} = \beta_I S^I_{\mu\nu} + \rho_I g_{\mu\nu} S^{\alpha I}_\alpha + e_{(\mu\nu)} + g_{\mu\nu} e^\alpha_\alpha \tag{2.21}$$

where

$$\beta_I = \beta_i \Omega^i_I, \quad \rho_I = \rho_i \Omega^i_I \tag{2.22}$$

and

$$e_{(\mu\nu)} = z_{(\mu\nu)} + s_{(\mu\nu)} + t_{(\mu\nu)} + u_{(\mu\nu)} \tag{2.23}$$

or

$$e_{(\mu\nu)} = e_{(IJ)} G_{\mu I} G_{\nu J} \tag{2.24}$$

with

$$e_{(IJ)} = \gamma_{(IJ)} + \sum_{K,L=1}^{N} \gamma_{K(I}\gamma_{LJ)} + \sum_{i,j=2}^{N} \Omega_{(iI}\Omega_{jJ)} + \sum_{i,k}^{N} \Omega_{i(I}\gamma_{KJ)} \tag{2.25}$$

and

$$e^\alpha_\alpha = \omega^\alpha_\alpha + t^\alpha_\alpha + s^\alpha_\alpha + u^\alpha_\alpha \tag{2.26}$$

$$e^\alpha_\alpha = \left(\tau_{IJ} + \tau_{KI}\tau_{LJ} + \Omega_{iI}\Omega_{jJ} + \Omega_{iI}\delta_{KJ}\right) G_{\alpha I} G^\alpha_J \tag{2.27}$$

Studying on gauge invariance, the first term is given by

$$\beta_I S'^I_{\mu\nu} = \beta_I \left(\partial_\mu G^I_\nu + \partial_\nu G^I_\mu\right) + 2\cancelto{0}{\beta_i \Omega^i_I \Omega_1^{-1I}} \partial_\mu \partial_\nu \alpha, \tag{2.28}$$

$$\beta_I S'^I_{\mu\nu} = \beta_I S^I_{\mu\nu} \tag{2.29}$$

which means that $S^I_{\mu\nu}$ alone is not gauge invariant, but $\beta_I S^I_{\mu\nu}$ is. Writing that $\beta_I S^I_{\mu\nu} = \beta_i X^i_{(\mu\nu)}$, one notices $\beta_I S^I_{\mu\nu}$ is gauge invariant and constituted by $(N-1)$ gauge invariant terms. There is also the relationship

$$\left(\Omega_{iI} S^I_{\mu\nu}\right)' = \Omega_{iI} S^I_{\mu\nu} \tag{2.30}$$

Given $\rho_I S^{\alpha I}_\alpha = 2\rho_I \partial_\alpha G^{\alpha I}$, one gets

$$\begin{aligned} \left(\rho_I S^{\alpha I}_\alpha\right)' &= 2\rho_I \partial_\alpha G^{\alpha I} + 2\partial_\alpha \cancelto{0}{\left(\rho_i \Omega^i_I \Omega_1^{-1I} \partial^\alpha \alpha\right)} \\ \left(\rho_I S^{\alpha I}_\alpha\right)' &= \rho_I S^{\alpha I}_\alpha \end{aligned} \tag{2.31}$$

For the collective symmetric field strength, $z_{(\mu\nu)}$:

$$\begin{aligned} z'_{(\mu\nu)} &= \left(\gamma_{(IJ)} G^I_\mu G^J_\nu\right)' \\ z'_{(\mu\nu)} &= z_{(\mu\nu)} + \gamma_{(ij)} \cancelto{0}{\Omega^i_I \Omega^j_J \Omega_1^{-1J}} G^I_\mu \partial_\nu \alpha + \gamma_{(ij)} \cancelto{0}{\Omega^i_I \Omega^j_J \Omega_1^{-1I}} G^J_\nu \partial_\mu \alpha + \\ &+ \gamma_{(ij)} \cancelto{0}{\Omega^i_I \Omega^j_J \Omega_1^{-1I} \Omega_1^{-1J}} \partial_\mu \alpha \partial_\nu \alpha \end{aligned} \tag{2.32}$$

and so, given that $\Omega_I^i \Omega_1^{-1I} = \delta_1^i = 0$, one proves that

$$z'_{(\mu\nu)} = z_{(\mu\nu)}$$

Studying the similar term ω_α^α:

$$\omega_\alpha^{\alpha\prime} = \tau_{(IJ)} G_\alpha^{I\prime} G^{\alpha I\prime}$$
$$\omega_\alpha^{\alpha\prime} = \tau_{(ij)} G_\alpha^I G^{\alpha J} + 2\tau_{(ij)} \Omega_I^i \Omega_J^j \Omega_1^{-1I} G^{\alpha J} \partial_\alpha \alpha + \tau_{(ij)} \Omega_I^i \Omega_J^j \Omega_1^{-1I} \Omega_1^{-1J} \partial_\alpha \alpha \partial^\alpha \alpha$$

One gets,

$$\omega_\alpha^{\alpha\prime} = \omega_\alpha^\alpha \tag{2.33}$$

Consequently, one obtains the generic gauge invariant symmetric field strength given by eq (2.21)

$$Z'_{(\mu\nu)} = Z_{(\mu\nu)} \tag{2.34}$$

which contains 10 gauge invariants fields strengths.

Notice also the generic relationship

$$\Omega_i^I z'_{\mu\nu,IJ} = \Omega_i^I z_{\mu\nu,IJ} \tag{2.35}$$

Similarly, for extended $e_{\mu\nu}$

Thus, given the fields set $\{G_{\mu I}\}$, one derives physical entities associated to gauge invariance and reality. In terms of fields association there are $(3N-1)$ field conglomerates $\Omega_{iI} G_\mu^I$, $\Omega_{[IJ]} G_\mu^J$, $\Omega_{(IJ)} G_\mu^J$. In terms of fields strengths, the antisymmetric sector contains N-$G_{\mu\nu I}$ granular terms and four collectives $e_{[\mu\nu]}$. For the symmetric sector, one gets two granular fields strengths $\beta_I S_{\mu\nu}^I = \beta_i X_{(\mu\nu)}^i$, $\rho_I S_\alpha^{I\alpha} = \rho_i X_\alpha^{i\alpha}$ and four collective fields $e_{(\mu\nu)}$ and ω_α^α, t_α^α, s_α^α, u_α^α. The model provides $(N+4)$ granular and collective fields strenghts. In total, $(4N+13)$ gauge invariant structures which will be responsible for the $U(1)$-electromagnetic flux.

A new physicality obtained from fields theories associating different fields in a same group is the meaning of circumstance. Appendix A relates on the matrix Ω_{IJ} parameters. They are $d, \alpha_i, \beta_i, \rho_i, \gamma_{ij}, \tau_{ij}, m_{ij}, \eta$ which are free coefficients that can take any value without violating gauge invariance. They generate a so-called volume of circumstances. A total of $\frac{1}{4}[3N^4 - 8N^3 + 13N^2 - 12N + 8]$ free coefficients which stipulate the circumstantial properties for the model [24]. Preserving gauge invariance circumstantial conditions are derived. Variables embedded at Ω_{IJ} matrices are responsible for the opportunities that the model provides. For instance, individual fields strengths are obtained by adjusting parameters.

Thus, analyzing individually $\beta_I S_{\mu\nu I}$ (I is not a repeated indice), it yields the circumstantial gauge invariant condition

$$\beta_I S'_{\mu\nu I} = \beta_I S_{\mu\nu I} \quad \text{for} \quad \beta_i \Omega_I^i \Omega_{iI}^{-1} = 0 \tag{2.36}$$

Also, one gets individually

$$\rho_I S_{\alpha\,I}^{\alpha} (\text{I is not repeated over}) \tag{2.37}$$

under the circumstantial condition

$$\rho_i \Omega_I^i \Omega_{iI}^{-1} = 0 \tag{2.38}$$

A further development is to consider individual collective fields. That ones where just two-to-two fields are associated singly. Separating

$$z_{\mu\nu} = \sum_{I,J=1}^{N} z_{\mu\nu,IJ} \tag{2.39}$$

we have,

$$z_{\mu\nu,IJ} = \gamma_{IJ} G_{\mu\,I} G_{\nu\,J} \text{ (I,J not repeated indices)} \tag{2.40}$$

Studying on its gauge invariance, where eq (2.40) represent the composition with individual two-by-two collective fields, it gives

$$z^{'}_{\mu\nu,IJ} = z_{\mu\nu,IJ} + \gamma_{IJ}\Omega^{-1}_{I1} G_{\nu J}\partial_\mu\alpha + \gamma_{IJ}\Omega^{-1}_{J1} G_{\mu I}\partial_\nu\alpha + \gamma_{IJ}\Omega^{-1}_{I1}\Omega^{-1}_{J1}\partial_\mu\alpha\partial_\nu\alpha \tag{2.41}$$

Thus, considering the antisymmetric decomposition, it yields

$$\begin{aligned} z^{'}_{[\mu\nu],IJ} &= z_{[\mu\nu],IJ} + \gamma_{[IJ]}\Omega^{-1}_{I1}\left(G_{\nu J}\partial_\mu\alpha - G_{\mu I}\partial_\nu\alpha\right) + \\ &+ \gamma_{[IJ]}\cancelto{0}{\Omega^{-1}_{I1}\Omega^{-1}_{J1}}\partial_\mu\alpha\partial_\nu\alpha \end{aligned} \tag{2.42}$$

with the following condition for every individual collective fields be invariant

$$\gamma_{[ij]}\Omega^{i}_{I}\Omega^{j}_{J}\Omega^{-1}_{I} = 0, \tag{2.43}$$

where eq.(2.43) means the circumstantial gauge symmetry condition for

$$z^{'}_{[\mu\nu],IJ} = z_{[\mu\nu],IJ} \tag{2.44}$$

Analyzing on individual symmetric two-by-two collective fields, we have

$$\begin{aligned} z^{'}_{(\mu\nu),IJ} &= z_{(\mu\nu),IJ} + \gamma_{(IJ)}\Omega^{-1}_{I1}\left(G_{\nu J}\partial_\mu\alpha + G_{\mu I}\partial_\nu\alpha\right) + \\ &+ \gamma_{(IJ)}\Omega^{-1}_{I1}\Omega^{-1}_{J1}\partial_\mu\alpha\partial_\nu\alpha \end{aligned} \tag{2.45}$$

which yields,

$$z^{'}_{(\mu\nu),IJ} = z_{(\mu\nu),IJ} \tag{2.46}$$

under circumstantials conditions

$$\gamma_{(ij)}\Omega^{i}_{I}\Omega^{j}_{J}\Omega^{-1}_{I1} = 0 \quad \text{and} \quad \gamma_{(ij)}\Omega^{i}_{I}\Omega^{j}_{J}\Omega^{-1}_{I1}\Omega^{-1}_{J1} = 0 \tag{2.47}$$

Similar conditions are obtained for

$$e'_{[\mu\nu]IJ} = e_{[\mu\nu],IJ} \tag{2.48}$$

and

$$e'_{(\mu\nu)IJ} = e_{(\mu\nu),IJ} \tag{2.49}$$

Thus there are two situations to be considered for understanding the fields strengths physicity. They are natural and circumstantial symmetries. For instance, while $G_{\mu\nu I}$, $\beta_I S^I_{\mu\nu}$ are naturally gauge invariant, however, a term as $\beta_1 S_{\mu\nu 1}$ may be gauge invariant under eq.(2.37) restriction. Consequently, under the natural symmetry there are $(N+8)$ fields strengths; under circumstances view, $(2N+16)$ fields strengths are viable.

The next sector to be explored is the mass term. From $\{D, X_i\}$ basis, we can write down the gauge invariant mass term

$$\mathcal{L}_m = m^2_{ij} X^i_\mu X^{j\,\mu} \tag{2.50}$$

Inserting eq.(2.4), one gets at the physical basis $\{G^I_\mu\}$ the expression

$$\mathcal{L}_m = m^2_{II} G^I_\mu G^{\mu I} \tag{2.51}$$

where

$$m^2_{II} = m^2_{ij}\Omega^i_I \Omega^j_I, \quad \text{and} \quad i \in \{2, ..., N\} \qquad I \in \{1, ..., N\}$$

We should now verify on the mass term gauge invariance

$$\begin{aligned} \mathcal{L}'_m &= m^2_{II} G^I_\mu{}' G^{\mu I}{}' \\ \mathcal{L}'_m &= \mathcal{L}_m + 2m^2_{(ij)}\Omega^i_I \Omega^j_I \Omega^{-1I}_1 G^{\mu I} \partial_\mu \alpha + \\ &+ m^2_{(ij)}\Omega^i_I \Omega^j_I \Omega^{-1I}_1 \Omega^{-1I}_1 \partial_\mu \alpha \partial_\nu \alpha \end{aligned} \tag{2.52}$$

Using eq.(2.5), it yields

$$\mathcal{L}'_m = \mathcal{L}_m \tag{2.53}$$

Eq. (2.53) provides a result which brings an alternative to the Higgs mechanism [25].

Another aspect is to analyze on each mass term separately

$$\mathcal{L}_m = m^2_{II} G_{\mu I} G^\mu_I \text{ (I is not a summatory indice)} \tag{2.54}$$

For instance, take $\mathcal{L}_m = m^2_{22} G_{\mu 2} G^\mu_2$ and ask on its gauge invariance. Consider

$$\mathcal{L}'_{m_{II}} = m^2_{ij}\Omega^i_I \Omega^j_I G'_{\mu I} G^{\mu'}_I$$

it gives,

$$\begin{aligned} \mathcal{L}'_m &= \mathcal{L}_m + 2m^2_{ij}\Omega^i_I \Omega^j_I \Omega^{-1}_{I1} G_{\mu I} \partial^\mu \alpha + \\ &+ m^2_{ij}\Omega^i_I \Omega^j_I \Omega^{-1}_{I1} \Omega^{-1}_{I1} \partial_\mu \alpha \partial^\mu \alpha \end{aligned} \tag{2.55}$$

which relates two conditions for eq.(2.55) be constitued by terms individually gauge invariants

$$m_{ij}^2 \Omega_I^i \Omega_I^j \Omega_{I1}^{-1} = 0 \quad \text{and} \quad m_{ij}^2 \Omega_I^i \Omega_I^j \left(\Omega_{I1}^{-1}\right)^2 = 0 \tag{2.56}$$

Thus, the model provides a directive and a circumstantial physics derived from eq.(2.6). Besides the natural $U(1)$-kinetic sector, one has to understand on the model circumstances. Investigate how far these free coefficients satisfy conditions as eqs.(2.43), (2.36), (2.38), (2.47), (2.56) without losing any Lagrangian term. This mean without cancelling the basic physics, which means to propagate spin-1 and spin-0 and interactions without violating gauge invariance.

Next step is to systematize the territories where the U(1)-EM energy flows. The EM domains conducting the EM energy. The physical sectors in the Lagrangian where the EM flux propagates potential fields conglomerates granular and collective fields. For this, the Lagrangian is separated as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{2.57}$$

where

$$\mathcal{L}_0 = \mathcal{L}_K + \mathcal{L}_m + \mathcal{L}_{GF} \tag{2.58}$$

Considering the kinetic sector

$$\mathcal{L}_K = \mathcal{L}_K^A + \mathcal{L}_K^S \tag{2.59}$$

one gets,

$$\mathcal{L}_K^A = a_I^2 G_{\mu\nu}^I G^{\mu\nu I} \tag{2.60}$$

and

$$\mathcal{L}_K^S = \beta_I \beta_J S_{\mu\nu}^I S^{\mu\nu J} \delta_{IJ} + 2\left(\beta_I \rho_J + 2\rho_I \rho_J\right) S_\alpha^{\alpha I} S_\beta^{\beta J} \tag{2.61}$$

which terms are separately gauge invariant. The mass term is given by eq.(2.51)

A further aspect on gauge theory is the gauge fixing. Considering the $\{D, X_i\}$ basis, the Lorentz gauge can be achieved by the following term

$$\mathcal{L}_{GF} = \frac{1}{2\xi}\left(\partial_\mu\left(D^\mu + \sigma_i X^{\mu i}\right)\right)^2 \tag{2.62}$$

which gives at physical basis:

$$\mathcal{L}_{GF} = \xi_{(IJ)}\left(\partial_\mu G^{\mu I}\right)\left(\partial_\nu G^{\nu J}\right) \tag{2.63}$$

where

$$\xi_{(IJ)} = \frac{1}{2\xi}\left[\Omega_{1I}\Omega_{1J} + \sigma_i\sigma_j\Omega_I^i\Omega_J^j + 2\sigma_i\Omega_{1I}\Omega_J^i\right] \tag{2.64}$$

We should also add to the Lagrangian the entitled semi-topological term [26]. It gives,

$$\begin{aligned}\mathcal{L}^{st} = Z_{\mu\nu}\tilde{Z}^{\mu\nu} = \partial_\mu\{2\epsilon^{\mu\nu\rho\sigma}\alpha_I G_\nu^I Z_{[\rho\sigma]}\} + \\ +\alpha_I e_{[KL]}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\rho^K \partial_\nu G_\sigma^L + \\ +2\alpha_K e_{[IJ]}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\nu^J \partial_\rho G_\sigma^K + \\ +e_{[IJ]}e_{[KL]}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\nu^J G_\rho^K G_\sigma^L\end{aligned} \tag{2.65}$$

where the first three terms build up a gauge invariant block and the last another one.

The interaction sector becomes

$$\mathcal{L}_I = \mathcal{L}_3^I + \mathcal{L}_4^I + \mathcal{L}_3^{\text{st}} + \mathbf{L}_4^{st} \tag{2.66}$$

The three-vertex contribution can be split as

$$\mathcal{L}_3^I = \mathcal{L}_3^A + \mathcal{L}_3^S + \mathcal{L}_3^{st,A} \tag{2.67}$$

where each of them is separately gauge invariant. They are

$$\mathcal{L}_3^A = 2a_I G_{\mu\nu}^I e^{[\mu\nu]} \tag{2.68}$$

and

$$\mathcal{L}_3^S = 2\beta_I S_{\mu\nu}^I e^{(\mu\nu)} + 2\left(\beta_I + 5\rho_I\right) S_\alpha^{\alpha I}(e_\beta^\beta) \tag{2.69}$$

and

$$\mathcal{L}_3^{st} = \theta_{IJK}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\nu^J \partial_\rho G_\sigma^K \tag{2.70}$$

where

$$\theta_{IJK} = -\alpha_I\gamma_{[JK]} + 2\alpha_K\gamma_{[IJ]} \tag{2.71}$$

The four-vertex contribution is

$$\mathcal{L}_4^I = \mathcal{L}_4^A + \mathcal{L}_4^S + +\mathcal{L}_4^{st,A} \tag{2.72}$$

where each of them is separately gauge invariant. They are

$$\mathcal{L}_4^A = e_{[\mu\nu]}e^{[\mu\nu]} \tag{2.73}$$

and

$$\mathcal{L}_4^S = e_{(\mu\nu)}e^{(\mu\nu)} + 2e_\alpha^\alpha \omega_\beta^\beta + 6(e_\alpha^\alpha + e_{\beta)}^\beta) \tag{2.74}$$

and

$$\mathcal{L}_4^{st,A} = e_{[IJ]}e_{[KL]}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\nu^J G_\rho^K G_\sigma^L \tag{2.75}$$

Thus the so-called U(1)-electromagnetic domains derived from the four bosons EM model are regions obtained by cutting off the Lagrangian in real and gauge invariants sectors. They correspond to the physical regions where the EM flux flows. From kinetic sector, there are: N-domains from $\mathcal{L}_K^A$ and $N(2N+1)$ from $\mathcal{L}_K^S$; one-from mass term. From three-vertex sector: N-from $\mathcal{L}_3^A$ and $4N$-from $\mathcal{L}_3^S$. From four-vertex terms: one-from $\mathcal{L}_4^A$ and three-from $\mathcal{L}_4^S$. From semi-topological term: one from $\mathcal{L}_3^{\mathbf{st}A}$ and other from $\mathcal{L}_4^{\mathrm{st}A}$. In total the model stipulates $(4N^2+4N+6)$ interconnected physical regions or domains. Through them, it will flow an EM energy with $(3N-1)$ fields conglomerates and $(2N+6)$ fields strengths.

Summarizing, from the $U(1)$-gauge symmetry given by (2.1), one develops at eq.(2.57) an antireductionist non-linear abelian Lagrangian. This Lagrangian contains different real and gauge invariant blocks to be studied. Each of them will be responsible for physical sectors identified as $U(1)$-electromagnetic domains. They are territories where the EM flux makes a specific physics by flowing its corresponding fields conglomerates and fields strengths. A physics which energy flux does not depend on electric charge and medium physical constants.

The corresponding N-equations of motion for every $G_{\mu I}$ fields are:

$$\partial_\mu\Big(a_I Z^{[\mu\nu]} + \eta a_I \tilde{Z}^{[\mu\nu]} + \beta_I Z^{(\mu\nu)} + \rho_I g^{\mu\nu} Z^{(\alpha}{}_{\alpha)}\Big) + \frac{1}{2}m_I^2 G_I^\nu = J_I^\nu(G) \tag{2.76}$$

with

$$J_I^\nu G = \gamma_{[IJ]}G_\mu^J Z^{[\mu\nu]} + \eta\gamma_{[IJ]}G_\mu^J \tilde{Z}^{[\mu\nu]} + \gamma_{(IJ)}G_\mu^J Z^{(\mu\nu)} + \tau_{(IJ)}G_\mu^J Z^{(\mu}{}_{\mu)} \tag{2.77}$$

Eq (2.76) builds up the $\{\frac{1}{2},\frac{1}{2}\}$ fields set whole dynamics derived from eq (2.6). It contains a system with N-coupled equations. Carrying granular and collective fields strenghts and fields conglomerates. Physically, it produces for every $G_{\mu I}$ field a dynamics where the LHS contains five physical entities and the RHS other four non-linear physical sources. Each term is separately gauge invariant and covariant. Notice that the RHS current is a coupling between fields conglomerates and fields strengths.

For expressing the spin-1 and spin-0 being propagated by the above N-equations of motion, one takes the projection operators, $\Theta_{\mu\nu}$ and $\omega_{\mu\nu}$. It yields,

Spin-1:

$$\partial_\mu\Big((a_I+\beta_I)\,Z^{[\mu\nu]} + \eta a_I \tilde{Z}^{[\mu\nu]}\Big) + \frac{1}{2}m_I^2 G_{I\,T}^\nu = J_{I\,T}^\nu(G) \tag{2.78}$$

Spin-0:

$$\partial^\nu\Big((\rho_I+\beta_I)\,Z^{(\alpha}{}_{\alpha)}\Big) + \frac{1}{2}m_I^2 G_{I\,L}^\nu = J_{I\,L}^\nu(G) \tag{2.79}$$

Adding to fields equations we have the Bianchi identities to be considered. First is the N-granular Bianchi identities. They are

$$\partial_\mu G^I_{\nu\rho} + \partial_\nu G^I_{\rho\mu} + \partial_\rho G^I_{\mu\nu} = 0 \tag{2.80}$$

The collective antisymmetric Bianchi identity is

$$\begin{aligned}&\partial_\mu z_{[\nu\rho]} + \partial_\rho z_{[\mu\nu]} + \partial_\nu z_{[\rho\mu]} = \\ &= \gamma_{[IJ]} G^I_\mu G^J_{\rho\nu} + \gamma_{[IJ]} G^I_\nu G^J_{\mu\rho} + \gamma_{[IJ]} G^I_\rho G^J_{\nu\mu}\end{aligned} \tag{2.81}$$

The collective symmetric Bianchi identity is

$$\begin{aligned}&\partial_\mu z_{(\nu\rho)} + \partial_\rho z_{(\mu\nu)} + \partial_\nu z_{(\rho\mu)} = \\ &= \gamma_{(IJ)} G^I_\mu G^J_{\rho\nu} + \gamma_{(IJ)} G^I_\nu G^J_{\mu\rho} + \gamma_{(IJ)} G^I_\rho G^J_{\nu\mu}\end{aligned} \tag{2.82}$$

and, also

$$\partial_\mu z^{\nu)}_{(\nu} + 2\partial_\nu z^{\nu)}_{(\mu} = \gamma_{(IJ)} G^I_\mu S^{\nu J}_\nu + 2\gamma_{(IJ)} G^I_\nu S^{\nu J}_\mu \tag{2.83}$$

Similarly to eqs (2.81), (2.82), (2.83), one gets for $s_{\mu\nu}, t_{\mu\nu}$ and $u_{\mu\nu}$. It yields equations where γ_{IJ} are replaced by e_{IJ} coefficients at eq (2.15) . At Appendix D the circumstantial Bianchi identities for individual two-by-two collective fields are studied.

3 SO(2) symmetry

Next step is to extend the U(1) model to four bosons fields model. Consider the quadruplet $A_{\mu I} \equiv \{A_\mu, U_\mu, V^{\pm}_\mu\}$ as the condition for transmitting the electric charge set $\{+, 0, -\}$. A configuration where two of them are charged. Then, we should remember that particle quantum numbers as mass, charge, spin and others must be derived from symmetry principles. Therefore, instead of introducing electric charge heuristically, as Maxwell did in the past, we should do now based on symmetry. As an additive quantum number the electric charge be associated to a commutative group. The simplest is the abelian case.

Considering SO(2) symmetry,

$$R(\theta) = \begin{pmatrix} cos\theta & sen\theta \\ -sen\theta & cos\theta \end{pmatrix} \tag{3.1}$$

or

$$R(\theta) = e^{i\theta\Sigma}, \quad \Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma^2 = 1 \tag{3.2}$$

one gets that, eq.(3.2) shows a commutative abelian group

$$R(\theta_{1,2}) = \begin{pmatrix} cos\theta_{1,2} & sen\theta_{1,2} \\ -sen\theta_{1,2} & cos\theta_{1,2} \end{pmatrix}$$
$$R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2) = R(\theta_2+\theta_1) = R(\theta_2)R(\theta_1)$$
$$e^{-i\Sigma\theta_1}e^{-i\Sigma\theta_2} = e^{-i(\Sigma\theta_1+\Sigma\theta_2)} = e^{-i\Sigma\theta_2}e^{-i\Sigma\theta_1} \tag{3.3}$$

where eq.(3.1) depends on just one parameter, which means dim $SO(2) = 1$

Thus, we can introduce from first symmetry principles the electric charge meaning based on the abelian $SO(2)$ rotation,

$$\begin{pmatrix} G'_{\mu 3} \\ G'_{\mu 4} \end{pmatrix} = \begin{pmatrix} \cos(q\alpha) & \sin(q\alpha) \\ -\sin(q\alpha) & \cos(q\alpha) \end{pmatrix} \begin{pmatrix} G_{\mu 3} \\ G_{\mu 4} \end{pmatrix} \tag{3.4}$$

Eq.(3.4) transforms granular fields strengths as

$$G^{3}_{\mu\nu}{}' = G^{3}_{\mu\nu}\cos(q\alpha) + G^{4}_{\mu\nu}\sin(q\alpha)$$
$$G^{4}_{\mu\nu}{}' = -G^{3}_{\mu\nu}\sin(q\alpha) + G^{4}_{\mu\nu}\cos(q\alpha) \tag{3.5}$$

and

$$S^{3}_{\mu\nu}{}' = \cos(q\alpha)S^{3}_{\mu\nu} + \sin(q\alpha)S^{4}_{\mu\nu}$$
$$S^{4}_{\mu\nu}{}' = -\sin(q\alpha)S^{3}_{\mu\nu} + \cos(q\alpha)S^{4}_{\mu\nu}$$

At Appendix B one studies a list of SO(2) invariants.

A further identification with electric charge is to redefine SO(2) symmetry in terms of $V^{\pm}_{\mu}$ fields. For a more explicit vision on electric charge presence, one makes the basis change $\{G_{\mu 3}, G_{\mu 4}\}$ for $\{V^{+}_{\mu}, V^{-}_{\mu}\}$. Defining $G^{3}_{\mu} = \frac{1}{\sqrt{2}}\left(V^{+}_{\mu} + V^{-}_{\mu}\right), \quad G^{4}_{\mu} = \frac{i}{\sqrt{2}}\left(V^{+}_{\mu} - V^{-}_{\mu}\right)$, it gives the SO(2) global transformations

$$V^{\pm'}_{\mu} = e^{\pm iq\alpha}V^{\pm}_{\mu} \tag{3.6}$$

which yields the granular fields strengths

$$V^{\pm'}_{\mu\nu} = \partial_\mu V^{\pm'}_{\nu} - \partial_\nu V^{\pm'}_{\mu} = e^{\pm iq\alpha}V_{\mu\nu} \tag{3.7}$$

For collective fields, from Appendix C, it yields the following $SO(2)$ invariants

$$z_{[\mu\nu]} = \gamma_{[12]}\left(G^{1}_{\mu}G^{2}_{\nu} - G^{1}_{\nu}G^{2}_{\mu}\right) + \gamma_{[34]}\left(G^{3}_{\mu}G^{4}_{\nu} - G^{4}_{\mu}G^{3}_{\nu}\right) \tag{3.8}$$

and

$$z_{(\mu\nu)} = \gamma_{(11)} G^1_\mu G^1_\nu + \gamma_{(22)} G^2_\mu G^2_\nu + \gamma_{(12)} \left(G^1_\mu G^2_\nu + G^2_\mu G^1_\nu \right) + \\ + \gamma_{(33)} \left(G^3_\mu G^3_\nu + G^4_\mu G^4_\nu \right) \tag{3.9}$$

Given that eqs (3.8) and (3.9) explicit two collective fields $SO(2)$ invariants we should understand their correspondent Bianchi Identities. For the antisymmetric part, eq (3.8) is written at Appendix C as

$$z_{[\mu\nu]} = 2 \overset{[12]}{z}_{[\mu\nu]} + 2 \overset{[+-]}{z}_{[\mu\nu]} \tag{3.10}$$

It yields the following Bianchi identity

$$\partial_\mu z_{[\nu\rho]} + \partial_\nu z_{[\rho\mu]} + \partial_\rho z_{[\mu\nu]} = \\ = \gamma_{[12]} \{ U_\mu F_{\nu\rho} + U_\nu F_{\rho\mu} + U_\rho F_{\mu\nu} - A_\mu U_{\nu\rho} - A_\nu U_{\rho\mu} - A_\rho U_{\mu\nu} \} + \\ + 2\gamma_{[34]} Im \{ V^+_\mu V^-_{\nu\rho} + V^-_\nu V^+_{\rho\mu} + V^-_\rho V^+_{\mu\nu} \} \tag{3.11}$$

For the symmetric part, eq (3.9), Appendix C writes

$$z_{(\mu\nu)} = \overset{(11)}{z}_{\mu\nu} + \overset{(22)}{z}_{\mu\nu} + 2 \overset{(12)}{z}_{(\mu\nu)} + 2 \overset{+-3}{z}_{(\mu\nu)} \tag{3.12}$$

which yields,

$$\partial_\mu z_{(\nu\rho)} + \partial_\nu z_{(\rho\mu)} + \partial_\rho z_{(\mu\nu)} = \gamma_{(11)} \left(A_\mu S^1_{\nu\rho} + A_\nu S^1_{\rho\mu} + A_\rho S^1_{\mu\nu} \right) + \\ + \gamma_{(22)} \left(U_\mu S^2_{\nu\rho} + U_\nu S^2_{\rho\mu} + U_\rho S^2_{\mu\nu} \right) + \\ + \gamma_{(12)} \left(U_\mu S^1_{\nu\rho} + U_\nu S^1_{\rho\mu} + U_\rho S^1_{\mu\nu} + A_\mu S^2_{\nu\rho} + A_\nu S^2_{\rho\mu} + A_\rho S^2_{\mu\nu} \right) + \\ + 2\gamma_{(33)} Re \left(V^-_\mu S^+_{\nu\rho} + V^-_\nu S^+_{\rho\mu} + V^-_\rho S^+_{\mu\nu} \right) \tag{3.13}$$

where relations eq (3.10) - (3.13) are extended to $e_{\mu\nu}$.

Another equations to be included on the Noether identities. It gives,

$$\partial_\mu J^\mu_N = 0, \\ K_I \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu G_{\mu I}} \right) = 0 \\ K_{(\mu\nu)} \equiv K_I \frac{\partial \mathcal{L}}{\partial (\partial_\mu G_{\nu I})} = 0.$$

where

$$J^\mu_N = iq \left(V^+_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu V^+_\nu)} - V^-_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu V^-_\nu)} \right) \tag{3.14}$$

4 $\mathbf{U}(1) \times \mathbf{SO}(2)_{\mathbf{global}}$: first gauge invariance method

A nonlinear abelian antireductionist EM is being developed through the quadruplet $\{A_\mu, U_\mu, V_\mu^\pm\}$. The physical motivation is to consider the electric charges at $\{+,0,-\}$ being transmitted through a four bosons quadruplet $\{A_\mu, U_\mu, V_\mu^\pm\}$. There is a new EM energy originated from three charges exchange. Its physical meaning is to extend Maxwell to a model derived from $U(1) \times SO(2)_{\text{global}}$ transformations under the same gauge parameter.

Given the quadruplet the first two fields are associated under U(1) and the last two charged fields under $U(1) \times SO(2)_{\text{global}}$. There are the two methods for establishing the gauge invariance. The first under fields strengths; the second, by potential fields.

For granular antisymmetric field strengths, one gets from eq (1.1)

$$G_{\mu\nu 1,2} \xrightarrow{U(1)\times SO(2)_{\text{global}}} G'_{\mu\nu 1,2} = G_{\mu\nu 1,2} \tag{4.1}$$

$$V^{\pm}_{\mu\nu 1,2} \xrightarrow{U(1)\times SO(2)_{\text{global}}} V^{\pm'}_{\mu\nu 1,2} = e^{\pm iq\alpha} V^{\pm}_{\mu\nu} \tag{4.2}$$

For collective antisymmetric field strengths, generalizing eq.() to eq(), one gets

$$e^{[12]'}_{\mu\nu} + e^{[\pm]'}_{[\mu\nu]} \xrightarrow{U(1)\times SO(2)_{\text{global}}} e^{[12]}_{\mu\nu} + e^{[\pm]}_{[\mu\nu]} \tag{4.3}$$

under the conditions

$$\begin{aligned} e^{[12]}_{\mu\nu}\Omega^{-1}_{11} &= 0 \qquad & e^{[12]}_{\mu\nu}\Omega^{-1}_{12} &= 0 \\ e^{[34]}_{\mu\nu}\Omega^{-1}_{31} &= 0 \qquad & e^{[34]}_{\mu\nu}\Omega^{-1}_{41} &= 0 \end{aligned} \tag{4.4}$$

and

$$e^{[\pm 1]'}_{[\mu\nu]} + e^{[\pm 2]'}_{[\mu\nu]} \xrightarrow{U(1)\times SO(2)_{\text{global}}} e^{iq\alpha}(e^{[\pm 1]}_{\mu\nu} + e^{[\pm 2]}_{[\mu\nu]}) \tag{4.5}$$

under the conditions

The above expressions produce the following gauge invariant antisymmetric Lagrangian under eq (1.1)

$$L^A = L^A_K + L^A_3 + L^A_4 \tag{4.6}$$

where

$$L^A_K = a_1 F_{\mu\nu}F^{\mu\nu} + a_2 U_{\mu\nu}U^{\mu\nu} + a_3 V^+_{\mu\nu}V^{\mu\nu -} \tag{4.7}$$

$$\begin{aligned} L^A_3 = {} & (F_{\mu\nu} + U_{\mu\nu})(e^{[12]}_{[\mu\nu]} + e^{[\pm]}_{[\mu\nu]}) + V^+_{\mu\nu}(e^{[-1]}_{[\mu\nu]} + e^{[-2]}_{[\mu\nu]}) + \\ & + V^-_{\mu\nu}(e^{[+1]}_{[\mu\nu]} + e^{[+2]}_{[\mu\nu]}) + (F_{\mu\nu} + U_{\mu\nu})(\tilde{e}^{[12]}_{[\mu\nu]} + \tilde{e}^{[\pm]}_{[\mu\nu]}) + \\ & + V^+_{[\mu\nu]}(\tilde{e}^{[-1]}_{[\mu\nu]} + \tilde{e}^{[-2]}_{[\mu\nu]}) + V^-_{[\mu\nu]}(\tilde{e}^{[+1]}_{[\mu\nu]} + \tilde{e}^{[+2]}_{[\mu\nu]}) \end{aligned} \tag{4.8}$$

with a new notation

$$F_{\mu\nu} \equiv G_{\mu\nu 1} \quad , \quad U_{\mu\nu} \equiv G_{\mu\nu 2} \tag{4.9}$$

which will be studied in more detail in a further work.

5 $\mathbf{U(1) \times SO(2)_{global}}$: second gauge invariance method

We should study the second method now. Understand eq (1.1) directly on potential fields. Given that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ where

$$\mathcal{L}_0 = \mathcal{L}_K + \mathcal{L}_m + \mathcal{L}_{GF} \tag{5.1}$$

we are going first to study eq.(5.1) under $U(1) \times SO(2)_{\text{global}}$ invariance.

Considering the kinetic term

$$\mathcal{L}_K = \mathcal{L}_K^A + \mathcal{L}_K^S \tag{5.2}$$

one gets,

$$\mathcal{L}_K^A = a_1^2 G_{\mu\nu}^1 G^{1\,\mu\nu} + a_2^2 G_{\mu\nu}^2 G^{2\,\mu\nu} + a_3^2 \big(G_{\mu\nu}^3 G^{3\,\mu\nu} + G_{\mu\nu}^4 G^{4\,\mu\nu}\big) \tag{5.3}$$

and

$$\begin{aligned}\mathcal{L}_K^S = {} & \beta_1^2 S_{\mu\nu}^1 S^{1\,\mu\nu} + \beta_2^2 S_{\mu\nu}^2 S^{2\,\mu\nu} + \beta_3^2 \big(S_{\mu\nu}^3 S^{3\,\mu\nu} + S_{\mu\nu}^4 S^{4\,\mu\nu}\big) + \\ & + \rho_1^2 S_\alpha^{1\,\alpha} S_\beta^{1\,\beta} + \rho_2^2 S_\alpha^{2\,\alpha} S_\beta^{2\,\beta} + \rho_1\rho_2 S_\alpha^{1\,\alpha} S_\beta^{2\,\beta} + \rho_3^2 \big(S_\alpha^{3\,\alpha} S_\beta^{3\,\beta} + S_\alpha^{4\,\alpha} S_\beta^{4\,\beta}\big)\end{aligned} \tag{5.4}$$

For the mass term

$$\mathcal{L}_m = \cancelto{0}{m_{11}^2 G_{\mu\,1} G_1^\mu} + m_{22}^2 G_{\mu\,2} G_2^\mu + m_{33}^2 \big(G_{\mu\,3} G_3^\mu + G_{\mu\,4} G_4^\mu\big) \tag{5.5}$$

Eq. (5.5) shows that the model necessarily contains one mass zero. This is due to the initial gauge transformation $D_\mu^{'} = D_\mu + \partial_\mu \alpha$. It says that the model is just an extension to QED plus gauge invariants Proca fields.

Thus, given $U(1) \times SO(2)_{\text{global}}$ symmetry, $\mathcal{L}_K^A$ contains three gauge invariant blocks, $\mathcal{L}_K^S$ two. Consequently, the corresponding eight physical spin-1 and spin-0 quanta inserted in the model propagate through five physical kinetic regions. They correspond to the EM domains through whose the EM energy flows interdependently.

The $U(1) \times SO(2)_{\text{global}}$-kinetic sector is:

$$\mathcal{L}_K^A = a_1^2 G_{\mu\nu}^1 G^{1\,\mu\nu} + a_2^2 G_{\mu\nu}^2 G^{2\,\mu\nu} + a_3^2 V_{\mu\nu}^+ V^{-\,\mu\nu} \tag{5.6}$$

and

$$\begin{aligned}\mathcal{L}_K^S = {} & \beta_1^2 S_{\mu\nu}^1 S^{1\,\mu\nu} + \beta_2^2 S_{\mu\nu}^2 S^{2\,\mu\nu} + 2\beta_3^2 S_{\mu\nu}^+ S^{\mu\nu\,-} + \\ & + \rho_1^2 S_\alpha^{1\,\alpha} S_\beta^{1\,\beta} + \rho_2^2 S_\alpha^{2\,\alpha} S_\beta^{2\,\beta} + \rho_1\rho_2 S_\alpha^{1\,\alpha} S_\beta^{2\,\beta} + 2\rho_3^2 S_\alpha^{+\,\alpha} S_\beta^{\beta\,-}\end{aligned} \tag{5.7}$$

and the mass term is

$$\mathcal{L}_m = m_{22}^2 G_{\mu\,2} G_2^{\mu} + m_{33}^2 V_{\mu\,+} V^{\mu\,-} \tag{5.8}$$

Eqs (5.6) contain five EM domains. This means three regions propagating three spin-1 particles and ten regions propagating three spin-0 particles.

6 U(1) × SO(2): Three linear interaction sector

Considering the interaction sector we are going first to look at the $L_3^{I\,A}$ term written in the $\{D, X\}$ basis:

$$L^A_{3\,\{D,X\}} = 2dD_{\mu\nu}\gamma_{[ij]}X^{\mu i}X^{\nu j} + 2\alpha_i X^i_{[\mu\nu]}\gamma_{[jk]}X^{\mu j}X^{\nu k}, \tag{6.1}$$

One notices that eq.(6.1) contains $\frac{(N-1)(N-2)}{2} + \frac{(N-1)^2(N-2)}{2} = \frac{N(N-1)(N-2)}{2}$ gauge invariant blocks.

Considering the following relationships:

$$\begin{aligned}
&D_{[\mu\nu]} = \Omega_{1I}G^I_{\mu\nu} \quad , X^i_{\mu\nu} = \Omega^i_I G^I_{\mu\nu}\\
&\gamma_{[ij]}X^i_\mu X^j_\nu = \gamma_{[IJ]}G^I_\mu G^J_\nu \quad \text{where}\, \gamma_{[IJ]} = \gamma_{[ij]}\Omega^{iI}\Omega^{jJ}\\
&S^i_{\mu\nu} = \Omega^i_I S^I_{\mu\nu}, \quad S^{\alpha i}_\alpha = \Omega^i_I S^{\alpha i}_\alpha\\
&\gamma_{(ij)}X^i_\mu X^j_\nu = \gamma_{(IJ)}G_{\mu I}G_{\nu J} \quad \text{where}\, \gamma_{(IJ)} = \gamma_{(ij)}\Omega^{iI}\Omega^{jJ}
\end{aligned} \tag{6.2}$$

one gets at $\{G_I\}$ basis:

$$\begin{aligned}
L^A_{3\{G_I\}} &= 2d\Omega_{1I}G^I_{\mu\nu}\gamma_{[jk]}\Omega^{jJ}\Omega^{kK}G^\mu_J G^\nu_K + 2\alpha_i\Omega^i_I G^I_{\mu\nu}\gamma_{[jk]}\Omega^{jJ}\Omega^{kK}G^\mu_J G^\nu_K\\
L^A_{3\{G_I\}} &= 2b_{1I}G^I_{\mu\nu}\gamma_{[JK]}G^{\mu\,J}G^{\nu\,K} + 2c_I G^I_{\mu\nu}\gamma_{[JK]}G^{\mu\,J}G^{\nu\,K}\\
L^A_{3\{G_I\}} &= 2b_{1I}\gamma_{[JK]}G^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K} + 2c_I\gamma_{[JK]}G^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K}\\
L^A_{3\{G_I\}} &= b_{1I[JK]}G^I_{\mu\nu}G^{\mu\,I}G^{\nu\,J} + c_{I[JK]}G^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K}
\end{aligned} \tag{6.3}$$

where

$$b_{1I[JK]} = 2b_{1I}\gamma_{[JK]},\; c_{I[JK]} = 2c_I\gamma_{[JK]} \tag{6.4}$$

Including the semi topological contribution, one gets from eq.(2.65)

$$\mathcal{L}^A_{3,st} = \eta_{I[JK]}\epsilon^{\mu\nu\rho\sigma}G^K_\rho \cdot (4G^I_\mu\partial_\nu G^J_\sigma + 2G^J_\mu\partial_\nu G^I_\sigma) \tag{6.5}$$

where

$$\eta_{I[JK]} = \alpha_I\gamma_{JK} \tag{6.6}$$

Or

$$L_3^{st} = \theta_{IJK}(\partial_\mu G_\nu^I)G^{\mu I}G^{\nu K} \tag{6.7}$$

with

$$\theta_{IJK} = 4\eta_{K[IJ]} - 2\eta_{I[JK]} \tag{6.8}$$

Thus, the trilinear anyisymmetric contribution

$$\mathcal{L}_3^A = \mathcal{L}_{3\{G_I\}}^A + \mathcal{L}_{3,st}^A \tag{6.9}$$

contains three basic gauge invariant blocks. Analyzing specifically $L^3_{A\,\{G_I\}}$ leads to $\frac{(N-1)(N-2)}{2}$ and $\frac{(N-1)^2(N-2)}{2}$ gauge invariant blocks respectively mixing granular and collective fields strengths flowing the EM energy.

Now, looking at $L^I_{3\,S}$ term written in the $\{D, X\}$ basis:

$$L^S_{3\,\{D,X\}} = 2\beta_i S^i_{\mu\nu}\gamma_{(jk)}X^{\mu j}X^{\nu k} + \{2\rho_i\gamma_{(jk)} + (8\rho_i + 2\beta_i)\,\tau_{(jk)}\}S^{\mu i}_\mu X^j_\nu X^{\nu k} \tag{6.10}$$

which yields $3N^2\frac{N-1}{2}$ gauge invariant blocks. Passing to the $\{G_I\}$ basis:

$$\begin{aligned} L^S_{3\,\{G_I\}} &= 2\beta_i\Omega^i_I S^I_{\mu\nu}\gamma_{(jk)}\Omega^j_J\Omega^k_K G^{\mu\,J}G^{\nu\,K} + \Omega^i_I S^{\mu I}_\mu\{2\rho_i\gamma_{(jk)} + (8\rho_i + 2\beta_i)\,\tau_{(jk)}\}\Omega^j_J\Omega^k_K G^J_\nu G^{\nu\,K} \\ L^S_{3\,\{G_I\}} &= 2\beta_I\gamma_{(JK)}S^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K} + \{2\rho_I\gamma_{(JK)} + (8\rho_I + 2\beta_I)\,\tau_{(JK)}\}S^{\mu I}_\mu G^J_\nu G^{\nu\,K} \end{aligned} \tag{6.11}$$

which gives

$$L^S_{3\,\{G_I\}} = d_{I(JK)}S^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K} + e_{I(JK)}S^{\mu I}_\mu G^J_\nu G^{\nu\,K} \tag{6.12}$$

with

$$\begin{aligned} d_{I(JK)} &= 2\beta_I\gamma_{(JK)} \\ e_{I(JK)} &= 2\{\rho_I\gamma_{(JK)} + (4\rho_I + \beta_I)\,\tau_{(JK)}\} \end{aligned} \tag{6.13}$$

leading to $\frac{N(N-1)}{2}$ and $\frac{3N(N-1)^2}{2}$ gauge invariant blocks respectively

Thus, in total, we have the $\mathcal{L}_3$ term with five types of gauge invariant blocks:

$$L_{3\{G_I\}} = b_{1I[JK]}G^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K} + c_{I[JK]}G^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K} + d_{I(JK)}S^I_{\mu\nu}G^{\mu\,J}G^{\nu\,K} + e_{I(JK)}S^{\mu I}_\mu G^J_\nu G^{\nu\,K} \tag{6.14}$$

which provides four types of physical sectors. Each of them contains respectively

$$\frac{(N-1)(N-2)}{2}, \frac{(N-1)^2(N-2)}{2}, \frac{N(N-1)}{2}, \frac{3N(N-1)^2}{2} \tag{6.15}$$

gauge invariant blocks or EM domains. In total it yields $2N(N-1)^2$ independent terms for the EM flux do physics.

Writing the above expression in a more compact form:

$$L_3 = (\alpha_{IJK} + \Theta_{IJK}) \left(\partial_\mu G^I_\nu\right) G^{\mu\, J} G^{\nu\, K} + \beta_{I(JK)} \left(\partial_\mu G^{\mu\, I}\right) G^J_\nu G^{\nu\, K} \tag{6.16}$$

where

$$\begin{aligned}
&\alpha_{IJK} = 2b_{1I[JK]} + 2c_{I[JK]} + 2d_{I(JK)} = 4b_{1I}\gamma_{[JK]} + 4c_I\gamma_{[JK]} + 4\beta_I\gamma_{(JK)} \\
&\beta_{I(JK)} = 2e_{I(JK)} = 4\{\rho_I\gamma_{(JK)} + (4\rho_I + \beta_I)\,\tau_{(JK)}\} \\
&\eta_{I[JK]} = \alpha_I\gamma_{[JK]} \\
&\Theta_{IJK} = 4\eta_{K[IJ]} - 2\eta_{I[JK]} \\
&\alpha_{IJK} + \theta_{IJK} = 4(b_{1I} + c_I - \frac{1}{2}\alpha_I)\gamma_{[JK]} + 4\alpha_K\gamma_{[IJ]} + 4\beta_I\gamma_{(JK)}
\end{aligned} \tag{6.17}$$

Next step is to study (6.1) Lagrangian under SO(2) invariance. Rewrite the expressions above in terms of four fields as $(A_\mu = G_{\mu 1}, U_\mu = G_{\mu 2}; G_{\mu 3}; G_{\mu 4})$. Consider the matricial transformation

$$G_{\mu I} \overset{SO(2)}{\longrightarrow} G'_{\mu I} = R_{IK} G^K_\mu \tag{6.18}$$

it gives

$$R = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & cos\alpha & sin\alpha \\ 0 & 0 & -sin\alpha & cos\alpha \end{array}\right) = e^{t\alpha} \quad , \quad t = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) \tag{6.19}$$

infinitesimally,

$$R_{im} = \delta_{im} + \alpha(t)_{im} \tag{6.20}$$

Substituting eq(6.20) in eq(6.16), we get from Appendix E the trilinear SO(2) invariance

$$
\begin{aligned}
\mathcal{L}_3 = & \alpha_{111}\left(\partial_\mu A_\nu\right) A^\mu A^\nu+\alpha_{112}\left(\partial_\mu A_\nu\right) A^\mu U^\nu+\alpha_{121}\left(\partial_\mu U_\nu\right) U^\mu A^\nu+ \\
& +\alpha_{122}\left(\partial_\mu A_\nu\right) U^\mu U^\nu+\alpha_{211}\left(\partial_\mu U_\nu\right) A^\mu A^\nu+\alpha_{212}\left(\partial_\mu U_\nu\right) A^\mu U^\nu+ \\
& +\alpha_{221}\left(\partial_\mu U_\nu\right) U^\mu U^\nu+\alpha_{222}\left(\partial_\mu U_\nu\right) U^\mu U^\nu+ \\
& +\left(\alpha_{331} A^\nu+\alpha_{332} U^\nu\right)\left[\left(\partial_\mu G_\nu^3\right) G^{\mu 3}+\left(\partial_\mu G_\nu^4\right) G^{\mu 4}\right]+ \\
& +\left(\alpha_{341} A^\nu+\alpha_{342} U^\nu\right)\left[\left(\partial_\mu G_\nu^3\right) G^{\mu 4}-\left(\partial_\mu G_\nu^4\right) G^{\mu 3}\right]+ \\
& +\left(\alpha_{313} A^\nu+\alpha_{323} U^\nu\right)\left[\left(\partial_\mu G_\nu^3\right) G^{\nu 3}+\left(\partial_\mu G_\nu^4\right) G^{\nu 4}\right]+ \\
& +\left(\alpha_{314} A^\nu+\alpha_{324} U^\nu\right)\left[\left(\partial_\mu G_\nu^3\right) G^{\nu 4}-\left(\partial_\mu G_\nu^4\right) G^{\nu 3}\right]+ \\
& +\left[\alpha_{133}\left(\partial_\mu A^\nu\right)+\alpha_{323}\left(\partial_\mu U^\nu\right)\right]\left[G^{\mu 3} G^{\nu 3}+G^{\mu 4} G^{\nu 4}\right]+ \\
& +\left[\alpha_{134}\left(\partial_\mu A^\nu\right)+\alpha_{234}\left(\partial_\mu U^\nu\right)\right]\left[G^{\mu 3} G^{\nu 3}-G^{\mu 4} G^{\nu 3}\right]+ \\
& +\beta_{1(11)}\left(\partial_\mu A^\mu\right) A_\nu A^\nu+2 \beta_{1(12)}\left(\partial_\mu A^\mu\right) A_\nu U^\nu+ \\
& +\beta_{1(22)}\left(\partial_\mu A^\mu\right) U_\nu U^\nu+2 \beta_{2(11)}\left(\partial_\mu U^\mu\right) A_\nu A^\nu+ \\
& +2 \beta_{2(12)}\left(\partial_\mu U^\mu\right) A_\nu U^\nu+2 \beta_{2(22)}\left(\partial_\mu U^\mu\right) U_\nu U^\nu+ \\
& +2\left(\beta_{3(13)} A^\nu+\beta_{3(23)} U^\nu\right)\left[\left(\partial_\mu G^{\mu 3}\right) G_\nu^3+\left(\partial_\mu G^{\mu 4}\right) G_\nu^4\right]+ \\
& +2\left(\beta_{3(14)} A^\nu+\beta_{3(24)} U^\nu\right)\left[\left(\partial_\mu G^{\mu 3}\right) G_\nu^4-\left(\partial_\mu G^{\mu 4}\right) G_\nu^3\right]+ \\
& +\left[\beta_{1(33)}\left(\partial_\mu A^\mu\right)+\beta_{2(33)}\left(\partial_\mu U^\mu\right)\right]\left[G_\nu^3 G^{\nu 3}+G_\nu^4 G^{\nu 4}\right]= \\
& +\epsilon^{\mu\nu\rho\sigma}\Big\{\theta_{211}(\partial_\rho A_\sigma) U_\mu A_\nu+\theta_{121}(\partial_\rho A_\sigma) A_\mu U_\nu+ \\
& +\theta_{212}(\partial_\rho U_\sigma) U_\mu A_\nu+\theta_{122}(\partial_\rho U_\sigma) A_\mu A_\nu+ \\
& +(\theta_{133} A_\mu+\theta_{233} U_\mu)[(\partial_\rho G_\sigma^3) G_\nu^3+(\partial_\rho G_\sigma^4) G_\nu^4]+ \\
& +(\theta_{313} A_\nu+\theta_{323} U_\nu)[(\partial_\rho G_\sigma^3) G_\mu^3+(\partial_\rho G_\sigma^4) G_\mu^4]+ \\
& +(\theta_{134} A_\mu+\theta_{234} U_\mu)[(\partial_\rho G_\sigma^4) G_\nu^3-(\partial_\rho G_\sigma^3) G_\nu^4]+ \\
& +(\theta_{314} A_\nu+\theta_{324} U_\nu)[(\partial_\rho G_\sigma^4) G_\mu^3-(\partial_\rho G_\sigma^3) G_\mu^4]+ \\
& +[\theta_{341}(\partial_\rho A_\sigma)+\theta_{342}(\partial_\rho U_\sigma)][G_\mu^3 G_\nu^4-G_\mu^4 G_\nu^3]\Big\}
\end{aligned} \tag{6.21}
$$

Thus one has to study on $\mathcal{L}_3$ under $U(1) \times SO(2)_{\mathbf{global}}$ gauge invariance. According to Appendix H it will depend on three coefficients relationships.

Expressing $\mathcal{L}_3$ in terms of V_μ^+ e V_μ^-, $G_\mu^3=\frac{1}{\sqrt{2}}\left(V_\mu^++V_\mu^-\right)$, $G_\mu^4=\frac{i}{\sqrt{2}}\left(V_\mu^+-V_\mu^-\right)$, we have:

- $\left(\partial_\mu G_\nu^3\right) G^{\mu 3}+\left(\partial_\mu G_\nu^4\right) G^{\nu 4}=\left(\partial_\mu V_\nu^+\right) V^{\mu -}+\left(\partial_\mu V_\nu^-\right) V^{\mu +}$
- $\left(\partial_\mu G_\nu^3\right) G^{\mu 4}-\left(\partial_\mu G_\nu^4\right) G^{\nu 3}=-i\left(\partial_\mu V_\nu^+\right) V^{\mu -}+i\left(\partial_\mu V_\nu^-\right) V^{\mu +}$
- $\left(\partial_\mu G_\nu^3\right) G^{\nu 4}+\left(\partial_\mu G_\nu^4\right) G^{\nu 3}=\left(\partial_\mu V_\nu^+\right) V^{\nu -}+\left(\partial_\mu V_\nu^-\right) V^{\nu +}$
- $\left(\partial_\mu G_\nu^3\right) G^{\nu 4}-\left(\partial_\mu G_\nu^4\right) G^{\nu 3}=-i\left(\partial_\mu V_\nu^+\right) V^{\nu -}+i\left(\partial_\mu V_\nu^-\right) V^{\nu +}$
- $G^{\mu 3} G^{\nu 3}+G^{\mu 4} G^{\nu 4}=V^{\mu +} V^{\nu -}+V^{\mu -} V^{\nu +}$
- $G^{\mu 3} G^{\nu 3}-G^{\mu 4} G^{\nu 4}=-i V^{\mu +} V^{\nu -}+i V^{\mu -} V^{\nu +}$
- $\left(\partial_\mu G^{\mu 3}\right) G_\nu^3+\left(\partial_\mu G^{\mu 4}\right) G_\nu^4=\left(\partial_\mu V^{\mu +}\right) V_\nu^-+\left(\partial_\mu V^{\mu -}\right) V_\nu^+$
- $\left(\partial_\mu G^{\mu 3}\right) G_\nu^4+\left(\partial_\mu G^{\mu 4}\right) G_\nu^3=-i\left(\partial_\mu V^{\mu +}\right) V_\nu^-+\left(\partial_\mu V^{\mu -}\right) V_\nu^+$

- $G^3_\nu G^{\nu\,3} - G^4_\nu G^{\nu\,4} = 2V^+_\nu V^{\nu\,-}$

Finally, we write $\mathcal{L}_3$ in terms of $\{A_\mu, U_\mu, V^+_\mu, V^-_\mu\}$. It yields the following expression for the four bosons three linear interactions under $U(1) \times SO(2)_{global}$

$$
\begin{aligned}
\mathcal{L}_3 =\, &\alpha_{111}\left(\partial_\nu A_\nu\right)A^\mu A^\nu + \alpha_{112}\left(\partial_\nu A_\nu\right)A^\mu U^\nu + \alpha_{121}\left(\partial_\nu U_\nu\right)U^\mu A^\nu + \\
&+\alpha_{122}\left(\partial_\mu A_\nu\right)U^\mu U^\nu + \alpha_{211}\left(\partial_\mu U_\nu\right)A^\mu A^\nu + \alpha_{212}\left(\partial_\mu U_\nu\right)A^\mu U^\nu + \\
&+\alpha_{221}\left(\partial_\nu U_\nu\right)U^\mu U^\nu + \alpha_{222}\left(\partial_\mu U_\nu\right)U^\mu U^\nu + \\
&+\left(\alpha_{331}A^\nu + \alpha_{332}U^\nu\right)\left[\left(\partial_\mu V^+_\nu\right)V^{\mu\,-} + \left(\partial_\mu V^-_\nu\right)V^{\mu\,+}\right] + \\
&-i\left(\alpha_{341}A^\nu + \alpha_{342}U^\nu\right)\left[\left(\partial_\mu V^+_\nu\right)V^{\mu\,-} - \left(\partial_\mu V^-_\nu\right)V^{\mu\,+}\right] + \\
&+\left(\alpha_{313}A^\nu + \alpha_{323}U^\nu\right)\left[\left(\partial_\mu V^+_\nu\right)V^{\nu\,-} + \left(\partial_\mu V^-_\nu\right)V^{\nu\,+}\right] + \\
&-i\left(\alpha_{314}A^\nu + \alpha_{324}U^\nu\right)\left[\left(\partial_\mu V^+_\nu\right)V^{\nu\,-} - \left(\partial_\mu V^-_\nu\right)V^{\nu\,+}\right] + \\
&+\left[\alpha_{133}\left(\partial_\mu A^\nu\right) + \alpha_{323}\left(\partial_\mu U^\nu\right)\right]\left[V^{\mu+}V^{\nu\,-} + V^{\mu-}V^{\nu\,+}\right] + \\
&-i\left[\alpha_{134}\left(\partial_\mu A^\nu\right) + \alpha_{234}\left(\partial_\mu U^\nu\right)\right]\left[V^{\mu+}V^{\nu\,-} - V^{\mu\,-}V^{\nu\,+}\right] + \\
&+\beta_{1(11)}\left(\partial_\mu A^\mu\right)A_\nu A^\nu + 2\beta_{1(12)}\left(\partial_\mu A^\mu\right)A_\nu U^\nu + \\
&+\beta_{1(22)}\left(\partial_\mu A^\mu\right)U_\nu U^\nu + 2\beta_{2(11)}\left(\partial_\mu U^\mu\right)A_\nu A^\nu + \\
&+2\beta_{2(12)}\left(\partial_\mu U^\mu\right)A_\nu U^\nu + 2\beta_{2(22)}\left(\partial_\mu U^\mu\right)U_\nu U^\nu + \\
&+2\left(\beta_{3(13)}A^\nu + \beta_{3(23)}U^\nu\right)\left[\left(\partial_\mu V^{\mu+}\right)V^-_\nu + \left(\partial_\mu V^{\mu-}\right)V^+_\nu\right] + \\
&+2i\left(\beta_{3(14)}A^\nu + \beta_{3(24)}U^\nu\right)\left[\left(\partial_\mu V^{\mu\,+}\right)V^-_\nu - \left(\partial_\mu V^{\mu\,-}\right)V^+_\nu\right] + \\
&+2\left[\beta_{1(33)}\left(\partial_\mu A^\mu\right) + \beta_{2(33)}\left(\partial_\mu U^\mu\right)\right]V^+_\nu V^{\nu\,-} + \\
&\epsilon^{\mu\nu\rho\sigma}\Big\{\theta_{121}(\partial_\rho A_\sigma)A_\mu U_\nu + \theta_{122}(\partial_\rho U_\sigma)A_\mu U_\nu + \\
&+\theta_{211}(\partial_\rho A_\sigma)U_\mu A_\nu + \theta_{212}(\partial_\rho U_\sigma)U_\mu A_\nu + \\
&+\Big[(\theta_{133} - \theta_{313})A_\mu + (\theta_{233} - \theta_{323})U_\mu\Big]\Big[(\partial_\rho V^+_\sigma)V-_\nu + (\partial_\rho V^-_\sigma)V^+_\nu\Big] \\
&+i\Big[(\theta_{134} - \theta_{314})A_\mu + (\theta_{234} - \theta_{324})U_\mu\Big]\Big[(\partial_\rho V^+_\sigma)V-_\nu - (\partial_\rho V^-_\sigma)V^+_\nu\Big] \\
&+i\Big[\theta_{341}(\partial_\rho A_\sigma) + \theta_{342}(\partial_\rho U_\sigma)\Big]\Big[V-_\mu V^+_\nu - V^+_\mu V^-_\nu\Big]\Big\}
\end{aligned}
\tag{6.22}
$$

Given eq (6.22) we have to understand $\mathcal{L}_3$ not only in terms of the fields set $\{A_\mu, U_\mu, V^\pm_\mu\}$ but through granular and collective fields strengths. Study eq.(6.22) by decomposing the 4 invariant blocks in terms of granular and collective terms:

A. Terms with α_{IJK}

We know that $\alpha_{IJK} = 2a_{I[JK]} + 2b_{I[JK]} + 2c_{I(JK)}$ and

$$\begin{aligned} b_{I[JK]} &= 2b_I\gamma_{[JK]} \\ c_{I[JK]} &= 2c_I\gamma_{[JK]} \\ d_{I(JK)} &= 2\beta_I\gamma_{(JK)} \end{aligned} \tag{6.23}$$

which means that

$$\alpha_{IJK} = 4b_I\gamma_{[JK]} + 4c_I\gamma_{[JK]} + 4\beta_I\gamma_{(JK)} \tag{6.24}$$

Let´s consider the $\mathcal{L}_{3\,\alpha}^{A,U}$:

1. $\alpha_{111}\left(\partial_\mu A_\nu\right)A^\mu A^\nu$

$$\begin{aligned} &= \left(4b_1\gamma_{[11]} + 4c_1\gamma_{[11]} + 4\beta_1\gamma_{(11)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S^1_{\mu\nu}\right)A^\mu A^\nu \\ &= \left(4b_1\cancelto{0}{\gamma_{[11]}} + 4c_1\cancelto{0}{\gamma_{[11]}} + 4\beta_1\gamma_{(11)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S^1_{\mu\nu}\right)A^\mu A^\nu \\ &= 4\beta_1\gamma_{(11)}\left(\cancelto{0}{\frac{1}{2}F_{\mu\nu}A^\mu A^\nu} + \frac{1}{2}S^1_{\mu\nu}A^\mu A^\nu\right) \\ &= 2\beta_1 S^1_{\mu\nu}\left(\gamma_{(11)}A^\mu A^\nu\right) = 2\beta_1 S^1_{\mu\nu}\overset{(11)}{z}{}^{(\mu\nu)} \end{aligned} \tag{6.25}$$

2. $\alpha_{112}\left(\partial_\mu A_\nu\right)A^\mu U^\nu$

$$\begin{aligned} &= \left(4b_1\gamma_{[12]} + 4c_1\gamma_{[12]} + 4\beta_1\gamma_{(12)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S_{\mu\nu}\right)A^\mu U^\nu \\ &= 2b_1F_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + 2c_1F_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + \\ &+2b_1S^1_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + 2c_1S^1_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + \\ &+2\beta_1F_{\mu\nu}\left(\gamma_{(12)}A^\mu U^\nu\right) + 2\beta_1S^1_{\mu\nu}\left(\gamma_{(12)}A^\mu U^\nu\right) \\ &= 2b_1F_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]} + 2c_1F_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]} + \\ &+2b_1S^1_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)} + 2c_1S^1_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)} + \\ &+2\beta_1F_{\mu\nu}\overset{(12)}{z}{}^{[\mu\nu]} + 2\beta_1S^1_{\mu\nu}\overset{(12)}{z}{}^{(\mu\nu)} \end{aligned} \tag{6.26}$$

3. $\alpha_{121}\left(\partial_\mu A_\nu\right)U^\mu A^\nu$

$$
\begin{aligned}
&= \left(4b_1\gamma_{[21]} + 4c_1\gamma_{[21]} + 4\beta_1\gamma_{(21)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S_{\mu\nu}\right)U^\mu A^\nu \\
&= \left(-4b_1\gamma_{[12]} - 4c_1\gamma_{[12]} + 4\beta_1\gamma_{(12)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S_{\mu\nu}\right)U^\mu A^\nu \\
&= -2b_1F_{\mu\nu}\left(\gamma_{[12]}U^\mu A^\nu\right) - 2c_1F_{\mu\nu}\left(\gamma_{[12]}U^\mu A^\nu\right) + \\
&-2b_1S^1_{\mu\nu}\left(\gamma_{[12]}U^\mu A^\nu\right) - 2c_1S^1_{\mu\nu}\left(\gamma_{[12]}U^\mu A^\nu\right) + \\
&+2\beta_1F_{\mu\nu}\left(\gamma_{(12)}U^\mu A^\nu\right) + 2\beta_1S^1_{\mu\nu}\left(\gamma_{(12)}U^\mu A^\nu\right) \\
&= 2b_1F_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]} + 2c_1F_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]} + \\
&-2b_1S^1_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)} - 2c_1S^1_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)} + \\
&+2\beta_1F_{\mu\nu}\overset{(12)}{z}{}^{[\mu\nu]} + 2\beta_1S^1_{\mu\nu}\overset{(12)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.27}
$$

4. $\alpha_{122}\left(\partial_\mu A_\nu\right)U^\mu U^\nu$

$$
\begin{aligned}
&= \left(4b_1\gamma_{[22]} + 4c_1\gamma_{[22]} + 4\beta_1\gamma_{(22)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S^1_{\mu\nu}\right)U^\mu U^\nu \\
&= \left(4b_1\cancelto{0}{\gamma_{[22]}} + 4c_1\cancelto{0}{\gamma_{[22]}} + 4\beta_1\gamma_{(22)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S^1_{\mu\nu}\right)U^\mu U^\nu \\
&= 2\beta_1S^1_{\mu\nu}\left(\gamma_{(22)}U^\mu U^\nu\right) = 2\beta_1S^1_{\mu\nu}\overset{(22)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.28}
$$

5. $\alpha_{211}\left(\partial_\mu U_\nu\right)A^\mu A^\nu$

$$
\begin{aligned}
&= \left(4b_2\gamma_{[11]} + 4c_1\gamma_{[11]} + 4\beta_1\gamma_{(11)}\right)\left(\frac{1}{2}U_{\mu\nu} + \frac{1}{2}S^2_{\mu\nu}\right)A^\mu A^\nu \\
&= \left(4b_2\cancelto{0}{\gamma_{[11]}} + 4c_2\cancelto{0}{\gamma_{[11]}} + 4\beta_2\gamma_{(11)}\right)\left(\frac{1}{2}U_{\mu\nu} + \frac{1}{2}S^2_{\mu\nu}\right)U^\mu U^\nu \\
&= 2\beta_2S^2_{\mu\nu}\left(\gamma_{(22)}A^\mu A^\nu\right) = 2\beta_2S^2_{\mu\nu}\overset{(11)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.29}
$$

6. $\alpha_{212}\left(\partial_\mu A_\nu\right)A^\mu U^\nu$

$$
\begin{aligned}
&= \left(4b_2\gamma_{[12]} + 4c_2\gamma_{[12]} + 4\beta_2\gamma_{(12)}\right)\left(\frac{1}{2}U_{\mu\nu} + \frac{1}{2}S^2_{\mu\nu}\right)A^\mu U^\nu \\
&= 2b_2U_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + 2c_2U_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + \\
&+2b_2S^2_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + 2c_2S^2_{\mu\nu}\left(\gamma_{[12]}A^\mu U^\nu\right) + \\
&+2\beta_2U_{\mu\nu}\left(\gamma_{(12)}A^\mu U^\nu\right) + 2\beta_2S^2_{\mu\nu}\left(\gamma_{(12)}A^\mu U^\nu\right) \\
&= 2b_2U_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]} + 2c_2U_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]} + \\
&+2b_2S^2_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)} + 2c_2S^2_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)} + \\
&+2\beta_2U_{\mu\nu}\overset{(12)}{z}{}^{[\mu\nu]} + 2\beta_2S^2_{\mu\nu}\overset{(12)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.30}
$$

7. $\alpha_{221}\left(\partial_{\mu}U_{\nu}\right)U^{\mu}A^{\nu}$

$$
\begin{aligned}
&= \left(4b_2\gamma_{[21]}+4c_2\gamma_{[21]}+4\beta_2\gamma_{(21)}\right)\left(\frac{1}{2}U_{\mu\nu}+\frac{1}{2}S^2_{\mu\nu}\right)U^{\mu}A^{\nu}\\
&= \left(-4b_2\gamma_{[12]}-4c_2\gamma_{[12]}+4\beta_2\gamma_{(12)}\right)\left(\frac{1}{2}U_{\mu\nu}+\frac{1}{2}S^2_{\mu\nu}\right)U^{\mu}A^{\nu}\\
&= -2b_2U_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)-2c_2U_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)+\\
&-2b_2S^2_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)-2c_2S^2_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)+\\
&+2\beta_2U_{\mu\nu}\left(\gamma_{(12)}U^{\mu}A^{\nu}\right)+2\beta_2S^2_{\mu\nu}\left(\gamma_{(12)}U^{\mu}A^{\nu}\right)\\
&= 2b_2U_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]}+2c_2U_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]}+\\
&-2b_2S^2_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)}-2c_2S^2_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)}+\\
&+2\beta_2U_{\mu\nu}\overset{(12)}{z}{}^{[\mu\nu]}+2\beta_2S^2_{\mu\nu}\overset{(12)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.31}
$$

8. $\alpha_{221}\left(\partial_{\mu}U_{\nu}\right)U^{\mu}A^{\nu}$

$$
\begin{aligned}
&= \left(4b_2\gamma_{[21]}+4c_2\gamma_{[21]}+4\beta_2\gamma_{(21)}\right)\left(\frac{1}{2}U_{\mu\nu}+\frac{1}{2}S^2_{\mu\nu}\right)U^{\mu}A^{\nu}\\
&= \left(-4b_2\gamma_{[12]}-4c_2\gamma_{[12]}+4\beta_2\gamma_{(12)}\right)\left(\frac{1}{2}U_{\mu\nu}+\frac{1}{2}S^2_{\mu\nu}\right)U^{\mu}A^{\nu}\\
&= -2b_2U_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)-2c_2U_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)+\\
&-2b_2S^2_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)-2c_2S^2_{\mu\nu}\left(\gamma_{[12]}U^{\mu}A^{\nu}\right)+\\
&+2\beta_2U_{\mu\nu}\left(\gamma_{(12)}U^{\mu}A^{\nu}\right)+2\beta_2S^2_{\mu\nu}\left(\gamma_{(12)}U^{\mu}A^{\nu}\right)\\
&= 2b_2U_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]}+2c_2U_{\mu\nu}\overset{[12]}{z}{}^{[\mu\nu]}+\\
&-2b_2S^2_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)}-2c_2S^2_{\mu\nu}\overset{[12]}{z}{}^{(\mu\nu)}+\\
&+2\beta_2U_{\mu\nu}\overset{(12)}{z}{}^{[\mu\nu]}+2\beta_2S^2_{\mu\nu}\overset{(12)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.32}
$$

9. $\alpha_{222}\left(\partial_{\mu}U_{\nu}\right)U^{\mu}U^{\nu}$

$$
\begin{aligned}
&= \left(4b_2\gamma_{[22]}+4c_2\gamma_{[22]}+4\beta_2\gamma_{(22)}\right)\left(\frac{1}{2}U_{\mu\nu}+\frac{1}{2}S^1_{\mu\nu}\right)U^{\mu}U^{\nu}\\
&= \left(4b_2\cancelto{0}{\gamma_{[22]}}+4c_2\cancelto{0}{\gamma_{[22]}}+4\beta_2\gamma_{(22)}\right)\left(\frac{1}{2}U_{\mu\nu}+\frac{1}{2}S^2_{\mu\nu}\right)U^{\mu}U^{\nu}\\
&= 4\beta_2\gamma_{(22)}\left(\cancelto{0}{\frac{1}{2}U_{\mu\nu}U^{\mu}U^{\nu}}+\frac{1}{2}S^2_{\mu\nu}U^{\mu}U^{\nu}\right)\\
&= 2\beta_1S^2_{\mu\nu}\left(\gamma_{(11)}U^{\mu}U^{\nu}\right)=2\beta_2S^2_{\mu\nu}\overset{(22)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.33}
$$

In the end, we have:

$$
\begin{aligned}
&\mathcal{L}^{A,U}_{3\,\alpha}=4\left(b_1F_{\mu\nu}+b_2U_{\mu\nu}\right)\overset{[12]}{z}{}^{[\mu\nu]}+\\
&+2\left(\beta_1S^1_{\mu\nu}+\beta_2S^2_{\mu\nu}\right)\left(\overset{(11)}{z}{}^{(\mu\nu)}+2\overset{(12)}{z}{}^{(\mu\nu)}+\overset{(22)}{z}{}^{(\mu\nu)}\right)
\end{aligned}
\tag{6.34}
$$

Let´s now take terms with the $\mathcal{L}_{3\,\alpha}^{V^+,V^-}$ part:

1. $(\alpha_{331} - i\alpha_{341})\,(\partial_\mu V_\nu^+)\, A^\nu V^{\mu\,-}$

$$
\begin{aligned}
&= \left(4b_3\gamma_{[31]} + 4c_3\gamma_{[31]} + 4\beta_3\gamma_{(31)} - 4ib_3\gamma_{[41]} - 4ic_3\gamma_{[41]} - 4i\beta_3\gamma_{(41)}\right)\left(\frac{1}{2}V^+_{\mu\nu} + \frac{1}{2}S^+_{\mu\nu}\right) A^\nu V^{\mu\,-}\\
&= \left(-4b_3\gamma_{[13]} - 4c_3\gamma_{[13]} + 4\beta_3\gamma_{(13)} + 4ib_3\gamma_{[14]} + 4ic_3\gamma_{[14]} - 4i\beta_3\gamma_{(14)}\right)\left(\frac{1}{2}V^+_{\mu\nu} + \frac{1}{2}S^+_{\mu\nu}\right) A^\nu V^{\mu\,-}\\
&= \left(2b_3\left(-\gamma_{[13]} + i\gamma_{[14]}\right) + 2c_3\left(-\gamma_{[13]} + i\gamma_{[14]}\right) + 2\beta_3\left(\gamma_{(13)} - i\gamma_{(14)}\right)\right)\left(V^+_{\mu\nu} + S^+_{\mu\nu}\right) A^\nu V^{\mu\,-}\\
&= -2b_3V^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,-}\right) - 2b_3S^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,-}\right) +\\
&-2c_3V^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,-}\right) - 2c_3S^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,-}\right) +\\
&+2\beta_3V^+_{\mu\nu}\left(\gamma_{(13)} - i\gamma_{(14)}\right)\left(A^\nu V^{\mu\,-}\right) + 2\beta_3S^+_{\mu\nu}\left(\gamma_{(13)} - i\gamma_{(14)}\right)\left(A^\nu V^{\mu\,-}\right)\\
&= 2b_3V^+_{\mu\nu}\,\overset{[-1]}{z}{}^{[\mu\nu]} - 2b_3S^+_{\mu\nu}\,\overset{[-1]}{z}{}^{(\mu\nu)} + 2c_3V^+_{\mu\nu}\,\overset{[-1]}{z}{}^{[\mu\nu]} - 2c_3S^+_{\mu\nu}\,\overset{[-1]}{z}{}^{(\mu\nu)} +\\
&-2\beta_3V^+_{\mu\nu}\,\overset{(-1)}{z}{}^{[\mu\nu]} + 2\beta_3S^+_{\mu\nu}\,\overset{(-1)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.35}
$$

2. $(\alpha_{313} - i\alpha_{314})\,(\partial_\mu V_\nu^+)\, A^\mu V^{\nu\,-}$

$$
\begin{aligned}
&= \left(4b_3\gamma_{[13]} + 4c_3\gamma_{[13]} + 4\beta_3\gamma_{(13)} - 4ib_3\gamma_{[14]} - 4ic_3\gamma_{[14]} - 4i\beta_3\gamma_{(14)}\right)\left(\frac{1}{2}V^+_{\mu\nu} + \frac{1}{2}S^+_{\mu\nu}\right) A^\mu V^{\nu\,-}\\
&= \left(2b_3\left(\gamma_{[13]} - i\gamma_{[14]}\right) + 2c_3\left(\gamma_{[13]} - i\gamma_{[14]}\right) + 2\beta_3\left(\gamma_{(13)} - i\gamma_{(14)}\right)\right)\left(V^+_{\mu\nu} + S^+_{\mu\nu}\right) A^\mu V^{\nu\,-}\\
&= 2b_3V^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,-}\right) + 2b_3S^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,-}\right) +\\
&+2c_3V^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,-}\right) + 2c_3S^+_{\mu\nu}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,-}\right) +\\
&+2\beta_3V^+_{\mu\nu}\left(\gamma_{(13)} - i\gamma_{(14)}\right)\left(A^\mu V^{\nu\,-}\right) + 2\beta_3S^+_{\mu\nu}\left(\gamma_{(13)} - i\gamma_{(14)}\right)\left(A^\mu V^{\nu\,-}\right)\\
&= 2b_3V^+_{\mu\nu}\,\overset{[-1]}{z}{}^{[\mu\nu]} + 2b_3S^+_{\mu\nu}\,\overset{[-1]}{z}{}^{(\mu\nu)} + 2c_3V^+_{\mu\nu}\,\overset{[-1]}{z}{}^{[\mu\nu]} + 2c_3S^+_{\mu\nu}\,\overset{[-1]}{z}{}^{(\mu\nu)} +\\
&+2\beta_3V^+_{\mu\nu}\,\overset{(-1)}{z}{}^{[\mu\nu]} + 2\beta_3S^+_{\mu\nu}\,\overset{(-1)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.36}
$$

3. $(\alpha_{331} + i\alpha_{341})\,(\partial_\mu V_\nu^-)\, A^\nu V^{\mu\,+}$

$$
\begin{aligned}
&= \left(4b_3\gamma_{[31]} + 4c_3\gamma_{[31]} + 4\beta_3\gamma_{(31)} + 4ib_3\gamma_{[41]} + 4ic_3\gamma_{[41]} + 4i\beta_3\gamma_{(41)}\right)\left(\frac{1}{2}V^-_{\mu\nu} + \frac{1}{2}S^-_{\mu\nu}\right) A^\nu V^{\mu\,+}\\
&= \left(-4b_3\gamma_{[13]} - 4c_3\gamma_{[13]} + 4\beta_3\gamma_{(13)} - 4ib_3\gamma_{[14]} - 4ic_3\gamma_{[14]} + 4i\beta_3\gamma_{(14)}\right)\left(\frac{1}{2}V^-_{\mu\nu} + \frac{1}{2}S^-_{\mu\nu}\right) A^\nu V^{\mu\,+}\\
&= \left(-2b_3\left(\gamma_{[13]} + i\gamma_{[14]}\right) - 2c_3\left(\gamma_{[13]} + i\gamma_{[14]}\right) + 2\beta_3\left(\gamma_{(13)} + i\gamma_{(14)}\right)\right)\left(V^-_{\mu\nu} + S^-_{\mu\nu}\right) A^\nu V^{\mu\,+}\\
&= -2b_3V^-_{\mu\nu}\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,+}\right) - 2b_3S^-_{\mu\nu}\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,+}\right) +\\
&-2c_3V^-_{\mu\nu}\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,+}\right) - 2c_3S^-_{\mu\nu}\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\nu V^{\mu\,+}\right) +\\
&+2\beta_3V^-_{\mu\nu}\left(\gamma_{(13)} + i\gamma_{(14)}\right)\left(A^\nu V^{\mu\,+}\right) + 2\beta_3S^-_{\mu\nu}\left(\gamma_{(13)} + i\gamma_{(14)}\right)\left(A^\nu V^{\mu\,+}\right) +\\
&= 2b_3V^-_{\mu\nu}\,\overset{[+1]}{z}{}^{[\mu\nu]} - 2b_3S^-_{\mu\nu}\,\overset{[+1]}{z}{}^{(\mu\nu)} + 2c_3V^-_{\mu\nu}\,\overset{[+1]}{z}{}^{[\mu\nu]} - 2c_3S^-_{\mu\nu}\,\overset{[+1]}{z}{}^{(\mu\nu)} +\\
&-2\beta_3V^-_{\mu\nu}\,\overset{(+1)}{z}{}^{[\mu\nu]} + 2\beta_3S^-_{\mu\nu}\,\overset{(+1)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.37}
$$

4. $(\alpha_{313} + i\alpha_{314})\,(\partial_\mu V_\nu^-)\,A^\mu V^{\nu\,+}$

$$
\begin{aligned}
&= \left(4b_3\gamma_{[13]} + 4c_3\gamma_{[13]} + 4\beta_3\gamma_{(13)} + 4ib_3\gamma_{[14]} + 4ic_3\gamma_{[14]} + 4i\beta_3\gamma_{(14)}\right)\left(\frac{1}{2}V_{\mu\nu}^- + \frac{1}{2}S_{\mu\nu}^-\right)A^\mu V^{\nu\,+}\\
&= \left(2b_3\left(\gamma_{[13]} + i\gamma_{[14]}\right) + 2c_3\left(\gamma_{[13]} + i\gamma_{[14]}\right) + 2\beta_3\left(\gamma_{(13)} + i\gamma_{(14)}\right)\right)\left(V_{\mu\nu}^- + S_{\mu\nu}^-\right)A^\mu V^{\nu\,+}\\
&= 2b_3V_{\mu\nu}^-\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,+}\right) + 2b_3S_{\mu\nu}^-\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,+}\right) +\\
&+2c_3V_{\mu\nu}^-\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,+}\right) + 2c_3S_{\mu\nu}^-\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A^\mu V^{\nu\,+}\right) +\\
&+2\beta_3V_{\mu\nu}^-\left(\gamma_{(13)} + i\gamma_{(14)}\right)\left(A^\mu V^{\nu\,+}\right) + 2\beta_3S_{\mu\nu}^-\left(\gamma_{(13)} + i\gamma_{(14)}\right)\left(A^\mu V^{\nu\,+}\right)\\
&= 2b_3V_{\mu\nu}^-\overset{[+1]}{z}{}^{[\mu\nu]} + 2b_3S_{\mu\nu}^-\overset{[+1]}{z}{}^{(\mu\nu)} + 2c_3V_{\mu\nu}^-\overset{[+1]}{z}{}^{[\mu\nu]} + 2c_3S_{\mu\nu}^-\overset{[+1]}{z}{}^{(\mu\nu)} +\\
&+2\beta_3V_{\mu\nu}^-\overset{(+1)}{z}{}^{[\mu\nu]} + 2\beta_3S_{\mu\nu}^-\overset{(+1)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.38}
$$

5. $(\alpha_{332} - i\alpha_{342})\,(\partial_\mu V_\nu^+)\,U^\nu V^{\mu\,-}$

$$
\begin{aligned}
&= \left(4b_3\gamma_{[32]} + 4c_3\gamma_{[32]} + 4\beta_3\gamma_{(32)} - 4ib_3\gamma_{[42]} - 4ic_3\gamma_{[42]} - 4i\beta_3\gamma_{(42)}\right)\left(\frac{1}{2}V_{\mu\nu}^+ + \frac{1}{2}S_{\mu\nu}^+\right)U^\nu V^{\mu\,-}\\
&= \left(-4b_3\gamma_{[23]} - 4c_3\gamma_{[23]} + 4\beta_3\gamma_{(23)} + 4ib_3\gamma_{[24]} + 4ic_3\gamma_{[24]} - 4i\beta_3\gamma_{(24)}\right)\left(\frac{1}{2}V_{\mu\nu}^+ + \frac{1}{2}S_{\mu\nu}^+\right)U^\nu V^{\mu\,-}\\
&= \left(2b_3\left(-\gamma_{[23]} + i\gamma_{[24]}\right) + 2c_3\left(-\gamma_{[23]} + i\gamma_{[24]}\right) + 2\beta_3\left(\gamma_{(23)} + i\gamma_{(24)}\right)\right)\left(V_{\mu\nu}^+ + S_{\mu\nu}^+\right)U^\nu V^{\mu\,-}\\
&= -2b_3V_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U^\nu V^{\mu\,-}\right) - 2b_3S_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U^\nu V^{\mu\,-}\right) +\\
&-2c_3V_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(A^\nu V^{\mu\,-}\right) - 2c_3S_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U^\nu V^{\mu\,-}\right) +\\
&+2\beta_3V_{\mu\nu}^+\left(\gamma_{(23)} - i\gamma_{(24)}\right)\left(U^\nu V^{\mu\,-}\right) + 2\beta_3S_{\mu\nu}^+\left(\gamma_{(23)} - i\gamma_{(24)}\right)\left(U^\nu V^{\mu\,-}\right) +\\
&= 2b_3V_{\mu\nu}^+\overset{[-2]}{z}{}^{[\mu\nu]} - 2b_3S_{\mu\nu}^+\overset{[-2]}{z}{}^{(\mu\nu)} + 2c_3V_{\mu\nu}^+\overset{[-2]}{z}{}^{[\mu\nu]} - 2c_3S_{\mu\nu}^+\overset{[-2]}{z}{}^{(\mu\nu)} +\\
&-2\beta_3V_{\mu\nu}^+\overset{(-2)}{z}{}^{[\mu\nu]} + 2\beta_3S_{\mu\nu}^+\overset{(-2)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.39}
$$

6. $(\alpha_{323} - i\alpha_{324})\,(\partial_\mu V_\nu^+)\,U^\mu V^{\nu\,-}$

$$
\begin{aligned}
&= \left(4b_3\gamma_{[23]} + 4c_3\gamma_{[23]} + 4\beta_3\gamma_{(23)} - 4ib_3\gamma_{[24]} - 4ic_3\gamma_{[24]} - 4i\beta_3\gamma_{(24)}\right)\left(\frac{1}{2}V_{\mu\nu}^+ + \frac{1}{2}S_{\mu\nu}^+\right)U^\mu V^{\nu\,-}\\
&= \left(2b_3\left(\gamma_{[23]} - i\gamma_{[24]}\right) + 2c_3\left(\gamma_{[23]} - i\gamma_{[24]}\right) + 2\beta_3\left(\gamma_{(23)} - i\gamma_{(24)}\right)\right)\left(V_{\mu\nu}^+ + S_{\mu\nu}^+\right)U^\mu V^{\nu\,-}\\
&= 2b_3V_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,-}\right) + 2b_3S_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,-}\right) +\\
&+2c_3V_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,-}\right) + 2c_3S_{\mu\nu}^+\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,-}\right) +\\
&+2\beta_3V_{\mu\nu}^+\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(U^\mu V^{\nu\,-}\right) + 2\beta_3S_{\mu\nu}^+\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(U^\mu V^{\nu\,-}\right)\\
&= 2b_3V_{\mu\nu}^+\overset{[-2]}{z}{}^{[\mu\nu]} + 2b_3S_{\mu\nu}^+\overset{[-2]}{z}{}^{(\mu\nu)} + 2c_3V_{\mu\nu}^+\overset{[-2]}{z}{}^{[\mu\nu]} + 2c_3S_{\mu\nu}^+\overset{[-2]}{z}{}^{(\mu\nu)} +\\
&+2\beta_3V_{\mu\nu}^+\overset{(-2)}{z}{}^{[\mu\nu]} + 2\beta_3S_{\mu\nu}^+\overset{(-2)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.40}
$$

7. $(\alpha_{332} + i\alpha_{342})\left(\partial_\mu V_\nu^-\right) A^\nu V^{\mu\,+}$

$$\begin{aligned}
&= \left(4b_3\gamma_{[32]} + 4c_3\gamma_{[32]} + 4\beta_3\gamma_{(32)} + 4ib_3\gamma_{[42]} + 4ic_3\gamma_{[42]} + 4i\beta_3\gamma_{(42)}\right)\left(\frac{1}{2}V^-_{\mu\nu} + \frac{1}{2}S^-_{\mu\nu}\right) U^\nu V^{\mu\,+} \\
&= \left(-4b_3\gamma_{[23]} - 4c_3\gamma_{[23]} + 4\beta_3\gamma_{(23)} - 4ib_3\gamma_{[24]} - 4ic_3\gamma_{[24]} + 4i\beta_3\gamma_{(24)}\right)\left(\frac{1}{2}V^-_{\mu\nu} + \frac{1}{2}S^-_{\mu\nu}\right) U^\nu V^{\mu\,+} \\
&= -2b_3\left(\gamma_{[23]} + i\gamma_{[24]}\right) - 2c_3\left(\gamma_{[23]} + i\gamma_{[24]}\right) + 2\beta_3\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(V^-_{\mu\nu} + S^-_{\mu\nu}\right) U^\nu V^{\mu\,+} \\
&= -2b_3V^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\nu V^{\mu\,+}\right) - 2b_3S^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\nu V^{\mu\,+}\right) + \\
&-2c_3V^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\nu V^{\mu\,+}\right) - 2c_3S^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\nu V^{\mu\,+}\right) + \\
&+2\beta_3V^-_{\mu\nu}\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(U^\nu V^{\mu\,+}\right) + 2\beta_3S^-_{\mu\nu}\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(U^\nu V^{\mu\,+}\right) \\
&= 2b_3V^-_{\mu\nu}\,\overset{[+2]}{z}{}^{[\mu\nu]} - 2b_3S^-_{\mu\nu}\,\overset{[+2]}{z}{}^{(\mu\nu)} + 2c_3V^-_{\mu\nu}\,\overset{[+2]}{z}{}^{[\mu\nu]} - 2c_3S^-_{\mu\nu}\,\overset{[+2]}{z}{}^{(\mu\nu)} + \\
&-2\beta_3V^-_{\mu\nu}\,\overset{(+2)}{z}{}^{[\mu\nu]} + 2\beta_3S^-_{\mu\nu}\,\overset{(+2)}{z}{}^{(\mu\nu)}
\end{aligned} \tag{6.41}$$

8. $(\alpha_{323} + i\alpha_{324})\left(\partial_\mu V_\nu^-\right) U^\mu V^{\nu\,+}$

$$\begin{aligned}
&= \left(4b_3\gamma_{[23]} + 4c_3\gamma_{[23]} + 4\beta_3\gamma_{(23)} + 4ib_3\gamma_{[24]} + 4ic_3\gamma_{[24]} + 4i\beta_3\gamma_{(24)}\right)\left(\frac{1}{2}V^-_{\mu\nu} + \frac{1}{2}S^-_{\mu\nu}\right) U^\mu V^{\nu\,+} \\
&= \left(2b_3\left(\gamma_{[23]} + i\gamma_{[24]}\right) + 2c_3\left(\gamma_{[23]} + i\gamma_{[24]}\right) + 2\beta_3\left(\gamma_{(23)} + i\gamma_{(24)}\right)\right)\left(V^-_{\mu\nu} + S^-_{\mu\nu}\right) U^\mu V^{\nu\,+} \\
&= 2b_3V^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,+}\right) + 2b_3S^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,+}\right) + \\
&+2c_3V^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,+}\right) + 2c_3S^-_{\mu\nu}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U^\mu V^{\nu\,+}\right) + \\
&+2\beta_3V^-_{\mu\nu}\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(U^\mu V^{\nu\,+}\right) + 2\beta_3S^-_{\mu\nu}\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(U^\mu V^{\nu\,+}\right) \\
&= 2b_3V^-_{\mu\nu}\,\overset{[+2]}{z}{}^{[\mu\nu]} + 2b_3S^-_{\mu\nu}\,\overset{[+2]}{z}{}^{(\mu\nu)} + 2c_3V^-_{\mu\nu}\,\overset{[+2]}{z}{}^{[\mu\nu]} + 2c_3S^-_{\mu\nu}\,\overset{[+2]}{z}{}^{(\mu\nu)} + \\
&+2\beta_3V^-_{\mu\nu}\,\overset{(+2)}{z}{}^{[\mu\nu]} + 2\beta_3S^-_{\mu\nu}\,\overset{(+2)}{z}{}^{(\mu\nu)}
\end{aligned} \tag{6.42}$$

9. $(\alpha_{133} - i\alpha_{134})\left(\partial_\mu A_\nu\right) V^{\mu\,+}V^{\nu\,-}$

$$\begin{aligned}
&= \left(4b_1\cancelto{0}{\gamma_{[33]}} + 4c_1\cancelto{0}{\gamma_{[33]}} + 4\beta_1\gamma_{(33)} - 4ib_1\gamma_{[34]} - 4ic_1\gamma_{[34]} - 4i\beta_1\gamma_{(34)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S^1_{\mu\nu}\right) V^{\mu\,+}V^{\nu\,-} \\
&= \left(2\beta_1\gamma_{(33)} - 2ib_1\gamma_{[34]} - 2ic_1\gamma_{[34]} - 2i\beta_1\gamma_{(34)}\right)\left(F_{\mu\nu} + S^1_{\mu\nu}\right) V^{\mu\,+}V^{\nu\,-} \\
&= 2b_1F_{\mu\nu}\,\overset{[+-]}{z}{}^{[\mu\nu]} + 2c_1F_{\mu\nu}\,\overset{[+-]}{z}{}^{[\mu\nu]} + 2b_1S^1_{\mu\nu}\,\overset{[+-]}{z}{}^{(\mu\nu)} + 2c_1S^1_{\mu\nu}\,\overset{[+-]}{z}{}^{(\mu\nu)} + \\
&+2\beta_1F_{\mu\nu}\,\overset{+-3}{z}{}^{[\mu\nu]} + 2\beta_1S^1_{\mu\nu}\,\overset{+-3}{z}{}^{(\mu\nu)} + 2\beta_1F_{\mu\nu}\,\overset{(+-)}{z}{}^{[\mu\nu]} + 2\beta_1S^1_{\mu\nu}\,\overset{(+-)}{z}{}^{(\mu\nu)}
\end{aligned} \tag{6.43}$$

10. $(\alpha_{133} + i\alpha_{134})\left(\partial_\mu A_\nu\right) V^{\mu\,-}V^{\nu\,+}$

$$\begin{aligned}
&= \left(4b_1\cancelto{0}{\gamma_{[33]}} + 4c_1\cancelto{0}{\gamma_{[33]}} + 4\beta_1\gamma_{(33)} + 4ib_1\gamma_{[34]} + 4ic_1\gamma_{[34]} + 4i\beta_1\gamma_{(34)}\right)\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}S^1_{\mu\nu}\right) V^{\mu\,-}V^{\nu\,+} \\
&= \left(2\beta_1\gamma_{(33)} + 2ib_1\gamma_{[34]} + 2ic_1\gamma_{[34]} + 2i\beta_1\gamma_{(34)}\right)\left(F_{\mu\nu} + S^1_{\mu\nu}\right) V^{\mu\,-}V^{\nu\,+} \\
&= 2b_1F_{\mu\nu}\,\overset{[+-]}{z}{}^{[\mu\nu]} + 2c_1F_{\mu\nu}\,\overset{[+-]}{z}{}^{[\mu\nu]} - 2b_1S^1_{\mu\nu}\,\overset{[+-]}{z}{}^{(\mu\nu)} - 2c_1S^1_{\mu\nu}\,\overset{[+-]}{z}{}^{(\mu\nu)} + \\
&-2\beta_1F_{\mu\nu}\,\overset{+-3}{z}{}^{[\mu\nu]} + 2\beta_1S^1_{\mu\nu}\,\overset{+-3}{z}{}^{(\mu\nu)} - 2\beta_1F_{\mu\nu}\,\overset{(+-)}{z}{}^{[\mu\nu]} + 2\beta_1S^1_{\mu\nu}\,\overset{(+-)}{z}{}^{(\mu\nu)}
\end{aligned} \tag{6.44}$$

11. $(\alpha_{233} - i\alpha_{234})\,(\partial_\mu A_\nu)\, V^{\mu+}V^{\nu-}$

$$
\begin{aligned}
&= \left(4b_2\cancelto{0}{\gamma_{[33]}} + 4c_2\cancelto{0}{\gamma_{[33]}} + 4\beta_2\gamma_{(33)} - 4ib_2\gamma_{[34]} - 4ic_2\gamma_{[34]} - 4i\beta_2\gamma_{(34)}\right)\left(\frac{1}{2}U_{\mu\nu} + \frac{1}{2}S^2_{\mu\nu}\right) V^{\mu+}V^{\nu-}\\
&= \left(2\beta_2\gamma_{(33)} - 2ib_2\gamma_{[34]} - 2ic_2\gamma_{[34]} - 2i\beta_2\gamma_{(34)}\right)\left(U_{\mu\nu} + S^2_{\mu\nu}\right) V^{\mu+}V^{\nu-}\\
&= 2b_2U_{\mu\nu}\ \overset{[+-]}{z}{}^{[\mu\nu]} + 2c_2U_{\mu\nu}\ \overset{[+-]}{z}{}^{[\mu\nu]} + 2b_2S^2_{\mu\nu}\ \overset{[+-]}{z}{}^{(\mu\nu)} + 2c_2S^2_{\mu\nu}\ \overset{[+-]}{z}{}^{(\mu\nu)} +\\
&+2\beta_2U_{\mu\nu}\ \overset{+-}{z}{}^{3[\mu\nu]} + 2\beta_2S^2_{\mu\nu}\ \overset{+-}{z}{}^{3(\mu\nu)} + 2\beta_2U_{\mu\nu}\ \overset{(+-)}{z}{}^{[\mu\nu]} + 2\beta_2S^2_{\mu\nu}\ \overset{(+-)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.45}
$$

12. $(\alpha_{133} + i\alpha_{134})\,(\partial_\mu A_\nu)\, V^{\mu-}V^{\nu+}$

$$
\begin{aligned}
&= \left(4b_2\cancelto{0}{\gamma_{[33]}} + 4c_2\cancelto{0}{\gamma_{[33]}} + 4\beta_2\gamma_{(33)} + 4ib_2\gamma_{[34]} + 4ic_2\gamma_{[34]} + 4i\beta_2\gamma_{(34)}\right)\left(\frac{1}{2}U_{\mu\nu} + \frac{1}{2}S^2_{\mu\nu}\right) V^{\mu-}V^{\nu+}\\
&= \left(2\beta_2\gamma_{(33)} + 2ib_2\gamma_{[34]} + 2ic_2\gamma_{[34]} + 2i\beta_2\gamma_{(34)}\right)\left(U_{\mu\nu} + S^2_{\mu\nu}\right) V^{\mu-}V^{\nu+}\\
&= 2b_2U_{\mu\nu}\ \overset{[+-]}{z}{}^{[\mu\nu]} + 2c_2U_{\mu\nu}\ \overset{[+-]}{z}{}^{[\mu\nu]} - 2b_2S^2_{\mu\nu}\ \overset{[+-]}{z}{}^{(\mu\nu)} - 2c_2S^2_{\mu\nu}\ \overset{[+-]}{z}{}^{(\mu\nu)} +\\
&-2\beta_2U_{\mu\nu}\ \overset{+-}{z}{}^{3[\mu\nu]} + 2\beta_2S^2_{\mu\nu}\ \overset{+-}{z}{}^{3(\mu\nu)} - 2\beta_2U_{\mu\nu}\ \overset{(+-)}{z}{}^{[\mu\nu]} + 2\beta_2S^2_{\mu\nu}\ \overset{(+-)}{z}{}^{(\mu\nu)}
\end{aligned}
\tag{6.46}
$$

Resulting in:

$$
\begin{aligned}
&\mathcal{L}^{V^+,V^-}_{3\,\alpha} = \left(4\,(b_1+c_1)\,F_{\mu\nu} + 4\,(b_2+c_2)\,U_{\mu\nu}\right)\overset{[+-]}{z}{}^{[\mu\nu]} + \left(4\beta_1F_{\mu\nu} + 4\beta_2U_{\mu\nu}\right)\overset{(+-)}{z}{}^{[\mu\nu]} +\\
&+4\,(b_3+c_3)\,V^+_{\mu\nu}\left(\overset{[-1]}{z}{}^{[\mu\nu]} + \overset{[-2]}{z}{}^{[\mu\nu]}\right) + 4\,(b_3+c_3)\,V^-_{\mu\nu}\left(\overset{[+1]}{z}{}^{[\mu\nu]} + \overset{[+2]}{z}{}^{[\mu\nu]}\right)
\end{aligned}
\tag{6.47}
$$

B. Terms with β_{IJK}

Moving forward, we consider the $\mathcal{L}^{A,U}_{3\,\beta}$

1. $\beta_{1(11)}\,(\partial_\mu A^\mu)\,A_\nu A^\nu$

$$
\begin{aligned}
&\beta_{1(11)}\,(\partial_\mu A^\mu)\,A_\nu A^\nu =\\
&= \Big[4\rho_1\gamma_{(11)} + 4(\beta_1+4\rho_1)\tau_{(11)}\Big]\frac{1}{2}S^{\mu 1}_{\mu}A_\nu A^\nu\\
&= 2\rho_1 S^{\mu 1}_{\mu}\ \overset{(11)}{z}{}^{\nu}_{\ \nu} + 2(\beta_1+4\rho_1)S^{\mu 1}_{\mu}\ \overset{(11)}{\omega}{}^{\nu}_{\ \nu}
\end{aligned}
\tag{6.48}
$$

2. $\beta_{1(12)}\,(\partial_\mu A^\mu)\,A_\nu U^\nu$

$$
\begin{aligned}
&\beta_{1(12)}(\partial_\mu A^\mu)A_\nu U^\nu\\
&= 2\Big[4\rho_1\gamma_{(12)} + 4(\beta_1+4\rho_1)\tau_{(12)}\Big]\frac{1}{2}S^{\mu 1}_{\mu}A_\nu U^\nu\\
&= 4\rho_1 S^{\mu 1}_{\mu}\ \overset{(12)}{z}{}^{\nu}_{\ \nu} + 4(\beta_1+4\rho_1)S^{\mu 1}_{\mu}\ \overset{(12)}{\omega}{}^{\nu}_{\ \nu}
\end{aligned}
\tag{6.49}
$$

3. $\beta_{1(22)}(\partial_\mu A^\mu)U_\nu U^\nu$

$$
\begin{aligned}
&\beta_{1(22)}(\partial_\mu A^\mu)U_\nu U^\nu =\\
&= \Big[4\rho_1\gamma_{(22)} + 4(\beta_1+4\rho_1)\tau_{(22)}\Big]\frac{1}{2}S^{\mu 1}_{\mu}U_\nu U^\nu\\
&= 2\rho_1 S^{\mu 1}_{\mu}\ \overset{(22)}{z}{}^{\nu}_{\ \nu} + 2(\beta_1+4\rho_1)S^{\mu 1}_{\mu}\ \overset{(22)}{\omega}{}^{\nu}_{\ \nu}
\end{aligned}
\tag{6.50}
$$

4. $\beta_{2(11)}(\partial_\mu U^\mu)A_\nu A^\nu$

$$\begin{aligned}
&\beta_{2(11)}(\partial_\mu U^\mu)A_\nu A^\nu \\
&= \Big[4\rho_2\gamma_{(11)} + 4(\beta_2+4\rho_2)\tau_{(11)}\Big]\frac{1}{2}S^{\mu 1}_{\mu}A_\nu A^\nu \\
&= 2\rho_2 S^{\mu 2}_{\mu}\, \overset{(11)}{z}{}^{\nu}_{\nu} + 2(\beta_2+4\rho_2)S^{\mu 2}_{\mu}\, \overset{(11)}{\omega}{}^{\nu}_{\nu}
\end{aligned} \tag{6.51}$$

5. $2\beta_{1(12)}(\partial_\mu U^\mu)A_\nu U^\nu$

$$\begin{aligned}
&2\beta_{2(12)}(\partial_\mu U^\mu)A_\nu U^\nu \\
&= 2\Big[4\rho_2\gamma_{(12)} + 4(\beta_2+4\rho_2)\tau_{(12)}\Big]\frac{1}{2}S^{\mu 2}_{\mu}A_\nu U^\nu \\
&= 4\rho_2 S^{\mu 2}_{\mu}\, \overset{(12)}{z}{}^{\nu}_{\nu} + 4(\beta_2+4\rho_2)S^{\mu 2}_{\mu}\, \overset{(12)}{\omega}{}^{\nu}_{\nu}
\end{aligned} \tag{6.52}$$

6. $\beta_{2(22)}(\partial_\mu U^\mu)U_\nu U^\nu$

$$\begin{aligned}
&\beta_{2(22)}(\partial_\mu U^\mu)U_\nu U^\nu \\
&= \Big[4\rho_2\gamma_{(22)} + 4(\beta_2+4\rho_2)\tau_{(22)}\Big]\frac{1}{2}S^{\mu 2}_{\mu}U_\nu U^\nu \\
&= 4\rho_2 S^{\mu 2}_{\mu}\, \overset{(22)}{z}{}^{\nu}_{\nu} + 2(\beta_2+4\rho_2)S^{\mu 2}_{\mu}\, \overset{(22)}{\omega}{}^{\nu}_{\nu}
\end{aligned} \tag{6.53}$$

So, we have:

$$\mathcal{L}^{A,U}_{3\,\beta} = (1)+(2)+...+(6) \tag{6.54}$$

$$\begin{aligned}
&= 2\rho_1 S^{\mu 1}_{\mu}\Big[\overset{(11)}{z}{}^{\nu}_{\nu} + 2\,\overset{(12)}{z}{}^{\nu}_{\nu} + \overset{(22)}{z}{}^{\nu}_{\nu}\Big] + \\
&+2(\beta_1+4\rho_1)S^{\mu 1}_{\mu}\Big[\overset{(11)}{\omega}{}^{\nu}_{\nu} + 2\,\overset{(12)}{\omega}{}^{\nu}_{\nu} + \overset{(22)}{\omega}{}^{\nu}_{\nu}\Big] + \\
&+2\rho_2 S^{\mu 2}_{\mu}\Big[\overset{(11)}{z}{}^{\nu}_{\nu} + 2\,\overset{(12)}{z}{}^{\nu}_{\nu} + \overset{(22)}{z}{}^{\nu}_{\nu}\Big] + \\
&+2(\beta_2+4\rho_2)S^{\mu 2}_{\mu}\Big[\overset{(11)}{\omega}{}^{\nu}_{\nu} + 2\,\overset{(12)}{\omega}{}^{\nu}_{\nu} + \overset{(22)}{\omega}{}^{\nu}_{\nu}\Big]
\end{aligned} \tag{6.55}$$

$$\begin{aligned}
\mathcal{L}^{A,U}_{3\,\beta} &= 2(\rho_1 S^{\mu 1}_{\mu} + \rho_2 S^{\mu 2}_{\mu})\Big[\overset{(11)}{z}{}^{\nu}_{\nu} + 2\,\overset{(12)}{z}{}^{\nu}_{\nu} + \overset{(22)}{z}{}^{\nu}_{\nu}\Big] + \\
&+2\Big[(\beta_1+4\rho_1)S^{\mu 1}_{\mu} + (\beta_2+4\rho_2)S^{\mu 2}_{\mu}\Big]\Big[\overset{(11)}{\omega}{}^{\nu}_{\nu} + 2\,\overset{(12)}{\omega}{}^{\nu}_{\nu} + \overset{(22)}{\omega}{}^{\nu}_{\nu}\Big]
\end{aligned} \tag{6.56}$$

Let´s consider the term $\mathcal{L}^{V^+,V^-}_{\beta,3}$

$$\begin{aligned}
\mathcal{L}^{V^+,V^-}_{\beta,3} &= 2\beta_{3(13)}(\partial_\mu V^{\mu+})A_\nu V^{-\,\nu} + 2\beta_{3(13)}(\partial_\mu V^{\mu-})A_\nu V^{+\,\nu} + \\
&+2\beta_{3(23)}(\partial_\mu V^{\mu+})U_\nu V^{-\,\nu} + 2\beta_{3(23)}(\partial_\mu V^{\mu-})U_\nu V^{+\,\nu} + \\
&-2i\beta_{3(14)}(\partial_\mu V^{\mu+})A_\nu V^{-\,\nu} + 2i\beta_{3(14)}(\partial_\mu V^{\mu-})A_\nu V^{+\,\nu} + \\
&-2i\beta_{3(24)}(\partial_\mu V^{\mu+})U_\nu V^{-\,\nu} + 2i\beta_{3(24)}(\partial_\mu V^{\mu-})U_\nu V^{+\,\nu} + \\
&+2\beta_{1(33)}(\partial_\mu A^\mu)V^+_\nu V^{\nu-} + 2\beta_{2(33)}(\partial_\mu U^\mu)V^+_\nu V^{\nu-}
\end{aligned} \tag{6.57}$$

Regrouping and taking in account that

$$\partial_\mu V^{\mu+} = \frac{1}{2}S^{\mu+}_{\mu}, \quad \partial_\mu V^{\mu-} = \frac{1}{2}S^{\mu-}_{\mu}, \quad \partial_\mu A^\mu = \frac{1}{2}S^{\mu 1}_{\mu}, \quad \partial_\mu U^\mu = \frac{1}{2}S^{\mu 2}_{\mu}, \tag{6.58}$$

we have

$$\mathcal{L}_{\beta,3}^{V^+,V^-} = (\beta_{3(13)} - i\beta_{3(14)})S_\mu^{\mu+} A_\nu V^{-\,\nu} + \tag{6.59}$$
$$+(\beta_{3(13)} + i\beta_{3(14)})S_\mu^{\mu-} A_\nu V^{+\,\nu} + \tag{6.60}$$
$$+(\beta_{3(23)} - i\beta_{3(24)})S_\mu^{\mu+} U_\nu V^{-\,\nu} + \tag{6.61}$$
$$+(\beta_{3(23)} + i\beta_{3(24)})S_\mu^{\mu-} U_\nu V^{+\,\nu} + \tag{6.62}$$
$$+(\beta_{1(33)} S_\mu^{\mu 1} + \beta_{2(33)} S_\mu^{\mu 2}) V_\nu^+ V^{\nu-} \tag{6.63}$$

Let´s remember that:

$$\beta_{I(JK)} = 4\rho_I \gamma_{(JK)} + (4\beta_I + 16\rho_I)\tau_{(JK)} \tag{6.64}$$

So:

$$\mathcal{L}_{\beta,3}^{V^+,V^-} = 4\rho_3\big(\gamma_{(13)} - i\gamma_{(14)}\big) S_\mu^{\mu+} A_\nu V^{-\,\nu} + \tag{6.65}$$
$$+(4\beta_3 + 16\rho_3)\big(\tau_{(13)} - i\tau_{(14)}\big) S_\mu^{\mu+} A_\nu V^{-\,\nu} + \tag{6.66}$$
$$+4\rho_3\big(\gamma_{(13)} + i\gamma_{(14)}\big) S_\mu^{\mu-} A_\nu V^{+\,\nu} + \tag{6.67}$$
$$+(4\beta_3 + 16\rho_3)\big(\tau_{(13)} - i\tau_{(14)}\big) S_\mu^{\mu-} A_\nu V^{+\,\nu} + \tag{6.68}$$
$$+4\rho_3\big(\gamma_{(23)} - i\gamma_{(24)}\big) S_\mu^{\mu+} U_\nu V^{-\,\nu} + \tag{6.69}$$
$$+(4\beta_3 + 16\rho_3)\big(\tau_{(23)} - i\tau_{(24)}\big) S_\mu^{\mu-} U_\nu V^{+\,\nu} + \tag{6.70}$$
$$+4\rho_3\big(\gamma_{(23)} + i\gamma_{(24)}\big) S_\mu^{\mu-} U_\nu V^{+\,\nu} + \tag{6.71}$$
$$+(4\beta_3 + 16\rho_3)\big(\tau_{(23)} + i\tau_{(24)}\big) S_\mu^{\mu-} U_\nu V^{+\,\nu} + \tag{6.72}$$
$$+4\rho_1\gamma_{(33)} S^{\mu 1} V_\nu^+ V^{\nu-} + 4\rho_2\gamma_{(33)} S_\mu^{\mu 2} V_\nu^+ V^{\nu-} \tag{6.73}$$
$$+(4\beta_1 + 16\rho_1)\tau_{(33)} S_\mu^{\mu 1} V_\nu^+ V^{\nu-} \tag{6.74}$$
$$+(4\beta_2 + 16\rho_2)\tau_{(33)} S_\mu^{\mu 2} V_\nu^+ V^{\nu-} \tag{6.75}$$

We will have at the end:

$$\begin{aligned}
\mathcal{L}_{\beta,3}^{V^+,V^-} &= 4\rho_3 \overset{(-1)}{z}{}^{\nu}_{\ \nu} S_\mu^{\ \mu+} + 4\big(\beta_3 + 4\rho_3\big) \overset{(-1)}{\omega}{}^{\nu}_{\ \nu} S_\mu^{\ \mu+} + \\
&+4\rho_3 \overset{(+1)}{z}{}^{\nu}_{\ \nu} S_\mu^{\ \mu-} + 4\big(\beta_3 + 4\rho_3\big) \overset{(+1)}{\omega}{}^{\nu}_{\ \nu} S_\mu^{\ \mu-} + \\
&+4\rho_3 \overset{(-2)}{z}{}^{\nu}_{\ \nu} S_\mu^{\ \mu+} + 4\big(\beta_3 + 4\rho_3\big) \overset{(-2)}{\omega}{}^{\nu}_{\ \nu} S_\mu^{\ \mu+} + \\
&+4\rho_3 \overset{(+2)}{z}{}^{\nu}_{\ \nu} S_\mu^{\ \mu-} + 4\big(\beta_3 + 4\rho_3\big) \overset{(+2)}{\omega}{}^{\nu}_{\ \nu} S_\mu^{\ \mu-} + \\
&+4 \overset{+-3}{z}{}^{\nu}_{\ \nu} \big(\rho_1 S_\mu^{\ \mu 1} + \rho_2 S_\mu^{\ \mu 2}\big) + \\
&+\big(4\beta_1 + 16\rho_1\big) \overset{+-3}{\omega}{}^{\nu}_{\ \nu} S_\mu^{\ \mu 1} + \big(4\beta_2 + 16\rho_2\big) \overset{+-3}{\omega}{}^{\nu}_{\ \nu} S_\mu^{\ \mu 2}
\end{aligned} \tag{6.76}$$

Four Gauge Invariant blocks in terms of Granular and Collective tensors

:

First Block:

$$\mathcal{L}_3^A = 4b_1 F_{\mu\nu}\Big(\overset{[12]}{z}{}^{[\mu\nu]} + \overset{[+-]}{z}{}^{[\mu\nu]}\Big) + 4b_2 U_{\mu\nu}\Big(\overset{[12]}{z}{}^{[\mu\nu]} + \overset{[+-]}{z}{}^{[\mu\nu]}\Big) +$$
$$+4b_3 V^+_{\mu\nu}\Big(\overset{[-1]}{z}{}^{[\mu\nu]} + \overset{[-2]}{z}{}^{[\mu\nu]}\Big) + 4b_3 V^-_{\mu\nu}\Big(\overset{[+1]}{z}{}^{[\mu\nu]} + \overset{[+2]}{z}{}^{[\mu\nu]}\Big) \tag{6.77}$$

Second Block

$$\mathcal{L}_3^A = 4c_1 F_{\mu\nu}\Big(\overset{[12]}{z}{}^{[\mu\nu]} + \overset{[+-]}{z}{}^{[\mu\nu]}\Big) + 4c_2 U_{\mu\nu}\Big(\overset{[12]}{z}{}^{[\mu\nu]} + \overset{[+-]}{z}{}^{[\mu\nu]}\Big) +$$
$$+4c_3 V^+_{\mu\nu}\Big(\overset{[-1]}{z}{}^{[\mu\nu]} + \overset{[-2]}{z}{}^{[\mu\nu]}\Big) + 4c_3 V^-_{\mu\nu}\Big(\overset{[+1]}{z}{}^{[\mu\nu]} + \overset{[+2]}{z}{}^{[\mu\nu]}\Big) \tag{6.78}$$

Third Block

$$\mathcal{L}_3 = 4\beta_1 F_{\mu\nu} \overset{(+-)}{z}{}^{[\mu\nu]} + 4\beta_2 U_{\mu\nu} \overset{(+-)}{z}{}^{[\mu\nu]} +$$
$$+2\beta_1 S^1_{\mu\nu}\Big(\overset{(11)}{z}{}^{\mu\nu} + 2\overset{(12)}{z}{}^{\mu\nu} + \overset{(22)}{z}{}^{\mu\nu} + 2\overset{+-3}{z}{}^{\mu\nu}\Big) +$$
$$+2\beta_2 S^2_{\mu\nu}\Big(\overset{(11)}{z}{}^{\mu\nu} + 2\overset{(12)}{z}{}^{\mu\nu} + \overset{(22)}{z}{}^{\mu\nu} + 2\overset{+-3}{z}{}^{\mu\nu}\Big) +$$
$$+4\beta_3 S^+_{\mu\nu}\Big(\overset{(-1)}{z}{}^{\mu\nu} + \overset{(-2)}{z}{}^{\mu\nu}\Big) + 4\beta_3 S^-_{\mu\nu}\Big(\overset{(+1)}{z}{}^{\mu\nu} + \overset{(+2)}{z}{}^{\mu\nu}\Big) + \tag{6.79}$$

Fourth Block

$$\mathcal{L}_3 = 2\rho_1 S_\mu^{\mu 1} \overset{(11)}{z}{}^\nu_\nu + 4\rho_1 S_\mu^{\mu 1} \overset{(12)}{z}{}^\nu_\nu + 2\rho_1 S_\mu^{\mu 1} \overset{(22)}{z}{}^\nu_\nu + 4\rho_1 S_\mu^{\mu 1} \overset{+-3}{z}{}^\nu_\nu +$$
$$+2(\beta_1 + 4\rho_1) S_\mu^{\mu 1} \overset{(11)}{\omega}{}^\nu_\nu + 4(\beta_1 + 4\rho_1) S_\mu^{\mu 1} \overset{(12)}{\omega}{}^\nu_\nu +$$
$$+2(\beta_1 + 4\rho_1) S_\mu^{\mu 1} \overset{(22)}{\omega}{}^\nu_\nu + 4(\beta_1 + 4\rho_1) S_\mu^{\mu 1} \overset{+-3}{\omega}{}^\nu_\nu +$$
$$+2\rho_2 S_\mu^{\mu 2} \overset{(11)}{z}{}^\nu_\nu + 4\rho_2 S_\mu^{\mu 2} \overset{(12)}{z}{}^\nu_\nu + 2\rho_2 S_\mu^{\mu 2} \overset{(22)}{z}{}^\nu_\nu + 4\rho_2 S_\mu^{\mu 2} \overset{+-3}{z}{}^\nu_\nu +$$
$$+2(\beta_2 + 4\rho_2) S_\mu^{\mu 2} \overset{(11)}{\omega}{}^\nu_\nu + 4(\beta_2 + 4\rho_2) S_\mu^{\mu 2} \overset{(12)}{\omega}{}^\nu_\nu +$$
$$+2(\beta_2 + 4\rho_2) S_\mu^{\mu 2} \overset{(22)}{\omega}{}^\nu_\nu + 4(\beta_2 + 4\rho_2) S_\mu^{\mu 2} \overset{+-3}{\omega}{}^\nu_\nu$$
$$+4(\beta_3 + 4\rho_3) S_\mu^{\mu +} \overset{(-1)}{\omega}{}^\nu_\nu + 4(\beta_3 + 4\rho_3) S_\mu^{\mu +} \overset{(-2)}{\omega}{}^\nu_\nu +$$
$$+4\rho_3 S_\mu^{\mu +} \overset{(-1)}{z}{}^\nu_\nu + 4\rho_3 S_\mu^{\mu +} \overset{(-2)}{z}{}^\nu_\nu +$$
$$+4(\beta_3 + 4\rho_3) S_\mu^{\mu -} \overset{(+1)}{\omega}{}^\nu_\nu + 4(\beta_3 + 4\rho_3) S_\mu^{\mu -} \overset{(+2)}{\omega}{}^\nu_\nu +$$
$$+4\rho_3 S_\mu^{\mu -} \overset{(+1)}{z}{}^\nu_\nu + 4\rho_3 S_\mu^{\mu -} \overset{(+2)}{z}{}^\nu_\nu + \tag{6.80}$$

7 $U(1) \times SO(2)$: Four nonlinear interaction sector

The interaction term, L_I^4 is expressed at basis$\{D, X_i\}$ as

$$\mathcal{L}_I^4 = \mathcal{L}_I^{4A} + \mathcal{L}_I^{4S} + \mathcal{L}_I^{4st} \tag{7.1}$$

where

$$L_I^{4A} = \gamma_{[ij]}\gamma_{[kl]}X_\mu^i X_\nu^j X^{\mu k}X^{\nu l} \tag{7.2}$$
$$L_I^{4A} = z_{[\mu\nu]}z^{[\mu\nu]}$$

and

$$L_I^{4S} = \{\gamma_{(il)} + \gamma_{(jm)} + 2\gamma_{(ij)}\tau_{(lm)} + 4\tau_{(ij)}\tau_{(lm)}\}X_\alpha^i X^{\alpha j}X_\beta^l X^{\beta m} \tag{7.3}$$
$$L_I^{4S} = z_{(\mu\nu)}z^{(\mu\nu)} + 2z_\alpha^\alpha \omega_\beta^\beta + 4\omega_\alpha^\alpha \omega_\beta^\beta$$

Considering the physical basis $\{G_J\}$,

$$\mathcal{L}_I^{4A} = z_{[\mu\nu]}z^{[\mu\nu]} = \gamma_{[IJ]}\gamma_{[KL]}G_\mu^I G_\nu^J G^{\mu K}G^{\nu L} \tag{7.4}$$

$$\begin{aligned}\mathcal{L}_I^{4S} &= z_{(\mu\nu)}z^{(\mu\nu)} + 2z_\alpha^\alpha \omega_\beta^\beta + 4\omega_\alpha^\alpha \omega_\beta^\beta = \\ &= \gamma_{(IJ)}\gamma_{(KL)}G_\mu^I G_\nu^J G^{\mu K}G^{\nu L} + 2\gamma_{(IJ)}\tau_{(KL)}G_\alpha^I G^{\alpha J}G_\beta^K G^{\beta L} + 4\tau_{(IJ)}\tau_{(KL)}G_\alpha^I G^{\alpha J}G_\beta^K G^{\beta L}\end{aligned} \tag{7.5}$$

We can express these two contributions in the following way:

$$\mathcal{L}_I^{4AS} = \Big(\gamma_{[IJ]}\gamma_{[KL]} + \gamma_{(IJ)}\gamma_{(KL)} + 2\gamma_{(IK)}\tau_{(JL)} + \tau_{(IK)}\tau_{(JL)}\Big)G_\mu^I G_\nu^J G^{\mu K}G^{\nu L} \tag{7.6}$$

Including the semitopological term from eq. (2.65), we have

$$\mathcal{L}_{st}^4 = \rho_{IJKL}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\nu^J G_\rho^K G_\sigma^L \tag{7.7}$$

where

$$\rho_{IJKL} = \gamma_{[IJ]}\gamma_{[KL]} \tag{7.8}$$

Finally, one obtains the reduced form:

$$\mathcal{L}_I^4 = a_{IJKL}G_\mu^I G_\nu^J G^{\mu K}G^{\nu L} \tag{7.9}$$

where

$$\begin{aligned}a_{IJKL} = \Big(&\gamma_{[IJ]}\gamma_{[KL]} + \gamma_{(IJ)}\gamma_{(KL)} + 2\gamma_{(IK)}\tau_{(JL)} + \\ &+\tau_{(IK)}\tau_{(JL)} + \eta\gamma_{[IJ]}\gamma_{[KL]}\eta\epsilon^{\mu\nu\rho\sigma}\Big)\end{aligned} \tag{7.10}$$

For the purpose of having a $\mathcal{L}_I^4$ that is SO(2) invariant, we get from Appendix 6 the following expression:

$$\begin{aligned}
\mathcal{L}_4 &= a_{1111}A_\mu A_\nu A^\mu A^\nu + a_{2222}U_\mu U_\nu U^\mu U^\nu + 4a_{1112}A_\mu A_\nu A^\mu U^\nu + \\
&+4a_{1222}A_\mu U_\nu U^\mu U^\nu + 2a_{1122}A_\mu A_\nu U^\mu U^\nu + 2a_{1212}A_\mu U_\nu A^\mu U^\nu + \\
&+2a_{1221}A_\mu U_\nu U^\mu A^\nu + \\
&+2a_{1133}A_\mu A_\nu\{G^{\mu 3}G^{\nu 3} + G^{\mu 4}G^{\nu 4}\} + 2a_{1331}A_\mu A^\nu\{G^{\mu 3}G^3_\nu + G^{\mu 4}G^4_\nu\} + \\
&+2a_{1313}A_\mu A^\mu\{G^3_\nu G^{\nu 3} + G^4_\nu G^{\nu 4}\} + 2a_{2233}U_\mu U_\nu\{G^{\mu 3}G^{\nu 3} + G^{\mu 4}G^{\nu 4}\} + \\
&+2a_{2332}U_\mu U\nu\{G^{\mu 3}G^3_\nu + G^{\mu 4}G^{\nu 4}\} + 2a_{2323}U_\mu U^\mu\{G^3_\nu G^{\nu 3} + G^4_\nu G^{\nu 4}\} + \\
&+4a_{1233}A_\mu U_\nu\{G^{\mu 3}G^{\nu 3} + G^{\mu 4}G^{\nu 4}\} + 2a_{1332}A_\mu U^\nu\{G^{\mu 3}G^3_\nu + G^{\mu 4}G^4_\nu\} + \\
&+4a_{1323}A_\mu U^\mu\{G^3_\nu G^{\nu 3} + G^4_\nu G^{\nu 4}\} + 2a_{1234}A_\mu U_\nu\{G^{\mu 3}G^{\nu 4} - G^{\mu 4}G^{\nu 3}\} + \\
&+4a_{1342}A_\mu U^\nu\{G^{\mu 4}G^3_\nu - G^{\mu 3}G^4_\nu\} + a_{3344}(G^3_\mu G^3_\nu + G^4_\mu G^4_\nu)(G^{\mu 3}G^{\nu 3} + G^{\mu 4}G^{\nu 4}) + \\
&+a_{3443}(G^3_\mu G^{\nu 3} + G^4_\mu G^{\nu 4})(G^{\mu 3}G^3_\nu + G^{\mu 4}G^4_\nu) + a_{3434}(G^3_\mu G^{\mu 3} + G^4_\mu G^{\mu 4})^2 \\
&+4\epsilon^{\mu\nu\rho\sigma}(\rho_{1234} + \rho_{1342})A_\mu U_\nu(G^3_\rho G^4_\sigma - G^4_\rho G^3_\sigma)
\end{aligned} \tag{7.11}$$

The associated $U(1) \times SO(2)_{\text{global}}$ gauge invariance to eq. (7.11) is studied at Appendix I.

Now we use:

$$G^3_\mu = \frac{1}{\sqrt{2}}(V^+_\mu + V^-_\mu), \qquad G^4_\mu = \frac{i}{\sqrt{2}}(V^+_\mu - V^-_\mu) \tag{7.12}$$

So:

$$\begin{aligned}
G^{\mu 3}G^{\nu 3} + G^{\mu 4}G^{\nu 4} &= V^{\mu +}V^{\nu -} + V^{\mu -}V^{\nu +} \\
G^{\mu 3}G^3_\nu + G^{\mu 4}G^4_\nu &= V^{\mu +}V^-_\nu + V^{\mu -}V^+_\nu \\
G^3_\nu G^{\nu 3} + G^4_\nu G^{\nu 4} &= 2W^+_\nu V^{\nu -} \\
G^{\mu 3}G^{\nu 4} - G^{\mu 4}G^{\nu 3} &= -iV^{\mu +}V^{\nu -} + iV^{\mu -}V^{\nu +} \\
G^{\mu 4}G^3_\nu - G^{\mu 3}G^4_\nu &= iV^{\mu +}V^-_\nu + iV^{\mu -}V^+_\nu
\end{aligned} \tag{7.13}$$

Substituing, we obtain the following expression in terms of the fields $\{A_\mu, U_\mu, V^+_\mu, V^-_\mu\}$:

$$\begin{aligned}
\mathcal{L}_4 &= a_{1111}A_\mu A_\nu A^\mu A^\nu + a_{2222}U_\mu U_\nu U^\mu U^\nu + 4a_{1112}A_\mu A_\nu A^\mu U^\nu + \\
&+4a_{1222}A_\mu U_\nu U^\mu U^\nu + 2a_{1122}A_\mu A_\nu U^\mu U^\nu + 2a_{1212}A_\mu U_\nu A^\mu U^\nu + \\
&+2a_{1221}A_\mu U_\nu U^\mu A^\nu + 2a_{1133}A_\mu A_\nu\{V^{\mu +}V^{\nu -} + V^{\mu -}V^{\nu +}\} + \\
&+2a_{1331}A_\mu A^\nu\{V^{\mu +}V^-_\nu + V^{\mu -}V^+_\nu\} + \\
&+4a_{1313}A_\mu A^\mu V^+_\nu V^{\nu -} + 2a_{2233}U_\mu U_\nu\{V^{\mu +}V^{\nu -} + V^{\mu -}V^{\nu -}\} + \\
&+2a_{2332}U_\mu U^\nu\{V^{\mu +}V^-_\nu + V^{\mu -}V^{\nu +}\} + 4a_{2323}U_\mu U^\mu V^+_\nu V^{\nu -} + \\
&+4a_{1233}A_\mu U_\nu\{V^{\mu +}V^{\nu -} + V^{\mu -}V^{\nu +}\} + 2a_{1332}A_\mu U^\nu\{V^{\mu +}V^-_\nu + V^{\mu -}V^+_\nu\} + \\
&+8a_{1323}A_\mu U^\mu V^+_\nu V^{\nu -} - 4ia_{1234}A_\mu U_\nu\{V^{\mu +}V^{\nu -} - V^{\mu -}V^{\nu +}\} + \\
&+4ia_{1342}A_\mu U^\nu\{V^{\mu +}V^-_\nu - V^{\mu -}V^+_\nu\} + a_{3344}(V^+_\mu V^-_\nu + V^-_\mu V^+_\nu)(V^{\mu +}V^{\nu -} + V^{\mu -}V^{\nu +}) + \\
&+4a_{3434}(V^+_\mu V^{\mu -})^2 + a_{3443}(V^+_\mu V^{\nu -} + V^-_\mu V^{\nu +})(V^{\mu +}V^-_\nu + V^{\mu -}V^+_\nu) \\
&+4i(\rho_{1234} - \rho_{1324} + \rho_{1342})\epsilon^{\mu\nu\rho\sigma}A_\mu U_\nu[V^-_\rho V^+_\sigma - V^+_\rho V^-_\sigma]
\end{aligned} \tag{7.14}$$

We can now write $\mathcal{L}_I^4$ as

$$\mathcal{L}_4 = \mathcal{L}_4^A + \mathcal{L}_4^S \tag{7.15}$$

where

$$\begin{aligned}
\mathcal{L}_4^A = {} & 2\overset{[12]}{z}_{[\mu\nu]}\overset{[12]}{z}^{[\mu\nu]} + 2\overset{[12]}{z}_{[\mu\nu]}\overset{[21]}{z}^{[\mu\nu]} + 4\overset{[13+]}{z}_{[\mu\nu]}\overset{[13-]}{z}^{[\mu\nu]} + \\
& +4\overset{[23+]}{z}_{[\mu\nu]}\overset{[23-]}{z}^{[\mu\nu]} + 8\overset{[13+]}{z}_{[\mu\nu]}\overset{[23-]}{z}^{[\mu\nu]} + 8\overset{[12]}{z}_{[\mu\nu]}\overset{[+-]}{z}^{[\mu\nu]} + \\
& -8\overset{(12)}{z}_{[\mu\nu]}\overset{(12)}{\omega}^{[\mu\nu]} - 16\overset{(12)}{\omega}_{[\mu\nu]}\overset{(12)}{\omega}^{[\mu\nu]} + 4\overset{(13+)}{z}_{[\mu\nu]}\overset{(13-)}{z}^{[\mu\nu]} + \\
& +4\overset{(23+)}{z}_{[\mu\nu]}\overset{(23-)}{z}^{[\mu\nu]} + 8\overset{(13+)}{z}_{[\mu\nu]}\overset{(23-)}{z}^{[\mu\nu]} - 16\overset{(+-)}{z}_{[\mu\nu]}\overset{(+-)}{\omega}^{[\mu\nu]} + \\
& -32\overset{(+-)}{\omega}_{[\mu\nu]}\overset{(+-)}{\omega}^{[\mu\nu]} - 4i\overset{(13+)}{z}_{[\mu\nu]}\overset{(24-)}{\omega}^{[\mu\nu]} + 4i\overset{(13-)}{z}_{[\mu\nu]}\overset{(24+)}{\omega}^{[\mu\nu]} + \\
& -4i\overset{(23+)}{z}_{[\mu\nu]}\overset{(14-)}{\omega}^{[\mu\nu]} + 4i\overset{(23-)}{z}_{[\mu\nu]}\overset{(14+)}{\omega}^{[\mu\nu]} - 16i\overset{(13+)}{\omega}_{[\mu\nu]}\overset{(24-)}{\omega}^{[\mu\nu]} + \\
& +16i\overset{(13-)}{\omega}_{[\mu\nu]}\overset{(24+)}{\omega}^{[\mu\nu]} - 2i\overset{(14+)}{z}_{[\mu\nu]}\overset{(23-)}{z}^{[\mu\nu]} + 2i\overset{(14-)}{z}_{[\mu\nu]}\overset{(23+)}{z}^{[\mu\nu]} + \\
& +16i\overset{(14+)}{\omega}_{[\mu\nu]}\overset{(23-)}{\omega}^{[\mu\nu]} - 16i\overset{(14-)}{\omega}_{[\mu\nu]}\overset{(23+)}{\omega}^{[\mu\nu]} + 2i\overset{(13+)}{z}_{[\mu\nu]}\overset{(24-)}{z}^{[\mu\nu]} + \\
& -2i\overset{(13-)}{z}_{[\mu\nu]}\overset{(24+)}{z}^{[\mu\nu]} + 4i\overset{(14+)}{z}_{[\mu\nu]}\overset{(23-)}{\omega}^{[\mu\nu]} - 4i\overset{(14-)}{z}_{[\mu\nu]}\overset{(23+)}{\omega}^{[\mu\nu]} + \\
& +4i\overset{(24+)}{z}_{[\mu\nu]}\overset{(13-)}{\omega}^{[\mu\nu]} - 4i\overset{(24-)}{z}_{[\mu\nu]}\overset{(13+)}{\omega}^{[\mu\nu]}
\end{aligned} \tag{7.16}$$

and

$$
\begin{aligned}
&\mathcal{L}_4^S = 2\overset{[12]}{z}_{(\mu\nu)}\overset{[12]}{z}^{(\mu\nu)} + 2\overset{[12]}{z}_{(\mu\nu)}\overset{[21]}{z}^{(\mu\nu)} + 4\overset{[13+]}{z}_{(\mu\nu)}\overset{[13-]}{z}^{(\mu\nu)}\\
&+4\overset{[23+]}{z}_{(\mu\nu)}\overset{[23-]}{z}^{(\mu\nu)} + 8\overset{[13+]}{z}_{(\mu\nu)}\overset{[23-]}{z}^{(\mu\nu)} + 4\overset{[+-]}{z}_{(\mu\nu)}\overset{[+-]}{z}^{(\mu\nu)}\\
&\overset{(11)}{z}_{(\mu\nu)}\overset{(11)}{z}^{(\mu\nu)} + \overset{(22)}{z}_{(\mu\nu)}\overset{(22)}{z}^{(\mu\nu)} + 2\overset{(11)}{z}_{(\mu\nu)}\overset{(11)}{\omega}^{(\mu\nu)} + 2\overset{(22)}{z}_{(\mu\nu)}\overset{(22)}{\omega}^{(\mu\nu)}\\
&+4\overset{(11)}{\omega}_{(\mu\nu)}\overset{(11)}{\omega}^{(\mu\nu)} + 4\overset{(22)}{\omega}_{(\mu\nu)}\overset{(22)}{\omega}^{(\mu\nu)} + 4\overset{(11)}{z}_{(\mu\nu)}\overset{(12)}{z}^{(\mu\nu)} + 4\overset{(12)}{z}_{(\mu\nu)}\overset{(22)}{z}^{(\mu\nu)}\\
&+8\overset{(12)}{z}_{(\mu\nu)}\overset{(11)}{\omega}^{(\mu\nu)} + 8\overset{(12)}{z}_{(\mu\nu)}\overset{(22)}{\omega}^{(\mu\nu)} + 16\overset{(11)}{\omega}_{(\mu\nu)}\overset{(12)}{\omega}^{(\mu\nu)} + 16\overset{(12)}{\omega}_{(\mu\nu)}\overset{(22)}{\omega}^{(\mu\nu)}\\
&+2\overset{(11)}{z}_{(\mu\nu)}\overset{(22)}{z}^{(\mu\nu)} + 4\big(\overset{(11)}{z}_{(\mu\nu)} + \overset{(22)}{z}_{(\mu\nu)}\big)\overset{+-3}{z}^{(\mu\nu)} + 4\overset{(12)}{z}_{(\mu\nu)}\overset{(12)}{z}^{(\mu\nu)} +\\
&+8\overset{(12)}{z}_{(\mu\nu)}\overset{(12)}{\omega}^{(\mu\nu)} + 16\overset{(12)}{\omega}_{(\mu\nu)}\overset{(12)}{\omega}^{(\mu\nu)} + 4\overset{(13+)}{z}_{(\mu\nu)}\overset{(13-)}{z}^{(\mu\nu)} +\\
&+4\overset{(23+)}{z}_{(\mu\nu)}\overset{(23-)}{z}^{(\mu\nu)} + 8\overset{(13+)}{z}_{(\mu\nu)}\overset{(23-)}{z}^{(\mu\nu)} + 8\overset{(12)}{z}_{(\mu\nu)}\overset{+-3}{z}^{(\mu\nu)} +\\
&4\overset{+-3}{z}_{(\mu\nu)}\overset{+-4}{z}^{(\mu\nu)} + 4i\overset{(13+)}{z}_{(\mu\nu)}\overset{(24-)}{\omega}^{(\mu\nu)} - 4i\overset{(13-)}{z}_{(\mu\nu)}\overset{(24+)}{\omega}^{(\mu\nu)} +\\
&+4i\overset{(23+)}{z}_{(\mu\nu)}\overset{(14-)}{\omega}^{(\mu\nu)} - 4i\overset{(23-)}{z}_{(\mu\nu)}\overset{(14+)}{\omega}^{(\mu\nu)} + 16i\overset{(13+)}{\omega}_{(\mu\nu)}\overset{(24-)}{\omega}^{(\mu\nu)} +\\
&-16i\overset{(13-)}{\omega}_{(\mu\nu)}\overset{(24+)}{\omega}^{(\mu\nu)} + 2i\overset{(14+)}{z}_{(\mu\nu)}\overset{(23-)}{z}^{(\mu\nu)} - 2i\overset{(14-)}{z}_{(\mu\nu)}\overset{(23+)}{z}^{(\mu\nu)} +\\
&-16i\overset{(14+)}{\omega}_{(\mu\nu)}\overset{(23-)}{\omega}^{(\mu\nu)} + 16i\overset{(14-)}{\omega}_{(\mu\nu)}\overset{(23+)}{\omega}^{(\mu\nu)} - 2i\overset{(13+)}{z}_{(\mu\nu)}\overset{(24-)}{z}^{(\mu\nu)} +\\
&+2i\overset{(13-)}{z}_{(\mu\nu)}\overset{(24+)}{z}^{(\mu\nu)} - 4i\overset{(14+)}{z}_{(\mu\nu)}\overset{(23-)}{\omega}^{(\mu\nu)} + 4i\overset{(14-)}{z}_{(\mu\nu)}\overset{(23+)}{\omega}^{(\mu\nu)} +\\
&-4i\overset{(24+)}{z}_{(\mu\nu)}\overset{(13-)}{\omega}^{(\mu\nu)} + 4i\overset{(24-)}{z}_{(\mu\nu)}\overset{(13+)}{\omega}^{(\mu\nu)} +\\
&-4\overset{[13+]}{z}{}^{\mu}_{\mu}\overset{[13-]}{z}{}^{\nu}_{\nu} - 4\overset{[23+]}{z}{}^{\mu}_{\mu}\overset{[23-]}{z}{}^{\nu}_{\nu} - 4\overset{[+-]}{z}{}^{\mu}_{\mu}\overset{[+-]}{z}{}^{\nu}_{\nu} +\\
&-4\overset{[13+]}{z}{}^{\mu}_{\mu}\overset{[23-]}{z}{}^{\nu}_{\nu} + 4\overset{[13-]}{z}{}^{\mu}_{\mu}\overset{[23+]}{z}{}^{\nu}_{\nu} + 4i\overset{[13-]}{z}{}^{\mu}_{\mu}\overset{[24+]}{z}{}^{\nu}_{\nu} - 4i\overset{[13+]}{z}{}^{\mu}_{\mu}\overset{[24-]}{z}{}^{\nu}_{\nu}\\
&+4\overset{(11)}{z}{}^{\mu}_{\mu}\overset{(22)}{\omega}{}^{\nu}_{\nu} + 8\overset{(11)}{\omega}{}^{\mu}_{\mu}\overset{(22)}{\omega}{}^{\nu}_{\nu} + 4\overset{(13+)}{z}{}^{\mu}_{\mu}\overset{(13-)}{z}{}^{\nu}_{\nu} + 4\overset{(23+)}{z}{}^{\mu}_{\mu}\overset{(23-)}{z}{}^{\nu}_{\nu} +\\
&+16\overset{(13+)}{z}{}^{\mu}_{\mu}\overset{(13-)}{\omega}{}^{\nu}_{\nu} + 16\overset{(23+)}{z}{}^{\mu}_{\mu}\overset{(23-)}{\omega}{}^{\nu}_{\nu} + 32\overset{(13+)}{\omega}{}^{\mu}_{\mu}\overset{(13-)}{\omega}{}^{\nu}_{\nu} +\\
&+32\overset{(23+)}{\omega}{}^{\mu}_{\mu}\overset{(23-)}{\omega}{}^{\nu}_{\nu} + 4\overset{(13+)}{z}{}^{\mu}_{\mu}\overset{(23-)}{z}{}^{\nu}_{\nu} - 4\overset{(13-)}{z}{}^{\mu}_{\mu}\overset{(23+)}{z}{}^{\nu}_{\nu}\\
&+16\overset{(13+)}{z}{}^{\mu}_{\mu}\overset{(13-)}{\omega}{}^{\nu}_{\nu} - 16\overset{(13-)}{z}{}^{\mu}_{\mu}\overset{(13+)}{\omega}{}^{\nu}_{\nu} + 32\overset{(13+)}{\omega}{}^{\mu}_{\mu}\overset{(23-)}{\omega}{}^{\nu}_{\nu} +\\
&-32\overset{(13-)}{\omega}{}^{\mu}_{\mu}\overset{(23+)}{\omega}{}^{\nu}_{\nu} - 4\overset{(+-)}{z}{}^{\mu}_{\mu}\overset{(+-)}{z}{}^{\nu}_{\nu} + 8\overset{+-3}{z}{}^{\mu}_{\mu}\overset{+-4}{\omega}{}^{\nu}_{\nu} + 16\overset{+-3}{\omega}{}^{\mu}_{\mu}\overset{+-4}{\omega}{}^{\nu}_{\nu} +\\
&+8\Big\{\overset{(11)}{z}{}^{\mu}_{\mu} + \overset{(22)}{z}{}^{\mu}_{\mu} + 2\overset{(11)}{\omega}{}^{\mu}_{\mu} + 2\overset{(22)}{\omega}{}^{\mu}_{\mu} + 2\overset{(12)}{z}{}^{\mu}_{\mu} + 4\overset{(12)}{\omega}{}^{\mu}_{\mu}\Big\}\overset{+-3}{\omega}{}^{\nu}_{\nu} +
\end{aligned}
\tag{7.17}
$$

8 Conclusion

Nature is not only structured by matter as by fields. Differently from 17^{th} century when Newton laws were edified in terms of matter, Maxwell equations introduced at 19^{th} century the meaning of fields. Physical behaviors as Faraday law become explained through lines of force. They became the new constitutions in order to manifest physical laws. Later on, Maxwell incorporated the displacement current composed by electric fields. Followed that, physical entities as EM energy, momentum, light propagation, and so on, become physically realistic through fields. Something saying that nature should no more be expressed just as straight matter. Aside to elements, as mass and electric charge, fields should be incorporated.

Maxwell equations mixed matter and fields. At one side, Coulomb and Ampère laws depending on current and charge; on the other hand, Faraday and Gauss magnetic laws on fields. Nevertheless, Maxwell legacy is fields generated by electric charge. It does not understand the fields energy flux as primordial. Maxwell theory shows the EM phenomena as determined by fields being derived by external sources.

Thus, a next physics challenge is to go beyond such Maxwell relationships between charges and fields. Einstein's equations gave a first step at this direction [27]. It established a nonlinear field theory development. The appearance of a nonlinear electromagnetism is a further step on this general relativity structure. A physicality where instead of charges and magnets, fields work as their own sources. The Salam-Shaw approach to non-abelian gauge theory was to look for selfinteracting photons[18]. This is the new fact coming from the various NLED models being proposed at literature.

A nonlinear EM changes our vision on EM phenomena. Another development happens. Nonlinearity makes nature more primitive than material sources. Their extension to Maxwell equations introduces a primary dynamics where fields precedes matter. Maxwell equations are also fields dependent, Faraday law gave the clue for electric and magnetic fields as source between themselves, however, nonlinear EM fields introduce a step forward. They show that fields do not come from charges and currents but from themselves.

Under this scope, eq.(1.1) develops a nonlinear electromagnetism where fields are their own sources. Something to be understood. It provides a new EM flux just depending on fields. New ingredients as fields conglomerates, granular and collective fields strengths, are performed together with nonlinear currents. There is an non-materialistic physics to be analyzed. Understand how fields surpass matter and their consequences.

Maxwell provides the EM flux simplest case. It is given by $\vec{E}$-$\vec{B}$ association and has only one EM domain which is the piece $\mathcal{L} = F_{\mu\nu}^2$. It is enough to show through Vector Poynting that its energy flux direction is not the same of the electric current. It shows how matters and EM transmission are moved by fields. As a next development, eq.(1.1) interlaces the photon with massive photon and charged photons. It shows that not only $\vec{E}$-$\vec{B}$ affect each other as the potential fields $\{A_\mu - U_\mu - V_\mu^\pm\}$ are interlined. Their integration build up electromagnetism regions constituted by different pieces which are called here as EM domains. They are interdependent physical regions where each one contains its own EM energy, and so, proper physics.

Thus, through Four Bosons, this work studies an antireductionist nonlinear microscopic electromagnetism. Based on charges set $\{+, 0, -\}$ a fields set $\{A_\mu, U_\mu, V_\mu^\pm\}$ is introduced. The efforts at this work became to clarify on the gauge invariance and electromagnetic domains running under this EM flux. Each of them contains its own physicity. Carrying its own energy, momentum and so on propitiating a physicality which precedes the electric charge. They also contain a separate and interconnected spin-1 and spin-0 physics.

Thus preserving the two basic Maxwell postulates which are light invariance and charge conservation, it enlarges an EM flux bigger than Maxwell flux. It not only expands the meaning of electric charge behavior in terms of conservation, conduction, transmission, interaction, but fundamentally, develops a new EM flux. It introduces a physics not depending on electric charge, permissibility ϵ and permeability μ constants, and with coupling constants beyond electric charge.

Thus the Four Bosons Electromagnetism proposes a health antireductionist nonlinear EM. It extends Maxwell to an EM energy being exchanged by three charges through a fields quadruplet. As consequence, it appears a new way of transmitting EM energy. It appears interlaced granular and collective EM fields. These fields are nonlinear and may react collectively. They provide an EM beyond electric charge and constitued by interdependent domains.

9 References

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Appendix A. Field Basis $\{D, X_i\}$

A.1. Field strength tensors

1. Granular type:

$$D_{\mu\nu} = \partial_\mu D_\nu - \partial_\nu D_\mu \tag{A.1}$$

$$X^i_{\mu\nu} = \partial_\mu X^i_\nu - \partial_\nu X^i_\mu \tag{A.2}$$

$$S^i_{\mu\nu} = \partial_\mu X^i_\nu + \partial_\nu X^i_\mu \tag{A.3}$$

So, under: $D'_\mu = D_\mu + N\partial_\mu\alpha$ and $X'^{i}_{\mu} = X^i_\mu$, we have:

$$D'_{\mu\nu} = D_{\mu\nu} \quad , \quad X'^i_{\mu\nu} = X^i_{\mu\nu} \quad , \quad S'^i_{\mu\nu} = S^i_{\mu\nu}. \tag{A.4}$$

$$\begin{aligned} D'_{\mu\nu} &= \partial_\mu D'_\nu - \partial_\nu D'_\mu \\ &= \partial_\mu(D_\nu + N\partial_\nu\alpha) - \partial_\nu(D_\mu + N\partial_\mu\alpha) \\ &= \partial_\mu D_\nu - \partial_\nu D_\mu + N\partial_\mu\partial_\nu\alpha - N\partial_\mu\partial_\nu\alpha = D_{\mu\nu} \end{aligned} \tag{A.5}$$

$$X'^{i}_{\mu\nu} = \partial_\mu X'^{\,i}_{\nu} - \partial_\nu X'^{\,i}_{\mu} = \partial_\mu X^i_\nu - \partial_\nu X^i_\mu = X^i_{\mu\nu} \tag{A.6}$$

2. Collective type

   We can construct the following gauge invariant compositions:

$$\begin{aligned} z_{[\mu\nu]} &= \gamma_{[ij]} X^i_\mu X^j_\nu \\ z_{(\mu\nu)} &= \gamma_{(ij)} X^i_\mu X^j_\nu \\ \omega_{(\mu\nu)} &= \tau_{(ij)} X^i_\mu X^j_\nu \end{aligned} \tag{A.7}$$

3. Collective fields 2-to-2

   Consider the expansion

$$z_{\mu\nu} = \gamma_{22} X^2_\mu X^2_\nu + \gamma_{23} X^2_\mu X^3_\nu + ... + \gamma_{NN} X^N_\mu X^N_\nu \tag{A.8}$$

   it yields that eq.(A.8) provides $(N-1)^2$ real and gauge invariants 2-to-2 collective fields.

4. Generic Tensors

   In this way, we can construct the following gauge invariant tensors of antisymmetric and symmetric nature. Using already the definition of collective tensors written above:

$$Z_{[\mu\nu]} = dD_{\mu\nu} + \alpha_i X^i_{\mu\nu} + z_{[\mu\nu]} \tag{A.9}$$

$$Z_{(\mu\nu)} = \beta_i S^i_{\mu\nu} + z_{(\mu\nu)} + g_{\mu\nu}\rho_i S^{\alpha i}_{\alpha} + g_{\mu\nu}\omega^\alpha_\alpha \tag{A.10}$$

   where $g_{\mu\nu}$ is the metric and $d, \alpha_i, \gamma_{[ij]}, \beta_i\, \rho_i, \gamma_{(ij)}\, \tau_{(ij)}$ are the free parameters (can assume any value without violate gauge invariance).

A.2. Lagrangian

We can construct the following Lagrangian:

$$\mathcal{L}(D, X_i) = Z_{\mu\nu}Z^{\mu\nu} - \eta Z_{\mu\nu}\tilde{Z}^{\mu\nu} - m^2_{ij} X^i_\mu X^{\mu j} + \frac{1}{2\xi}\Big[\partial_\mu\big(D^\mu + \sigma_i X^{\mu\, i}\big)\Big] \tag{A.11}$$

where:

$$Z_{\mu\nu}Z^{\mu\nu} = Z_{[\mu\nu]}Z^{[\mu\nu]} + Z_{(\mu\nu)}Z^{(\mu\nu)}$$
$$\tilde{Z}_{\mu\nu} = \varepsilon_{\mu\nu\rho\delta}Z^{\rho\delta}$$

Notice that the D_μ field only exists in the antisymmetric sector of the theory. Observe the possibility of include the mass term without Higgs. The next step is to separate the different sectors inside of by this systemic Lagrangian $\mathcal{L}$.

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_S + \mathcal{L}_{ST} + \mathcal{L}_M + \mathcal{L}_{GF} \tag{A.12}$$

1. Antisymmetric part of $\mathcal{L}$:

$$\mathcal{L}_A = Z_{[\mu\nu]}Z^{[\mu\nu]} = (dD_{\mu\nu} + \alpha_i X^i_{\mu\nu} + z_{[\mu\nu]}).(dD^{\mu\nu} + \alpha_k X^{\mu\nu k} + z^{[\mu\nu]})$$

   which gives

$$Z_{[\mu\nu]}Z^{[\mu\nu]} = (dD_{\mu\nu} + \alpha_i X^i_{\mu\nu})^2 + 2(dD_{\mu\nu} + \alpha_i X^i_{\mu\nu})z^{[\mu\nu]} + z_{[\mu\nu]}z^{[\mu\nu]} \tag{A.13}$$

   where each one of the terms is separately gauge invariant.

2. Symmetric sector of $\mathcal{L}$:

$$\mathcal{L}_S = Z_{(\mu\nu)}Z^{(\mu\nu)}$$
$$\mathcal{L}_S = \left(\beta_i S^i_{\mu\nu} + z_{(\mu\nu)} + g_{\mu\nu}\rho_i S^{\alpha i}_{\alpha} + g_{\mu\nu}\omega^{\alpha}_{\alpha}\right)\left(\beta_j S^{j\,\mu\nu} + z^{(\mu\nu)} + g^{\mu\nu}\rho_j S^{\alpha j}_{\alpha} + g^{\mu\nu}\omega^{\alpha}_{\alpha}\right) \tag{A.14}$$

   Simplifying and grouping terms:

$$Z_{(\mu\nu)}Z^{(\mu\nu)} = \beta_i\beta_j S^i_{\mu\nu}S^{\mu\nu i} + 2\beta_i S^i_{\mu\nu}z^{(\mu\nu)} + (2\beta_i\rho_j + 4\rho_i\rho_j)S^{\alpha i}_{\alpha}S^{\beta j}_{\beta} +$$
$$+z_{(\mu\nu)}z^{(\mu\nu)} + 2\beta_i S^{\alpha\,i}_{\alpha}\omega^{\beta}_{\beta} + 2\rho_i S^{\alpha\,i}_{\alpha}z^{\beta}_{\beta} + 8\rho_i S^{\alpha\,i}_{\alpha}\omega^{\beta}_{\beta} + 2z^{\alpha}_{\alpha}\omega^{\beta}_{\beta} + 4\omega^{\alpha}_{\alpha}\omega^{\beta}_{\beta} \tag{A.15}$$

   where each one of the terms is separately gauge invariant.

3. Semitopological Term

A.3. Free Coefficients

Eq.(A.11) Lagrangian contain parameters as $d, \alpha_i, \beta_i, \rho_i, m_{ij}$ and so on that can take any value without violating gauge invariance. They are called free coefficients. They are responsible for the circumstances that the model uses. (A.11) uses $\frac{1}{4}\left[3N^4 - 8N^3 - 12N + 8\right]$ free coefficients [].

A.3.1. Kinetic part:

$$\begin{aligned}
\mathcal{L}_K &= \mathcal{L}_K^A + \mathcal{L}_K^S \\
\mathcal{L}_K^A &= d^2 D_{\mu\nu} D^{\mu\nu} + 2d\alpha_i D_{\mu\nu} X^{\mu\nu i} + \alpha_i\alpha_j X^i_{\mu\nu} X^{\mu\nu j} \\
\mathcal{L}_K^S &= \beta_i\beta_j S^i_{\mu\nu} S^{\mu\nu j} + (4\rho_i\rho_j + 2\beta_i\rho_j) S^{\alpha i}_\alpha S^{\beta j}_\beta
\end{aligned} \tag{A.16}$$

A.3.2. Gauge Fixing:

$$\mathcal{L}_{GF} = \frac{1}{2\xi}\Big[\partial_\mu\big(D^\mu + \sigma_i X^{\mu\, i}\big)\Big] \tag{A.17}$$

A.3.3. Mass:

$$\mathcal{L}_m = m^2_{ij} X^i_\mu X^{\mu j} \tag{A.18}$$

A.3.4. Term of interaction $\mathcal{L}_I^3$:

$$\mathcal{L}^3 = \mathcal{L}_A^3 + \mathcal{L}_S^3 \tag{A.19}$$

$$\mathcal{L}_A^3 = 2dD_{\mu\nu} z^{[\mu\nu]} + 2\alpha_i X^i_{\mu\nu} z^{[\mu\nu]} \tag{A.20}$$

$$\mathcal{L}_S^3 = 2\beta_i S^i_{\mu\nu} z^{(\mu\nu)} + 2\rho_i S^{\alpha i}_\alpha z^\beta_\beta + (8\rho_i + 2\beta_i)\, S^{\alpha i}_\alpha \omega^\beta_\beta \tag{A.21}$$

A.3.5. Term of interaction $\mathcal{L}_I^4$:

$$\mathcal{L}_I^4 = \mathcal{L}_A^4 + \mathcal{L}_S^4 \tag{A.22}$$

$$\mathcal{L}_A^4 = z_{[\mu\nu]} z^{[\mu\nu]} \tag{A.23}$$

$$\mathcal{L}_S^4 = z_{(\mu\nu)} z^{(\mu\nu)} + 2z^\alpha_\alpha \omega^\beta_\beta + 4\omega^\alpha_\alpha \omega^\beta_\beta \tag{A.24}$$

A.3.6. Term of $\mathcal{L}^{st}$:

$$\begin{aligned}
&\mathcal{L}^{st} = 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu\Big\{\big(dD_\nu + \alpha_i X^i_\nu\big) Z_{[\rho\sigma]}\Big\} + \\
&-2d\big(dD_\nu + \alpha_i X^i_\nu\big)\gamma_{[jk]}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\big(X^j_\rho X^k_\sigma\big) + \epsilon^{\mu\nu\rho\sigma}\gamma_{[ij]} X^i_\mu X^j_\nu Z_{[\rho\sigma]}
\end{aligned} \tag{A.25}$$

Appendix B. SO(2) invariants

B.1. $\mathcal{L}_K^A$: Granular kinetic term

Considering the symmetry SO(2),

$$\begin{pmatrix} G'_{\mu 3} \\ G'_{\mu 4} \end{pmatrix} = \begin{pmatrix} \cos(q\alpha) & \sin(q\alpha) \\ -\sin(q\alpha) & \cos(q\alpha) \end{pmatrix} \begin{pmatrix} G_{\mu 3} \\ G_{\mu 4} \end{pmatrix} \tag{B.1}$$

one gets for $G^3_{\mu\nu}$,

$$\begin{aligned} &{G^3_{\mu\nu}}' = \partial_\mu G^{3\prime}_\nu - \partial_\nu G^{3\prime}_\mu \\ &\boldsymbol{{G^3_{\mu\nu}}' = \cos(q\alpha) G^3_{\mu\nu} + \sin(q\alpha) G^4_{\mu\nu}} \end{aligned} \tag{B.2}$$

Similarly, for ${G^4_{\mu\nu}}'$:

$$\boldsymbol{{G^4_{\mu\nu}}' = \sin(q\alpha) G^3_{\mu\nu} - \cos(q\alpha) G^4_{\mu\nu}} \tag{B.3}$$

It yields the following invariance

$$\begin{aligned} &1. {G^3_{\mu\nu}}' G^{\mu\nu 3\prime} + {G^4_{\mu\nu}}' G^{\mu\nu 4\prime} = \left(\cos(q\alpha) G^3_{\mu\nu} + \sin(q\alpha) G^4_{\mu\nu}\right)^2 + \left(-\sin(q\alpha) G^3_{\mu\nu} + \cos(q\alpha) G^4_{\mu\nu}\right)^2 \\ &{G^3_{\mu\nu}}' G^{\mu\nu 3\prime} + {G^4_{\mu\nu}}' G^{\mu\nu 4\prime} = \cos^2(q\alpha) G^3_{\mu\nu} G^{\mu\nu 3} + 2\cos(q\alpha)\sin(q\alpha) G^3_{\mu\nu 3} G^4_{\mu\nu 4} + \\ &+ \sin^2(q\alpha) G^4_{\mu\nu 4} G^4_{\mu\nu 4} \sin(q\alpha)^2 G^3_{\mu\nu 3} G^3_{\mu\nu 3} - 2\sin(q\alpha)\cos(q\alpha) G^3_{\mu\nu 3} G^4_{\mu\nu 4} + \cos^2(q\alpha) G^4_{\mu\nu} G^{\mu\nu 4} \end{aligned} \tag{B.4}$$

and so,

$$\boldsymbol{{G^3_{\mu\nu}}' G^{\mu\nu 3\prime} + {G^4_{\mu\nu}}' G^{\mu\nu 4\prime} = G^3_{\mu\nu} G^{\mu\nu 3} + G^4_{\mu\nu} G^{\mu\nu 4}} \tag{B.5}$$

B.2. $\mathcal{L}_K^S$: Granular kinetic terms

Similarly:

$$\begin{aligned} {S^3_{\mu\nu}}' &= \cos(q\alpha) S^3_{\mu\nu} + \sin(q\alpha) S^4_{\mu\nu} \\ {S^4_{\mu\nu}}' &= -\sin(q\alpha) S^3_{\mu\nu} + \cos(q\alpha) S^4_{\mu\nu} \end{aligned}$$

Working on the $(S^3_{\mu\nu})^2 + (S^4_{\mu\nu})^2$,

$$\begin{aligned} &({S^3_{\mu\nu}}')^2 + ({S^4_{\mu\nu}}')^2 = \\ &= (\cos(q\alpha) S^3_{\mu\nu} + \sin(q\alpha) S^4_{\mu\nu})^2 + (-\sin(q\alpha) S^3_{\mu\nu} + \cos(q\alpha) S^4_{\mu\nu})^2 = \\ &= (\cos(q\alpha) S^3_{\mu\nu})^2 + 2(\cos(q\alpha) S^3_{\mu\nu} \sin(q\alpha) S^4_{\mu\nu}) + (\sin(q\alpha) S^4_{\mu\nu})^2 + \\ &+ (-\sin(q\alpha) S^3_{\mu\nu})^2 + 2(-\sin(q\alpha) S^3_{\mu\nu} \cos(q\alpha) S^4_{\mu\nu}) + (\cos(q\alpha) S^4_{\mu\nu})^2 \\ &= (S^3_{\mu\nu})^2 (\cos(q\alpha) + \sin(q\alpha)) + (S^4_{\mu\nu})^2 (\cos(q\alpha) + \sin(q\alpha)) \\ &2. \boldsymbol{({S^3_{\mu\nu}}')^2 + ({S^4_{\mu\nu}}')^2 = (S^3_{\mu\nu})^2 + (S^4_{\mu\nu})^2} \end{aligned}$$

For the term $(S_{\alpha}^{\alpha 3})^2 + (S_{\alpha}^{\alpha 4})^2$:

$$S_{\alpha}^{\alpha 3} = 2\partial_{\alpha} G^{\alpha 3}$$
$$S_{\alpha}^{\alpha' 3} = 2\cos(q\alpha)\partial_{\alpha} G^{\alpha 3} + 2\sin(q\alpha)\partial_{\alpha} G^{\alpha 4}$$
$$(S_{\alpha}^{\alpha' 3})^2 = 4\cos^2(q\alpha)(\partial_{\alpha} G^{\alpha 3})(\partial_{\beta} G^{\beta 3}) + 4\sin^2(q\alpha)(\partial_{\alpha} G^{\alpha 4})(\partial_{\beta} G^{\beta 4}) + 8\sin(q\alpha)\cos(q\alpha)\partial_{\alpha} G^{\alpha 3}\partial_{\beta} G^{\beta 4}$$

$$S_{\alpha}^{\alpha 4} = 2\partial_{\alpha} G^{\alpha 4}$$
$$S_{\alpha}'^{\alpha 4} = 2\partial_{\alpha}\left(-\sin(q\alpha)G^{\alpha 3} + \cos(q\alpha)B^{\alpha 4}\right)$$
$$S_{\alpha}'^{\alpha 4} = -2\sin(q\alpha)\partial_{\alpha} G^{\alpha 3} + \cos(q\alpha)\partial_{\alpha} G^{\alpha 4}$$
$$(S_{\alpha}'^{\alpha 4})^2 = 4\sin^2(q\alpha)\partial_{\alpha} G^{\alpha 3}\partial_{\beta} G^{\beta 3} + 4\cos^2(q\alpha)\partial_{\alpha} G^{\alpha 4}\partial_{\beta} G^{\beta 4} - 8\sin(q\alpha)\cos(q\alpha)\partial_{\alpha} G^{\alpha 3}\partial_{\beta} G^{\beta 4}$$

And so

$$3.\boldsymbol{(S_{\alpha}'^{\alpha 3})^2 + (S_{\alpha}'^{\alpha 4})^2 = 4(\partial_{\alpha} G^{\alpha 3})^2 + 4(\partial_{\alpha} G^{\alpha 4})^2 \equiv (S_{\alpha}^{\alpha 3})^2 + (S_{\alpha}^{\alpha 4})^2} \tag{B.6}$$

B.3. Mass term

1. Term $(G_{\mu}^{3})^2 + (G_{\mu}^{4})^2$:

$$(G_{\mu}'^{3})^2 + (G_{\mu}'^{4})^2 = (\cos(q\alpha)G_{\mu}^{3} + \sin(q\alpha)G_{\mu}^{4})^2 + (-\sin(q\alpha)G_{\mu}^{3} + \cos(q\alpha)G_{\mu}^{4})^2$$
$$(G_{\mu}'^{3})^2 + (G_{\mu}'^{4})^2 = \cos^2(q\alpha)G_{\mu}^{3}G^{\mu 3} + \sin^2(q\alpha)G_{\mu}^{4}G^{\mu 4} + 2\sin(q\alpha)\cos(q\alpha)G_{\mu}^{3}G^{\mu 4} +$$
$$+\sin^2(q\alpha)G_{\mu}^{3}G^{\mu 4}\cos^2(q\alpha)G_{\mu}^{4}G^{\mu 3} - 2\sin(q\alpha)\cos(q\alpha)G_{\mu}^{3}G^{\mu 4} \tag{B.7}$$

$$\boldsymbol{(G_{\mu}'^{3})^2 + (G_{\mu}'^{4})^2 = (G_{\mu}^{3})^2 + (G_{\mu}^{4})^2} \tag{B.8}$$

B.4. Interaction Terms

1. Term $G_{\mu}'^{3}G_{\nu}'^{4} - G_{\mu}'^{4}G_{\nu}'^{3}$

$$G_{\mu}'^{3}G_{\nu}'^{4} - G_{\mu}'^{4}G_{\nu}'^{3} = \left(\cos(q\alpha)G_{\mu}^{3} + \sin(q\alpha)G_{\mu}^{4}\right)\left(-\sin(q\alpha)G_{\nu}^{3} + \cos(q\alpha)G_{\nu}^{4}\right) +$$
$$-\left(-\sin(q\alpha)G_{\mu}^{3} + \cos(q\alpha)G_{\mu}^{4}\right)\left(\cos(q\alpha)G_{\nu}^{3} + \sin(q\alpha)G_{\nu}^{4}\right)$$
$$G_{\mu}'^{3}G_{\nu}'^{4} - G_{\mu}'^{4}G_{\nu}'^{3} = -\sin(q\alpha)\cos(q\alpha)G_{\mu}^{3}G_{\nu}^{3} + \cos^2(q\alpha)G_{\mu}^{3}G_{\nu}^{4} - \sin^2(q\alpha)G_{\mu}^{4}G_{\nu}^{3} +$$
$$+\sin(q\alpha)\cos(q\alpha)G_{\mu}^{4}G_{\nu}^{4} - (-\sin(q\alpha)\cos(q\alpha)G_{\nu}^{3}G_{\nu}^{3} - \sin^2(q\alpha)G_{\mu}^{3}G_{\nu}^{4} +$$
$$+\cos(q\alpha)G_{\mu}^{4}G_{\nu}^{3} + \sin(q\alpha)\cos(q\alpha)G_{\nu}^{3}G_{\mu}^{4})$$
$$G_{\mu}'^{3}G_{\nu}'^{4} - G_{\mu}'^{4}G_{\nu}'^{3} = \left(\sin^2(q\alpha) + \cos^2(q\alpha)\right)\left(G_{\mu}^{3}G_{\nu}^{4} - G_{\nu}^{3}G_{\mu}^{4}\right) \tag{B.9}$$

and so,

$$\boldsymbol{G_{\mu}'^{3}G_{\nu}'^{4} - G_{\mu}'^{4}G_{\nu}'^{3} = G_{\mu}^{3}G_{\nu}^{4} - G_{\nu}^{3}G_{\mu}^{4}} \tag{B.10}$$

2. Term $\left(G_{\mu}'^{4}G_{\nu}'^{3} - G_{\mu}'^{3}G_{\nu}'^{4}\right)$, which is the term $-(G_{\mu}'^{3}G_{\nu}'^{4} - G_{\mu}'^{4}G_{\nu}'^{3})$, so also invariant.

$$\boldsymbol{\left(G_{\mu}'^{4}G_{\nu}'^{3} - G_{\mu}'^{3}G_{\nu}'^{4}\right) = \left(G_{\mu}^{4}G_{\nu}^{3} - G_{\mu}^{3}G_{\nu}^{4}\right)} \tag{B.11}$$

3.Term $\left(G'^3_\mu G'^3_\nu - G'^4_\mu G'^4_\nu\right)$:

$$\begin{aligned}
&\left(G'^3_\mu G'^3_\nu - G'^4_\mu G'^4_\nu\right) = \left(\cos(q\alpha)G^3_\mu + \sin(q\alpha)G^4_\mu\right)\left(\cos(q\alpha)G^3_\nu + \sin(q\alpha)G^4_\nu\right) + \\
&- \left(-\sin(q\alpha)G^3_\mu + \cos(q\alpha)G^4_\mu\right)\left(-\sin(q\alpha)G^3_\nu + \cos(q\alpha)G^4_\nu\right) \\
&= (\cos(q\alpha))^2 G^3_\mu G^3_\nu + \sin(q\alpha)\cos(q\alpha)G^3_\mu G^4_\nu + \sin(q\alpha)\cos(q\alpha)G^3_\nu G^4_\mu + (\sin(q\alpha))^2 G^4_\mu G^4_\nu + \\
&- \left((\sin(q\alpha))^2 G^3_\mu G^3_\nu - \sin(q\alpha)\cos(q\alpha)G^3_\mu G^4_\nu - \sin(q\alpha)\cos(q\alpha)G^3_\nu G^4_\mu + (\cos(q\alpha))^2 G^4_\mu G^4_\nu\right) \\
&= ((\cos(q\alpha))^2 - (\sin(q\alpha))^2)G^3_\mu G^3_\nu + 2\sin(q\alpha)\cos(q\alpha)G^3_\mu G^4_\nu + \\
&+2\sin(q\alpha)\cos(q\alpha)G^3_\nu G^4_\mu - ((\cos(q\alpha))^2 - (\sin(q\alpha))^2)G^4_\mu G^4_\nu \\
&= \cos(2q\alpha)G^3_\mu G^3_\nu + \sin(2q\alpha)G^3_\mu G^4_\nu + \sin(2q\alpha)G^3_\nu G^4_\mu - \cos(2q\alpha)G^4_\mu G^4_\nu \\
&\left(\boldsymbol{G'^3_\mu G'^3_\nu - G'^4_\mu G'^4_\nu}\right) : \textbf{Not Invariant}
\end{aligned}$$

4. Term $\left(G'^3_\mu G'^3_\nu + G'^4_\mu G'^4_\nu\right)$

$$\begin{aligned}
&\left(G'^3_\mu G'^3_\nu + G'^4_\mu G'^4_\nu\right) = \left(\cos(q\alpha)G^3_\mu + \sin(q\alpha)G^4_\mu\right)\left(\cos(q\alpha)G^3_\nu + \sin(q\alpha)G^4_\nu\right) + \\
&+ \left(-\sin(q\alpha)G^3_\mu + \cos(q\alpha)G^4_\mu\right)\left(-\sin(q\alpha)G^3_\nu + \cos(q\alpha)G^4_\nu\right) \\
&= (\cos(q\alpha))^2 G^3_\mu G^3_\nu + \sin(q\alpha)\cos(q\alpha)G^3_\mu G^4_\nu + \sin(q\alpha)\cos(q\alpha)G^3_\nu G^4_\mu + (\sin(q\alpha))^2 G^4_\mu G^4_\nu + \\
&+(\sin(q\alpha))^2 G^3_\mu G^3_\nu - \sin(q\alpha)\cos(q\alpha)G^3_\mu G^4_\nu - \sin(q\alpha)\cos(q\alpha)G^3_\nu G^4_\mu + (\cos(q\alpha))^2 G^4_\mu G^4_\nu \\
&= ((\cos(q\alpha))^2 + (\sin(q\alpha))^2)G^3_\mu G^3_\nu + ((\cos(q\alpha))^2 + (\sin(q\alpha))^2)G^4_\mu G^4_\nu \\
&= G^3_\mu G^3_\nu + G^4_\mu G^4_\nu
\end{aligned} \tag{B.12}$$

and so,

$$.\boldsymbol{G'^3_\mu G'^3_\nu + G'^4_\mu G'^4_\nu = G^3_\mu G^3_\nu + G^4_\mu G^4_\nu} \tag{B.13}$$

5. Term $\partial_\mu G'^3_\nu \cdot G'^{\mu 3} + \partial_\mu G'^4_\nu \cdot G'^{\mu 4}$

$$\begin{aligned}
&\partial_\mu G'^3_\nu \cdot G'^{\mu 3} + \partial_\mu G'^4_\nu \cdot G'^{\mu 4} = \\
&= \partial_\mu\left[(G^3_\mu\cos(q\alpha) + \sin(q\alpha)G^4_\nu)\right](\cos(q\alpha)G^{\mu 3} + \sin(q\alpha)G^{\mu 4}) + \\
&+\partial_\mu[(-\sin(q\alpha)G^3_\nu + \cos(q\alpha)G^4_\nu)] \cdot \left(G^{\mu 3}(\sin(q\alpha)) + \cos(q\alpha)G^{\mu 4}\right) = \\
&= \cos^2(q\alpha)(\partial_\mu G^3_\nu)G^{\mu 3} + \sin(q\alpha)\cos(q\alpha)(\partial_\mu G^3_\nu)G^{\mu 4} + \\
&+\sin(q\alpha)\cos(q\alpha)(\partial_\mu G^4_\nu)G^{\mu 3} + \sin^2(q\alpha)(\partial_\mu G^4_\nu)G^{\mu 4} + \\
&+\sin^2(q\alpha)(\partial_\mu G^3_\nu)G^{\mu 3} - \sin(q\alpha)\cos(q\alpha)(\partial_\mu G^3_\nu)G^{\mu 4} \\
&-\sin(q\alpha)\cos(q\alpha)(\partial_\mu G^4_\nu)G^{\mu 3} + \cos^2(q\alpha)(\partial_\mu G^4_\nu)G^{\mu 4} =
\end{aligned} \tag{B.14}$$

and so,

$$\boldsymbol{\partial_\mu G'^3_\nu \cdot G'^{\mu 3} + \partial_\mu G'^4_\nu \cdot G'^{\mu 4} = \partial_\mu G^3_\nu \cdot G^{\mu 3} + \partial_\mu G^4_\nu \cdot G^{\mu 4}} \tag{B.15}$$

6. Term $(\partial_\mu G'^3_\nu)G^{\mu 4'} - (\partial_\mu G'^4_\nu)G^{\mu 3'}$

$$\begin{aligned}
&(\partial_\mu G'^3_\nu)G^{\mu 4'} - (\partial_\mu G'^4_\nu)G^{\mu 3'} = \\
&= \partial_\mu(G^3_\nu \cos(q\alpha) + G^4_\nu \sin(q\alpha)) \cdot (G^{\mu 3}(-\sin(q\alpha)) + G^{\mu 4}(\cos(q\alpha))) + \\
&-\partial_\mu(-G^3_\nu \sin(q\alpha) + G^4_\nu \cos(q\alpha)) \cdot (G^{\mu 3}(\cos(q\alpha)) + G^{\mu 4}(\sin(q\alpha))) = \\
&= -(\partial_\mu G^3_\nu)G^{\mu 3} \sin(q\alpha)\cos(q\alpha) + (\partial_\mu G^3_\nu)G^{\mu 4}\cos^2(q\alpha) + \\
&-\sin^2(q\alpha)(\partial_\mu G^4_\nu)G^{\mu 3} + \cos^2(q\alpha)(\partial_\mu G^4_\nu)G^{\mu 4} + \\
&-(-\sin(q\alpha)\partial_\mu G^3_\mu G^{\mu 3}\cos(q\alpha) + \sin^2(q\alpha)(\partial_\mu G^3_\nu)G^{\mu 4} + \\
&+\partial_\mu G^4_\nu G^{\mu 3}\cos^2(q\alpha) + \partial_\mu G^4_\nu G^{\mu 4}\sin(q\alpha)\cos(q\alpha))
\end{aligned} \tag{B.16}$$

and so,

$$\boldsymbol{(\partial_\mu G'^3_\nu)G^{\mu 4'} - (\partial_\mu G'^4_\nu)G^{\mu 3'} = (\partial_\mu G^3_\nu)G^{\mu 4} - (\partial_\mu G^4_\nu)G^{\mu 3}}$$

7. Term $(\partial_\mu G'^3_\nu)G'^{\nu 3} - (\partial_\mu G'^4_\nu)G'^{\nu 4}$

$$\begin{aligned}
&(\partial_\mu G'^3_\nu)G'^{\nu 3} - (\partial_\mu G'^4_\nu)G'^{\nu 4} = \\
&= \partial_\mu(\sin(q\alpha)G^3_\nu + \cos(q\alpha)G^4_\nu) \cdot (\sin(q\alpha)G^{\nu 3} + \cos(q\alpha)G^{\nu 4}) + \\
&+\partial_\mu(-\cos(q\alpha)G^3_\nu + \sin(q\alpha)G^4_\nu) \cdot (-\cos(q\alpha)G^{\nu 3} + \sin(q\alpha)G^{\nu 4})
\end{aligned} \tag{B.17}$$

and so,

$$\boldsymbol{(\partial_\mu G'^3_\nu)G'^{\nu 3} - (\partial_\mu G'^4_\nu)G'^{\nu 4} = (\partial_\mu G^3_\nu)G^{\nu 3} - (\partial_\mu G^4_\nu)G^{\nu 4}}$$

8. Term $\partial_\mu G'^3_\nu G'^{\nu 4} - (\partial_\mu G^4_\nu)' G'^{\nu 3}$

$$\begin{aligned}
&\partial_\mu G'^3_\nu G'^{\nu 4} - (\partial_\mu G^4_\nu)' G'^{\nu 3} = \\
&= \partial_\mu(G^3_\nu \cos(q\alpha) + G^4_\nu \sin(q\alpha)) \cdot (G^{\nu 3}(-\sin(q\alpha)) + G^{\nu 4}(\cos(q\alpha))) + \\
&-\partial_\mu(-G^3_\nu \sin(q\alpha) + G^4_\nu \cos(q\alpha)) \cdot (G^{\nu 3}(\cos(q\alpha)) + G^{\nu 4}(\sin(q\alpha))) = \\
&= -(\partial_\mu G^3_\nu)G^{\nu 3}\sin(q\alpha)\cos(q\alpha) + (\partial_\mu G^3_\nu)G^{\nu 4}\cos^2(q\alpha) + \\
&-(\partial_\mu G^4_\nu)G^{\nu 3}\sin^2(q\alpha) + (\partial_\mu G^4_\nu)G^{\nu 4}\sin(q\alpha)\cos(q\alpha) + \\
&+(\partial_\mu G^3_\nu)G^{\nu 3}\sin(q\alpha)\cos(q\alpha) - (\partial_\mu G^3_\nu)G^{\nu 4}\sin^2(q\alpha) + \\
&-(\partial_\mu G^4_\nu)G^{\nu 3}\cos^2(q\alpha) - (\partial_\mu G^4_\nu)G^{\nu 4}\sin(q\alpha)\cos(q\alpha)
\end{aligned} \tag{B.18}$$

and so,

$$\boldsymbol{\partial_\mu G'^3_\nu G'^{\nu 4} - (\partial_\mu G^4_\nu)' G'^{\nu 3} = \partial_\mu G^3_\nu G^{\nu 4} - (\partial_\mu G^4_\nu)G^{\nu 3}} \tag{B.19}$$

9. Term $(\partial_\mu G^{\mu 3})' G'^4_\nu + (\partial_\mu G^{\mu 4})' G'^4_\nu$

$$\begin{aligned}
&(\partial_\mu G^{\mu 3})' G'^4_\nu + (\partial_\mu G^{\mu 4})' G'^4_\nu = \\
&= \partial_\mu (G^3_\mu \cos(q\alpha) + G^4_\mu \sin(q\alpha))(-G^3_\nu \sin(q\alpha) + G^4_\nu \cos(q\alpha)) + \\
&+\partial_\mu (-G^3_\mu \sin(q\alpha) + G^4_\mu \cos(q\alpha))(-G^3_\nu \sin(q\alpha) + G^4_\nu \cos(q\alpha)) \\
&= -\partial_\mu G^3_\mu G^3_\nu \sin(q\alpha)\cos(q\alpha) + \partial_\mu G^3_\mu G^4_\nu \cos^2(q\alpha) + \\
&-\partial_\mu G^4_\mu G^3_\nu \sin^2(q\alpha) + \partial_\mu G^4_\mu G^4_\nu \sin(q\alpha)\cos(q\alpha) \\
&+\partial_\mu G^3_\mu G^3_\nu \sin^2(q\alpha) - \partial_\mu G^3_\mu G^4_\nu \sin(q\alpha)\cos(q\alpha) \\
&-\partial_\mu G^4_\mu G^3_\nu \sin(q\alpha)\cos(q\alpha) + \partial_\mu G^4_\mu G^4_\nu \cos^2(q\alpha) \\
&= \partial_\mu G^3_\mu G^3_\nu (\sin^2(q\alpha) - \sin(q\alpha)\cos(q\alpha)) + \\
&+\partial_\mu G^3_\mu G^4_\nu (\cos^2(q\alpha) - \sin(q\alpha)\cos(q\alpha)) + \\
&+\partial_\mu G^4_\mu G^3_\nu (-\sin(q\alpha)\cos(q\alpha))
\end{aligned} \tag{B.20}$$

and so,

$$\boldsymbol{(\partial_\mu G^{\mu 3})' G'^4_\nu + (\partial_\mu G^{\mu 4})' G'^4_\nu = (\partial_\mu G^{\mu 3}) G^4_\nu + (\partial_\mu G^{\mu 4}) G^4_\nu} \tag{B.21}$$

Making the following change of basis $\{G_{\mu 3}, G_{\mu 4}\} \to \{V^+_\mu, V^-_\mu\}$.

$$V^+_\mu = \frac{1}{\sqrt{2}}(G^3_\mu - iG^4_\mu), \quad V^-_\mu = \frac{1}{\sqrt{2}}(G^3_\mu + iG^4_\mu) \tag{B.22}$$

or

$$G^3_\mu = \frac{1}{2}(V^+_\mu + V^-_\mu) \quad G^4_\mu = \frac{i}{\sqrt{2}}(V^+_\mu - V^-_\mu) \tag{B.23}$$

The corresponding SO(2) invariants written as $V^\pm_\mu$ are rewritten as

B.5. Kinetic term

The following real and gauge invariant expressions are obtained, trough eq.(3.6)

1. $G^3_{\mu\nu} G^{\mu\nu 3} + G^4_{\mu\nu} G^{\mu\nu 4}$

$$\begin{aligned}
&G^3_{\mu\nu} G^{\mu\nu 3} + G^4_{\mu\nu} G^{\mu\nu 4} = \frac{1}{2}(V^+_{\mu\nu} + V^-_{\mu\nu})^2 - \frac{1}{2}(V^+_{\mu\nu} - V^-_{\mu\nu})^2 \\
&= \frac{1}{2}(V^+_{\mu\nu} V^{\mu\nu +} + 2V^+_{\mu\nu} V^{\mu\nu -} + V^-_{\mu\nu} V^{\mu\nu -}) - \frac{1}{2}(V^+_{\mu\nu} V^{\mu\nu +} - 2V^+_{\mu\nu} V^{\mu\nu -} + V^-_{\mu\nu} V^{\mu\nu -}) \\
&= 2V^+_{\mu\nu} V^{\mu\nu -} \\
&G^3_{\mu\nu} G^{\mu\nu 3} + G^4_{\mu\nu} G^{\mu\nu 4} = 2V^+_{\mu\nu} V^{\mu\nu -}
\end{aligned}$$

2. $S^3_{\mu\nu}S^{\mu\nu 3} + S^4_{\mu\nu}S^{\mu\nu 4}$

$$S^3_{\mu\nu}S^{\mu\nu 3} + S^4_{\mu\nu}S^{\mu\nu 4} = \frac{1}{2}(S^+_{\mu\nu} + S^-_{\mu\nu})^2 - \frac{1}{2}(S^+_{\mu\nu} - S^-_{\mu\nu})^2$$
$$= \frac{1}{2}({S^+_{\mu\nu}}^2 + 2S^+_{\mu\nu}S^{\mu\nu -} + {S^-_{\mu\nu}}^2 - {S^+_{\mu\nu}}^2 + 2S^+_{\mu\nu}S^{\mu\nu -} - {S^-_{\mu\nu}}^2)$$
$$= 2S^+_{\mu\nu}S^{\mu\nu -}$$

3. $L^S_K : spin - 0$

$$S^{\mu 3}_{\mu}S^{\nu 3}_{\nu} + S^{\mu 4}_{\mu}S^{\nu 4}_{\nu} \equiv 2S^{\mu +}_{\mu}S^{\nu -}_{\nu} \tag{B.24}$$

B.6. Mass term

$$1.(G^3_\mu)^2 + (G^4_\mu)^2 \equiv 2V^+_\mu V^{-\mu} \tag{B.25}$$

$$2.G^3_\mu G^4_\nu - G^4_\mu G^3_\nu = -iV^+_\mu V^-_\nu + iV^-_\mu V^+_\nu \tag{B.26}$$

$$3.G^4_\mu G^3_\nu - G^3_\mu G^4_\nu = iV^+_\mu V^-_\nu - iV^-_\mu V^+_\nu \tag{B.27}$$

$$4.G^3_\mu G^3_\nu + G^4_\mu G^4_\nu = V^{+\mu}V^{-\nu} + V^{-\mu}V^{+\nu} \tag{B.28}$$

$$5.G^3_\nu G^{\nu 3} + G^4_\nu G^{\nu 4} = 2V^+_\mu V^{\nu -} \tag{B.29}$$

Mass terms

$$\mu_1 V^+_\mu V^{-\mu} + \mu_2\cancelto{0}{(-iV^+_\mu V^{-\mu} + iV^-_\mu V^{\mu +})} + \mu_3\cancelto{0}{(iV^+_\mu V^{-\mu} - iV^-_\mu V^{\mu +})} \tag{B.30}$$

Appendix C. Collective Fields

Defining

$$z_{\mu\nu} = \gamma_{IJ}G^I_\mu G^J_\nu \tag{C.1}$$

one gets

$$z_{\mu\nu} = z_{[\mu\nu]} + z_{(\mu\nu)}$$

- Antisymmetric part: $z_{[\mu\nu]}$

$$z_{[\mu\nu]} = \frac{1}{2}\left(z_{\mu\nu} - z_{\nu\mu}\right)$$
$$z_{[\mu\nu]} = \frac{1}{2}\left(\gamma_{IJ}G^I_\mu G^J_\nu - \gamma_{IJ}G^I_\nu G^J_\mu\right)$$
$$z_{[\mu\nu]} = \frac{1}{2}(\gamma_{IJ} - \gamma_{JI})G^I_\mu G^J_\nu$$
$$z_{[\mu\nu]} = \gamma_{[IJ]}G^I_\mu G^J_\nu$$

where

$$\gamma_{[IJ]} = \begin{pmatrix} 0 & \gamma_{[12]} & \gamma_{[13]} & \gamma_{[14]} \\ -\gamma_{[12]} & 0 & \gamma_{[23]} & \gamma_{[24]} \\ -\gamma_{[13]} & -\gamma_{[23]} & 0 & \gamma_{[34]} \\ -\gamma_{[14]} & -\gamma_{[24]} & -\gamma_{[34]} & 0 \end{pmatrix} \tag{C.2}$$

For four fields:

$$z_{[\mu\nu]} = \gamma_{[12]}G^1_\mu G^2_\nu - \gamma_{[12]}G^2_\mu G^1_\nu +$$
$$+\gamma_{[13]}G^1_\mu G^3_\nu - \gamma_{[13]}G^3_\mu G^1_\nu +$$
$$+\gamma_{[14]}G^1_\mu G^4_\nu - \gamma_{[14]}G^4_\mu G^1_\nu +$$
$$+\gamma_{[23]}G^2_\mu G^3_\nu - \gamma_{[23]}G^3_\mu G^2_\nu +$$
$$+\gamma_{[24]}G^2_\mu G^4_\nu - \gamma_{[24]}G^4_\mu G^2_\nu +$$
$$+\gamma_{[34]}G^3_\mu G^4_\nu - \gamma_{[34]}G^4_\mu G^3_\nu +$$

So, we will have six two-by-two antisymmetric collective fields:

$$z_{[\mu\nu]} = \begin{pmatrix} 0 & z^{[12]}_{[\mu\nu]} & z^{[13]}_{[\mu\nu]} & z^{[14]}_{[\mu\nu]} \\ -z^{[12]}_{[\mu\nu]} & 0 & z^{[23]}_{[\mu\nu]} & z^{[24]}_{[\mu\nu]} \\ -z^{[13]}_{[\mu\nu]} & -z^{[23]}_{[\mu\nu]} & 0 & z^{[34]}_{[\mu\nu]} \\ -z^{[14]}_{[\mu\nu]} & -z^{[24]}_{[\mu\nu]} & -z^{[34]}_{[\mu\nu]} & 0 \end{pmatrix} \tag{C.3}$$

Which together express the gauge invariant collective field.

$$z_{[\mu\nu]} = \gamma_{[12]}(G^1_\mu G^2_\nu - G^2_\mu G^1_\nu) + \gamma_{[13]}(G^1_\mu G^3_\nu - G^3_\mu G^1_\nu) +$$
$$\gamma_{[14]}(G^1_\mu G^4_\nu - G^4_\mu G^1_\nu) + \gamma_{[23]}(G^2_\mu G^3_\nu - G^3_\mu G^2_\nu) +$$
$$\gamma_{[24]}(G^2_\mu G^4_\nu - G^4_\mu G^2_\nu) + \gamma_{[34]}(G^3_\mu G^4_\nu - G^4_\mu G^3_\nu)$$

In order to $z_{[\mu\nu]}$ be $SO(2)$ invariant, we must have the conditions:

$$\gamma_{[13]} = \gamma_{[14]} = \gamma_{[23]} = \gamma_{[24]} = 0, \tag{C.4}$$

where

$$\gamma_{[IJ]} = \gamma_{[ij]}\Omega^i_I\Omega^j_J \tag{C.5}$$

Making the change of variables $G^1_\mu = A_\mu, G^2_\mu = U_\mu$ and $G^3_\mu = \frac{1}{\sqrt{2}}\left(V^+_\mu + V^-_\mu\right)$, $G^4_\mu = \frac{i}{\sqrt{2}}\left(V^+_\mu - V^-_\mu\right)$ one gets,

$$\begin{aligned} z_{[\mu\nu]} = {} & \gamma_{[12]}\left(A_\mu U_\nu - A_\nu U_\mu\right) + \frac{1}{\sqrt{2}}\left(\gamma_{[13]} + i\gamma_{[14]}\right)\left(A_\mu V^+_\nu - A_\nu V^+_\mu\right) + \\ & + \frac{1}{\sqrt{2}}\left(\gamma_{[13]} - i\gamma_{[14]}\right)\left(A_\mu V^-_\nu - A_\nu V^-_\mu\right) + \frac{1}{\sqrt{2}}\left(\gamma_{[23]} + i\gamma_{[24]}\right)\left(U_\mu V^+_\nu - U_\nu V^+_\mu\right) \\ & + \frac{1}{\sqrt{2}}\left(\gamma_{[23]} - i\gamma_{[24]}\right)\left(U_\mu V^-_\nu - U_\nu V^-_\mu\right) - i\gamma_{[34]}\left(V^+_\mu V^-_\nu - V^+_\nu V^-_\mu\right) \end{aligned} \tag{C.6}$$

Defining

$$\begin{aligned}
z_{\mu\nu}^{[12]} &= \gamma_{[12]} A_\mu U_\nu \\
z_{\mu\nu}^{[+1]} &= \left(\gamma_{[13]} + i\gamma_{[14]}\right) A_\mu V_\nu^+ \\
z_{\mu\nu}^{[-1]} &= \left(\gamma_{[13]} - i\gamma_{[14]}\right) A_\mu V_\nu^- \\
z_{\mu\nu}^{[+2]} &= \left(\gamma_{[23]} + i\gamma_{[24]}\right) U_\mu V_\nu^+ \\
z_{\mu\nu}^{[-2]} &= \left(\gamma_{[23]} - i\gamma_{[24]}\right) U_\mu V_\nu^- \\
z_{\mu\nu}^{[+-]} &= -i\gamma_{[34]} V_\mu^+ V_\nu^- \\
z_{\mu\nu}^{[-+]} &= i\gamma_{[34]} V_\mu^- V_\nu^+
\end{aligned} \tag{C.7}$$

$$z_{[\mu\nu]} = 2z_{[\mu\nu]}^{[12]} + \sqrt{2} z_{[\mu\nu]}^{[+1]} + \sqrt{2} z_{[\mu\nu]}^{[-1]} + \sqrt{2} z_{[\mu\nu]}^{[+2]} + \sqrt{2} z_{[\mu\nu]}^{[-2]} + z_{[\mu\nu]}^{[+-]} + z_{[\mu\nu]}^{[-+]} \tag{C.8}$$

which yields for SO(2) invariance,

$$z_{[\mu\nu]} = 2 \overset{[12]}{z}_{[\mu\nu]} + \overset{[+-]}{z}_{[\mu\nu]} + \overset{[-+]}{z}_{[\mu\nu]} \tag{C.9}$$

For $U(1) \times SO(2)$ symmetry, one gets that (C.9) is invariant under the following relationships between the free parameters.

$$\begin{aligned}
\gamma_{[12]} \Omega_{21}^{-1} = 0 \quad &, \quad \gamma_{[12]} \Omega_{11}^{-1} = 0 \\
\gamma_{[34]} \Omega_{31}^{-1} = 0 \quad &, \quad \gamma_{[34]} \Omega_{41}^{-1} = 0
\end{aligned} \tag{C.10}$$

Where

$$k_\pm = \Omega_{31}^{-1} + i\Omega_{41}^{-1} \tag{C.11}$$

- Symmetric part $z_{(\mu\nu)}$

$$\begin{aligned}
z_{(\mu\nu)} &= \frac{1}{2}(z_{\mu\nu} + z_{\nu\mu}) \\
z_{(\mu\nu)} &= \frac{1}{2}\left(\gamma_{IJ} G_\mu^I G_\nu^J + \gamma_{JI} G^J \mu G_\nu^I\right) \\
z_{(\mu\nu)} &= \frac{1}{2}(\gamma_{IJ} + \gamma_{JI}) G_\mu^I G_\nu^J \\
z_{(\mu\nu)} &= \gamma_{(IJ)} G_\mu^I G_\nu^J
\end{aligned}$$

$$\gamma_{(IJ)} = \begin{pmatrix} \gamma_{(11)} & \gamma_{(12)} & \gamma_{(13)} & \gamma_{(14)} \\ \gamma_{(12)} & \gamma_{(22)} & \gamma_{(23)} & \gamma_{(24)} \\ \gamma_{(13)} & \gamma_{(23)} & \gamma_{(33)} & \gamma_{(34)} \\ \gamma_{(34)} & \gamma_{(24)} & \gamma_{(34)} & \gamma_{(44)} \end{pmatrix} \tag{C.12}$$

Considering that any symmetric tensor can be written as

$$z_{(\mu\nu)} = \bar{z}_{(\mu\nu)} + \frac{1}{4}(trz)\eta_{\mu\nu} \tag{C.13}$$

where $\bar{z}_{(\mu\nu)}$ means a traceless matrix, one gets

$$z_{(\mu\nu)} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{12} & z_{22} & z_{23} & z_{24} \\ z_{13} & z_{23} & z_{33} & z_{34} \\ z_{14} & z_{24} & z_{34} & -(z_{11}+z_{22}+z_{33}) \end{bmatrix} + \\ +\frac{1}{4}(z_{11}+z_{22}+z_{33}+z_{44}) \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{C.14}$$

which gives 10 symmetric two-by-two collective fields.

$$\begin{aligned} z_{(\mu\nu)} &= \gamma_{(11)} G_{\mu 1} G_{\nu 1} + \gamma_{(22)} G_{\mu 2} G_{\nu 2} + \\ &+\gamma_{(33)} G_{\mu 3} G_{\nu 3} + \gamma_{(44)} G_{\mu 4} G_{\nu 4} + \\ &+\gamma_{(12)} G_{\mu 1} G_{\nu 2} + \gamma_{(13)} G_{\mu 1} G_{\nu 3} + \\ &+\gamma_{(23)} G_{\mu 2} G_{\nu 3} + \gamma_{(24)} G_{\mu 2} G_{\nu 4} + \\ &+\gamma_{(34)} G_{\mu 3} G_{\nu 4} \end{aligned} \tag{C.15}$$

Expressing in the same way that $z_{[\mu\nu]}$:

$$\begin{aligned} z_{(\mu\nu)} &= \gamma_{(11)} A_\mu A_\nu + \gamma_{(22)} U_\mu U_\nu + \gamma_{(12)} \left(A_\mu U_\nu + U_\mu A_\nu\right) + \\ &+\frac{1}{\sqrt{2}}\left(\gamma_{(13)} + i\gamma_{(14)}\right)\left(A_\mu V_\nu^+ + A_\nu V_\mu^+\right) + \\ &+\frac{1}{\sqrt{2}}\left(\gamma_{(13)} - i\gamma_{(14)}\right)\left(A_\mu V_\nu^- + A_\nu V_\mu^-\right) + \\ &+\frac{1}{\sqrt{2}}\left(\gamma_{(23)} + i\gamma_{(24)}\right)\left(U_\mu V_\nu^+ + U_\nu V_\mu^+\right) + \\ &+\frac{1}{\sqrt{2}}\left(\gamma_{(23)} - i\gamma_{(24)}\right)\left(U_\mu V_\nu^- + U_\nu V_\mu^-\right) + \\ &+\frac{1}{2}\left(\gamma_{(33)} - \gamma_{(44)}\right)\left(V_\mu^+ V_\nu^+ + V_\mu^- V_\nu^-\right) + \\ &+\frac{1}{2}\left(\gamma_{(33)} + \gamma_{(44)}\right)\left(V_\mu^+ V_\nu^- + V_\mu^- V_\nu^+\right) + \\ &+i\gamma_{(34)}\left(V_\mu^+ V_\nu^+ + V_\nu^- V_\mu^-\right) \end{aligned} \tag{C.16}$$

For $z_{(\mu\nu)}$ be SO(2) invariant, we must have the conditions

$$\begin{aligned} &\gamma_{(13)} = \gamma_{(14)} = \gamma_{(23)} = \gamma_{(24)} = \gamma_{(34)} = 0 \\ &\gamma_{(33)} = \gamma_{(44)} \end{aligned} \tag{C.17}$$

Defining

$$z_{\mu\nu}^{(-1)} = (\gamma_{(13)} - i\gamma_{(14)})A_\mu V_\nu^- \tag{C.18}$$
$$z_{\mu\nu}^{(+1)} = (\gamma_{(13)} + i\gamma_{(14)})A_\mu V_\nu^+$$
$$z_{\mu\nu}^{(-2)} = (\gamma_{(23)} - i\gamma_{(24)})U_\mu V_\nu^-$$
$$z_{\mu\nu}^{(+2)} = (\gamma_{(23)} - i\gamma_{(24)})U_\mu V_\nu^+$$
$$z_{\mu\nu}^{(+-)} = -i\gamma_{(34)}V_\mu^+ V_\nu^-$$
$$z_{\mu\nu}^{(-+)} = i\gamma_{(34)}V_\mu^- V_\nu^+$$
$$z_{\mu\nu}^{(+-3)} = \gamma_{(33)}V_\mu^+ V_\nu^-$$
$$z_{\mu\nu}^{(-+3)} = \gamma_{(33)}V_\mu^- V_\nu^+$$

where

$$\gamma_{(IJ)} = \gamma_{[ij]}\Omega_I^i \Omega_J^j \tag{C.19}$$

And so, we will have under $SO(2)$

$$z_{(\mu\nu)} = \overset{(11)}{z}_{(\mu\nu)} + \overset{(22)}{z}_{(\mu\nu)} + 2\overset{(12)}{z}_{(\mu\nu)} + 2\overset{+-3}{z}_{(\mu\nu)} \tag{C.20}$$

Studying now for $U(1)\times SO(2)$ symmetry, one gets the (C.20)

$$\gamma_{[11]}\Omega_{11}^{-1} = 0 \quad , \quad \gamma_{[22]}\Omega_{21}^{-1} = 0$$
$$\gamma_{(12)}\Omega_{11}^{-1} = 0 \quad , \quad \gamma_{(12)}\Omega_{21}^{-1} = 0$$
$$\gamma_{[33]}\Omega_{31}^{-1} = 0 \quad , \quad \gamma_{[33]}\Omega_{41}^{-1} = 0 \tag{C.21}$$

Appendix D. Two Collective Bianchi Identities

Given the quadruplete $\{A_\mu, U_\mu, V_\mu^\pm\}$ and the corresponding two fields associations given by eq.(2.40), one gets a set of antisymmetric two-collective fields

$$\{z_{[\mu\nu]}^{[12]}, z_{[\mu\nu]}^{(12)}, z_{[\mu\nu]}^{(+1)}, z_{[\mu\nu]}^{(+2)}, z_{[\mu\nu]}^{[+-]}, z_{[\mu\nu]}^{(+-)}, z_{[\mu\nu]}^{+-3}, z_{[\mu\nu]}^{+-4}\}$$

and the symmetric two-collective fields set

$$\{z_{(\mu\nu)}^{[12]}, z_{(\mu\nu)}^{(12)}, z_{(\mu\nu)}^{(+1)}, z_{(\mu\nu)}^{(+2)}, z_{(\mu\nu)}^{[+-]}, z_{(\mu\nu)}^{(+-)}, z_{(\mu\nu)}^{+-3}, z_{(\mu\nu)}^{+-4} \textbf{ and } z_{(\mu\nu)}^{11}, z_{(\mu\nu)}^{22}\}$$

.

Considering the antisymmetric fields strengths

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad U_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu \tag{D.1}$$
$$V_{\mu\nu}^\pm = \partial_\mu V_\nu^\pm - \partial_\nu V_\mu^\pm$$

and the symmetric ones

$$S_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu, \quad \tilde{U}_{\mu\nu} = \partial_\mu U_\nu + \partial_\nu U_\mu \tag{D.2}$$
$$\tilde{V}_{\mu\nu}^\pm = \partial_\mu V_\nu^\pm + \partial_\nu V_\mu^\pm$$

one derives the correponding Bianchi identities.

For $z^{[12]}_{\mu\nu} \equiv \gamma_{[12]} A_\mu U_\nu$:

Antisymmetric case,

$$\partial_\mu z^{[12]}_{[\nu\rho]} + \partial_\nu z^{[12]}_{[\rho\mu]} + \partial_\rho z^{[12]}_{[\mu\nu]} = \frac{1}{2}\gamma_{[12]}\{U_\mu F_{\nu\rho} + U_\nu F_{\rho\mu} + U_\rho F_{\mu\nu} + -A_\mu U_{\nu\rho} - A_\nu U_{\rho\mu} - A_\rho U_{\mu\nu}\} \tag{D.3}$$

Symmetric case,

$$\partial_\mu z^{[12]}_{(\nu\rho)} + \partial_\nu z^{[12]}_{(\rho\mu)} + \partial_\rho z^{[12]}_{(\mu\nu)} = \frac{1}{2}\gamma_{[12]}\{U_\mu S_{\nu\rho} + U_\nu S_{\rho\mu} + U_\rho S_{\mu\nu} + +A_\mu \tilde{U}_{\nu\rho} + A_\nu \tilde{U}_{\rho\mu} + A_\rho \tilde{U}_{\mu\nu}\} \tag{D.4}$$

For $z^{(12)}_{\mu\nu} \equiv \gamma_{12} A_\mu U_\nu$:

Antisymmetric case,

$$\partial_\mu z^{(12)}_{[\nu\rho]} + \partial_\nu z^{(12)}_{[\rho\mu]} + \partial_\rho z^{(12)}_{[\mu\nu]} = \frac{1}{2}\gamma_{(12)}\{U_\mu F_{\nu\rho} + U_\nu F_{\rho\mu} + U_\rho F_{\mu\nu} + -A_\mu U_{\nu\rho} - A_\nu U_{\rho\mu} - A_\rho U_{\mu\nu}\} \tag{D.5}$$

Symmetric case,

$$\partial_\mu z^{(12)}_{(\nu\rho)} + \partial_\nu z^{(12)}_{(\rho\mu)} + \partial_\rho z^{(12)}_{(\mu\nu)} = \frac{1}{2}\gamma_{(12)}\{U_\mu S_{\nu\rho} + U_\nu S_{\rho\mu} + U_\rho S_{\mu\nu} + +A_\mu \tilde{U}_{\nu\rho} + A_\nu \tilde{U}_{\rho\mu} + A_\rho \tilde{U}_{\mu\nu}\} \tag{D.6}$$

For $z^{(+1)}_{\mu\nu} \equiv (\gamma_{(13)} + i\gamma_{(14)}) A_\mu U^+_\nu$:

Antisymmetric case,

$$\partial_\mu z^{(+1)}_{[\nu\rho]} + \partial_\nu z^{(+1)}_{[\rho\mu]} + \partial_\rho z^{(+1)}_{[\mu\nu]} = \frac{1}{2}(\gamma_{(13)} + i\gamma_{(14)})\{V^+_\mu F_{\nu\rho} + V^+_\nu F_{\rho\mu} + V^+_\rho F_{\mu\nu} + -A_\mu V^+_{\nu\rho} - A_\nu V^+_{\rho\mu} - A_\rho V^+_{\mu\nu}\} \tag{D.7}$$

Symmetric case,

$$\partial_\mu z^{(+1)}_{(\nu\rho)} + \partial_\nu z^{(+1)}_{(\rho\mu)} + \partial_\rho z^{(+1)}_{(\mu\nu)} = \frac{1}{2}(\gamma_{(13)} + i\gamma_{(14)})\{V^+_\mu S_{\nu\rho} + V^+_\nu S_{\rho\mu} + V^+_\rho S_{\mu\nu} + +A_\mu \tilde{V}_{\nu\rho} + A_\nu \tilde{V}_{\rho\mu} + A_\rho \tilde{V}_{\mu\nu}\} \tag{D.8}$$

For $z^{(+2)}_{\mu\nu} \equiv (\gamma_{(23)} + i\gamma_{(24)}) U_\mu V^+_\nu$:

Antisymmetric case,

$$\partial_\mu z^{(+2)}_{[\nu\rho]} + \partial_\nu z^{(+2)}_{[\rho\mu]} + \partial_\rho z^{(+2)}_{[\mu\nu]} = \frac{1}{2}(\gamma_{(23)} + i\gamma_{(24)})\{V^+_\mu U_{\nu\rho} + V^+_\nu U_{\rho\mu} + V^+_\rho U_{\mu\nu} + -U_\mu V^+_{\nu\rho} - U_\nu V^+_{\rho\mu} - U_\rho V^+_{\mu\nu}\} \tag{D.9}$$

Symmetric case,

$$\partial_\mu z^{(+2)}_{(\nu\rho)} + \partial_\nu z^{(+2)}_{(\rho\mu)} + \partial_\rho z^{(+2)}_{(\mu\nu)} = \frac{1}{2}(\gamma_{(23)} + i\gamma_{(24)})\{V^+_\mu \tilde{U}_{\nu\rho} + V^+_\nu \tilde{U}_{\rho\mu} + V^+_\rho \tilde{U}_{\mu\nu} + \\ +U_\mu \tilde{V}_{\nu\rho} + U_\nu \tilde{V}_{\rho\mu} + U_\rho \tilde{V}_{\mu\nu}\} \tag{D.10}$$

For $z^{(+-)}_{\mu\nu} \equiv -i\gamma_{(34)} V^+_\mu V^-_\nu$:

Antisymmetric case,

$$\partial_\mu z^{(+-)}_{[\nu\rho]} + \partial_\nu z^{(+-)}_{[\rho\mu]} + \partial_\rho z^{(+-)}_{[\mu\nu]} = \frac{-i}{2}\gamma_{(34)}\{V^-_\mu V^+_{\nu\rho} + V^-_\nu V^+_{\rho\mu} + V^-_\rho V^+_{\mu\nu} + \\ -V^+_\mu V^-_{\nu\rho} - V^+_\nu V^-_{\rho\mu} - V^+_\rho V^-_{\mu\nu}\} \tag{D.11}$$

Symmetric case,

$$\partial_\mu z^{(+-)}_{(\nu\rho)} + \partial_\nu z^{(+-)}_{(\rho\mu)} + \partial_\rho z^{(+-)}_{(\mu\nu)} = \frac{-i}{2}\gamma_{(34)}\{V^-_\mu \tilde{V}^+_{\nu\rho} + V^-_\nu \tilde{V}^+_{\rho\mu} + V^-_\rho \tilde{V}^+_{\mu\nu} + \\ +V^+_\mu \tilde{V}^-_{\nu\rho} + V^+_\nu \tilde{V}^-_{\rho\mu} + V^+_\rho \tilde{V}^-_{\mu\nu}\} \tag{D.12}$$

For $z^{(+-3)}_{\mu\nu} \equiv \gamma_{(33)} V^+_\mu V^-_\nu$:

Antisymmetric case,

$$\partial_\mu z^{(+-3)}_{[\nu\rho]} + \partial_\nu z^{(+-3)}_{[\rho\mu]} + \partial_\rho z^{(+-3)}_{[\mu\nu]} = \frac{i}{2}\gamma_{(33)}\{V^-_\mu V^+_{\nu\rho} + V^-_\nu V^+_{\rho\mu} + V^-_\rho V^+_{\mu\nu} + \\ -V^+_\mu V^-_{\nu\rho} - V^+_\nu V^-_{\rho\mu} - V^+_\rho V^-_{\mu\nu}\} \tag{D.13}$$

Symmetric case,

$$\partial_\mu z^{(+-3)}_{(\nu\rho)} + \partial_\nu z^{(+-3)}_{(\rho\mu)} + \partial_\rho z^{(+-3)}_{(\mu\nu)} = \frac{-i}{2}\gamma_{(33)}\{V^-_\mu \tilde{V}^+_{\nu\rho} + V^-_\nu \tilde{V}^+_{\rho\mu} + V^-_\rho \tilde{V}^+_{\mu\nu} + \\ +V^+_\mu \tilde{V}^-_{\nu\rho} + V^+_\nu \tilde{V}^-_{\rho\mu} + V^+_\rho \tilde{V}^-_{\mu\nu}\} \tag{D.14}$$

For $z^{(+-4)}_{\mu\nu} \equiv \gamma_{(44)} V^+_\mu V^-_\nu$:

Antisymmetric case,

$$\partial_\mu z^{(+-4)}_{[\nu\rho]} + \partial_\nu z^{(+-4)}_{[\rho\mu]} + \partial_\rho z^{(+-4)}_{[\mu\nu]} = \frac{i}{2}\gamma_{(44)}\{V^-_\mu V^+_{\nu\rho} + V^-_\nu V^+_{\rho\mu} + V^-_\rho V^+_{\mu\nu} + \\ -V^+_\mu V^-_{\nu\rho} - V^+_\nu V^-_{\rho\mu} - V^+_\rho V^-_{\mu\nu}\} \tag{D.15}$$

Symmetric case,

$$\partial_\mu z^{(+-4)}_{(\nu\rho)} + \partial_\nu z^{(+-4)}_{(\rho\mu)} + \partial_\rho z^{(+-4)}_{(\mu\nu)} = \frac{-i}{2}\gamma_{(44)}\{V^-_\mu \tilde{V}^+_{\nu\rho} + V^-_\nu \tilde{V}^+_{\rho\mu} + V^-_\rho \tilde{V}^+_{\mu\nu} + \\ +V^+_\mu \tilde{V}^-_{\nu\rho} + V^+_\nu \tilde{V}^-_{\rho\mu} + V^+_\rho \tilde{V}^-_{\mu\nu}\} \tag{D.16}$$

Appendix E. $L_3 : SO(2)$ invariance

We are going to study the $SO(2)$ invariance for trilinear vertice. Rewriting eq (5.14),

$$L^3 = \alpha_{IJK}(\partial_\mu G^I_\nu)G^{\mu J}G^{\nu K} + \beta_{I(JK)}(\partial_\mu G^{\mu I})G^J_\nu G^{\nu K} + \\ + \left(4\eta_{J[IK]} + 2\eta_{I[JK]}\right)(\partial_\mu G^I_\nu)G^{\mu K}G^{\nu J} \tag{E.1}$$

and given transformations eqs (5.16-18), one gets that the condition for $L'_3 = L_3$ after some algebric manipulations is

$$(t)_{ip}\alpha_{iqr} + (t)_{iq}\alpha_{pir} + (t)_{ir}\alpha_{pqi} = 0 \tag{E.2}$$

$$(t)_{ip}\beta_{iqr} + (t)_{iq}\beta_{p(ir)} + (t)_{ir}\beta_{p()} \tag{E.3}$$

Considering first coefficients α_{IJK},

$$\alpha_{IJK} = 2b_1\gamma_{[JK]} + 4c_I\gamma_{[JK]} + 4\beta_I\gamma_{(JK)} \tag{E.4}$$

there are 64 coefficients to be studied

- Take α_{IJK} with $I, J \in 1, 2, K \in [3, 4)$. It yields 24 nulls coefficients. They are

$$\begin{array}{cccc} \alpha_{113} = 0 & \alpha_{114} = 0 & \alpha_{123} = 0 & \alpha_{124} = 0 \\ \alpha_{131} = 0 & \alpha_{132} = 0 & \alpha_{141} = 0 & \alpha_{142} = 0 \\ \alpha_{213} = 0 & \alpha_{214} = 0 & \alpha_{223} = 0 & \alpha_{224} = 0 \\ \alpha_{241} = 0 & \alpha_{242} = 0 & \alpha_{231} = 0 & \alpha_{232} = 0 \\ \alpha_{411} = 0 & \alpha_{412} = 0 & \alpha_{421} = 0 & \alpha_{422} = 0 \\ \alpha_{311} = 0 & \alpha_{312} = 0 & \alpha_{321} = 0 & \alpha_{322} = 0 \end{array} \tag{E.5}$$

- Taking α_{IJK} with $I, J, K \in 1, 2$. It yields 8 free coefficients.

$$\begin{array}{cccc} \alpha_{111} & \alpha_{112} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{221} & \alpha_{222} \end{array} \tag{E.6}$$

- Take α_{IJK} with $I, J, K \in 3, 4$. It yields 8 free coefficients.

$$\begin{array}{cccccccc} \alpha_{334} & = & \alpha_{343} & = & \alpha_{433} & = & \alpha_{444} & = 0 \\ \alpha_{333} & = & \alpha_{344} & = & \alpha_{434} & = & \alpha_{443} & = 0 \end{array} \tag{E.7}$$

- Take α_{IJK} under permutations where $I \in 1, 2, J, K \in 3, 4$.

$$\begin{aligned} \alpha_{133} &= \alpha_{144} & \alpha_{233} &= \alpha_{244} \\ \alpha_{134} &= -\alpha_{143} & \alpha_{234} &= -\alpha_{243} \end{aligned}$$

$$\begin{aligned} \alpha_{314} &= -\alpha_{413} & \alpha_{323} &= \alpha_{424} \\ \alpha_{313} &= \alpha_{414} & \alpha_{324} &= -\alpha_{423} \end{aligned}$$

$$\begin{aligned} \alpha_{431} &= -\alpha_{341} & \alpha_{441} &= \alpha_{331} \\ \alpha_{432} &= -\alpha_{342} & \alpha_{442} &= \alpha_{332} \end{aligned} \tag{E.8}$$

It yields 20 independent coefficients α_{IJK}

B. Considering $\beta_{I(JK)}$ There are 64 coefficients to be studied. Considering eq D.3 , one gets

- Take $\beta_{I(JK)}$ with $I, J \in 1, 2, K \in [3, 4)$. It yields 24 nulls coefficients.

$$\begin{aligned} &\beta_{1(13)} = 0 & &\beta_{1(14)} = 0 & &\beta_{1(23)} = 0 & &\beta_{1(24)} = 0 \\ &\beta_{1(31)} = 0 & &\beta_{1(32)} = 0 & &\beta_{1(41)} = 0 & &\beta_{1(42)} = 0 \\ &\beta_{2(13)} = 0 & &\beta_{2(14)} = 0 & &\beta_{2(23)} = 0 & &\beta_{2(24)} = 0 \\ &\beta_{2(41)} = 0 & &\beta_{2(42)} = 0 & &\beta_{2(31)} = 0 & &\beta_{2(32)} = 0 \\ &\beta_{4(11)} = 0 & &\beta_{4(12)} = 0 & &\beta_{4(21)} = 0 & &\beta_{4(22)} = 0 \\ &\beta_{3(11)} = 0 & &\beta_{3(12)} = 0 & &\beta_{3(21)} = 0 & &\beta_{3(22)} = 0 \end{aligned} \tag{E.9}$$

- Take $\beta_{I(JK)}$ with $I \in 1, 2$ and $I, K \in 3, 4$, one gets

$$\begin{aligned} \beta_{1(34)} &= -\beta_{1(43)} = 0 \\ \beta_{2(34)} &= -\beta_{2(43)} = 0 \end{aligned} \tag{E.10}$$

- Take $\beta_{I(JK)}$ with $I, J, K \in 1, 2$.

$$\begin{aligned} &\beta_{1(11)} & &\beta_{1(12)} = \beta_{1(21)} & &\beta_{1(22)} \\ &\beta_{2(11)} & &\beta_{2(12)} = \beta_{2(21)} & &\beta_{2(22)} \end{aligned} \tag{E.11}$$

  It yields $(8 - 2)$ free coefficients.

- Take $\beta_{I(JK)}$ with $I, J, K \in 3, 4$. It yields (8) null coefficients

$$\begin{aligned} \beta_{3(34)} &= \beta_{3(43)} = \beta_{4(33)} = \beta_{4(44)} = 0 \\ \beta_{3(33)} &= \beta_{3(44)} = \beta_{4(34)} = \beta_{4(43)} = 0 \end{aligned} \tag{E.12}$$

- Take $\beta_{I(J)}$ under permutations where $I \in 3,4$ and $J,K \in 1,2,3,4$

$$\begin{aligned} &\beta_{3(13)} = \beta_{4(14)}, \qquad \beta_{3(14)} = -\beta_{4(13)} \\ &\beta_{3(23)} = \beta_{4(24)}, \qquad \beta_{4(23)} = \beta_{3(24)} \\ &\beta_{3(41)} = -\beta_{4(31)}, \qquad \beta_{3(42)} = -\beta_{4(32)} \\ &\beta_{3(31)} = \beta_{4(41)}, \qquad \beta_{4(42)} = \beta_{3(32)} \end{aligned} \tag{E.13}$$

Appendix F. $\mathcal{L}_3^{st}$: $SO(2)$ invariance

Given that

$$L_3^{st} = \Theta_{IJK}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\nu^J \partial_\rho G_\sigma^K$$

where

$$\Theta_{IJK} \equiv 4\eta_{K[IJ]} - 2\eta_{I[JK]}, \qquad \eta_{I[JK]} = \alpha_I\gamma_{JK} \tag{F.1}$$

and considering $SO(2)$ transformations at eqs (5.16-18), one gets

$$\begin{aligned} L_3^{st} = &\Theta_{111}(\partial_\mu G_\nu^1)G^{\mu 1}G^{\nu 1} + \Theta_{211}(\partial_\mu G_\nu^2)G^{\mu 1}G^{\nu 1} + \\ &+\Theta_{121}(\partial_\mu G_\nu^1)G^{\mu 2}G^{\nu 1} + \Theta_{221}(\partial_\mu G_\nu^2)G^{\mu 2}G^{\nu 1} + \\ &+\Theta_{112}(\partial_\mu G_\nu^1)G^{\mu 1}G^{\nu 2} + \Theta_{212}(\partial_\mu G_\nu^2)G^{\mu 1}G^{\nu 2} + \\ &+\Theta_{122}(\partial_\mu G_\nu^1)G^{\mu 2}G^{\nu 2} + \Theta_{222}(\partial_\mu G_\nu^2)G^{\mu 2}G^{\nu 2} + \\ &+\Theta_{133}(\partial_\mu G_\nu^1)[G^{\mu 3}G^{\nu 3} + G^{\mu 4}G^{\nu 4}] \\ &+\Theta_{233}(\partial_\mu G_\nu^2)[G^{\mu 3}G^{\nu 3} + G^{\mu 4}G^{\nu 4}] \\ &+\Theta_{313}G^{\mu 1}[(\partial_\mu G_\nu^3)G^{\nu 3} + (\partial_\mu G_\nu^4)G^{\nu 4}] \\ &+\Theta_{323}G^{\mu 2}[(\partial_\mu G_\nu^3)G^{\nu 3} + (\partial_\mu G_\nu^4)G^{\nu 4}] \\ &+\Theta_{331}G^{\nu 1}[(\partial_\mu G_\nu^3)G^{\mu 3} + (\partial_\mu G_\nu^4)G^{\mu 4}] \\ &+\Theta_{332}G^{\nu 2}[(\partial_\mu G_\nu^3)G^{\mu 3} + (\partial_\mu G_\nu^4)G^{\mu 4}] \\ &+\Theta_{134}(\partial_\mu G_\nu^1)[G^{\mu 3}G^{\nu 4} - G^{\mu 4}G^{\nu 3}] \\ &+\Theta_{234}(\partial_\mu G_\nu^2)[G^{\mu 3}G^{\nu 4} - G^{\mu 4}G^{\nu 3}] \\ &+\Theta_{314}G^{\mu 1}[(\partial_\mu G_\nu^3)G^{\nu 4} - (\partial_\mu G_\nu^4)G^{\nu 3}] \\ &+\Theta_{324}G^{\mu 2}[(\partial_\mu G_\nu^3)G^{\nu 4} - (\partial_\mu G_\nu^4)G^{\nu 3}] \\ &+\Theta_{341}G^{\nu 1}[(\partial_\mu G_\nu^3)G^{\mu 4} - (\partial_\mu G_\nu^4)G^{\mu 3}] \\ &+\Theta_{342}G^{\nu 2}[(\partial_\mu G_\nu^3)G^{\mu 4} - (\partial_\mu G_\nu^4)G^{\mu 3}] \end{aligned} \tag{F.2}$$

under the conditions.

$$
\begin{array}{cccc}
\Theta_{113}=0 & \Theta_{114}=0 & \Theta_{123}=0 & \Theta_{124}=0 \\
\Theta_{131}=0 & \Theta_{132}=0 & \Theta_{141}=0 & \Theta_{142}=0 \\
\Theta_{213}=0 & \Theta_{214}=0 & \Theta_{223}=0 & \Theta_{224}=0 \\
\Theta_{231}=0 & \Theta_{232}=0 & \Theta_{241}=0 & \Theta_{242}=0 \\
\Theta_{311}=0 & \Theta_{312}=0 & \Theta_{321}=0 & \Theta_{322}=0 \\
\Theta_{411}=0 & \Theta_{412}=0 & \Theta_{421}=0 & \Theta_{422}=0
\end{array} \tag{F.3}
$$

and

$$
\begin{array}{cccc}
\Theta_{133}=\Theta_{144} & \Theta_{233}=\Theta_{244} & \Theta_{313}=\Theta_{414} & \Theta_{323}=\Theta_{424} \\
\Theta_{331}=\Theta_{441} & \Theta_{332}=\Theta_{442} & \Theta_{134}=-\Theta_{143} & \Theta_{234}=-\Theta_{243} \\
\Theta_{314}=-\Theta_{413} & \Theta_{324}=-\Theta_{423} & \Theta_{341}=-\Theta_{431} & \Theta_{342}=-\Theta_{432}
\end{array} \tag{F.4}
$$

and

$$
\begin{array}{cc}
\Theta_{333}=0 & \Theta_{433}=0 \\
\Theta_{334}=0 & \Theta_{434}=0 \\
\Theta_{343}=0 & \Theta_{443}=0 \\
\Theta_{344}=0 & \Theta_{444}=0
\end{array} \tag{F.5}
$$

It yields

$$
\begin{aligned}
\mathcal{L}_3^{st} = \big\{ & \theta_{121}(\partial_\rho G_\sigma^1)G_\mu^1 G_\nu^2 + \theta_{122}(\partial_\rho G_\sigma^2)G_\mu^1 G_\nu^2 + \\
& +\theta_{211}(\partial_\rho G_\sigma^1)G_\mu^2 G_\nu^1 + \theta_{212}(\partial_\rho G_\sigma^2)G_\mu^2 G_\nu^1 + \\
& +(\theta_{133}-\theta_{313})G_\mu^1\left[(\partial_\rho G_\sigma^3)G_\mu^3 + (\partial_\rho G_\sigma^4)G_\mu^4\right] + \\
& +(\theta_{233}-\theta_{323})G_\mu^2\left[(\partial_\rho G_\sigma^3)G_\mu^3 + (\partial_\rho G_\sigma^4)G_\mu^4\right] + \\
& +(\theta_{134}-\theta_{314})G_\mu^1\left[(\partial_\rho G_\sigma^4)G_\mu^3 + (\partial_\rho G_\sigma^3)G_\mu^4\right] + \\
& +(\theta_{234}-\theta_{324})G_\mu^2\left[(\partial_\rho G_\sigma^4)G_\mu^3 + (\partial_\rho G_\sigma^3)G_\mu^4\right] + \\
& +\left[\theta_{341}(\partial_\rho G_\sigma^1) + \theta_{342}(\partial_\rho G_\sigma^2)\right]\left[G_\mu^3 G_\nu^4 - G_\mu^4 G_\nu^3\right]\big\}\epsilon^{\mu\nu\rho\sigma}
\end{aligned} \tag{F.6}
$$

Introducing the electric charge, one gets

$$\begin{aligned}\mathcal{L}_3^{st} = &\{\theta_{121}(\partial_\rho A_\sigma)A_\mu U_\nu + \theta_{122}(\partial_\rho U_\sigma)A_\mu U_\nu + \\ &+\theta_{211}(\partial_\rho A_\sigma)U_\mu A_\nu + \theta_{212}(\partial_\rho U_\sigma)U_\mu A_\nu + \\ &+(\theta_{133} - \theta_{313})A_\mu[(\partial_\rho V_\sigma^+)V_\nu^- + (\partial_\rho V_\sigma^-)V_\nu^+] + \\ &+(\theta_{233} - \theta_{323})U_\mu[(\partial_\rho V_\sigma^+)V_\nu^- + (\partial_\rho V_\sigma^-)V_\nu^+] + \\ &+i(\theta_{134} - \theta_{314})A_\mu[(\partial_\rho V_\sigma^+)V_\nu^- - (\partial_\rho V_\sigma^-)V_\nu^+] + \\ &+i(\theta_{234} - \theta_{324})U_\mu[(\partial_\rho V_\sigma^+)V_\nu^- - (\partial_\rho V_\sigma^-)V_\nu^+] + \\ &+i[\theta_{341}(\partial_\rho A_\sigma) + \theta_{342}(\partial_\rho U_\sigma)][V_\mu^- V_\nu^+ - V_\mu^+ V_\nu^-]\}\end{aligned} \tag{F.7}$$

Appendix G. $\mathcal{L}_4^{st}$: $SO(2)$ invariance

The quadrilinear semitopological term is

$$\mathcal{L}_4^{st} = \rho_{IJKL}\epsilon^{\mu\nu\rho\sigma}G_\mu^I G_\nu^J G_\rho^K G_\sigma^L$$

where

$$\rho_{IJKL} = \gamma_{[IJ]}\gamma_{[KL]} \tag{G.1}$$

Imposing the SO(2) symmetry on the coefficients, one gets

$$\mathcal{L}_4^{st} = 8\left(\rho_{1234} - \rho_{1324} + \rho_{1342}\right) G_\mu^1 G_\nu^2 (G_\rho^3 G_\sigma^4)\epsilon^{\mu\nu\rho\sigma} \tag{G.2}$$

which gives,

$$\mathcal{L}_4^{st} = 8i\left(\rho_{1234} - \rho_{1324} + \rho_{1342}\right) A_\mu U_\nu \left[V_\rho^- V_\sigma^+\right] \epsilon^{\mu\nu\rho\sigma} \tag{G.3}$$

Appendix H. $U(1) \times SO(2)_{global}$ invariance: L_3

Considering eq(6.21) under $U(1) \times SO(2)_{\text{Global}}$ transformations, one gets the following gauge dependences

$$\begin{aligned}\mathcal{L}_1 = &\Big\{a_{111}k_1^3 + a_{222}k_2^3 + \left(a_{112} + a_{121} + a_{211}\right)k_1^2 k_2 + \left(a_{122} + a_{212} + a_{221}\right)k_1 k_2^2 + \\ &+\Big[\left(a_{331} + a_{313} + a_{133}\right)k_1 + \left(a_{332} + a_{323} + a_{233}\right)k_2\Big]\left(k_3^2 + k_4^2\right) + \\ &+\Big\}\boldsymbol{\partial_\mu\left(\partial_\nu\alpha\right)\partial^\mu\alpha\partial^\nu\alpha} +\end{aligned} \tag{H.1}$$

$$
\begin{aligned}
&+\Bigg\{\Big[2a_{111}k_1^2+\big(a_{112}+a_{121}+a_{212}\big)k_1k_2+\big(a_{212}+a_{221}\big)k_2^2+\big(a_{331}+a_{313}\big)\big(k_3^2+k_4^2\big)\Big]A^\mu+\\
&+\Big[2a_{222}k_2^2+\big(a_{112}+a_{121}\big)k_1^2+\big(2a_{122}+a_{212}+a_{221}\big)k_1k_2+\big(a_{332}+a_{323}\big)\big(k_3^2+k_4^2\big)\Big]U^\mu+\\
&\Big[\big(a_{331}+a_{313}+2a_{134}+2a_{133}\big)k_1k_3+\big(a_{332}+a_{323}+2a_{234}+2a_{233}\big)k_2k_3+\\
&-\big(a_{341}+a_{314}\big)k_1k_4-\big(a_{342}+a_{324}\big)k_2k_4\Big]G_3^\mu+\\
&\Big[\big(a_{331}+a_{313}+a_{133}-2a_{134}\big)k_1k_4+\big(a_{332}+a_{323}+2a_{233}-2a_{234}\big)k_2k_4+\\
&+\big(a_{341}+a_{314}\big)k_1k_3+\big(a_{342}+a_{324}\big)k_2k_3\Big]G_4^\mu\Bigg\}\boldsymbol{\partial_\mu\left(\partial_\nu\alpha\right)\partial^\mu\alpha}+
\end{aligned}
\tag{H.2}
$$

$$
\begin{aligned}
&+\Bigg[\Bigg(a_{111}k_1^2+\big(a_{112}+a_{121}\big)k_1k_2+a_{122}k_2^2+\big(a_{133}+a_{134}\big)k_3^2+\big(a_{133}-a_{134}\big)k_4^2\Bigg)\partial_\mu A_\nu+\\
&+\Bigg(a_{222}k_2^2+\big(a_{212}+a_{221}\big)k_1k_2+a_{211}k_1^2+\big(a_{233}+a_{234}\big)k_3^2+\big(a_{233}-a_{234}\big)k_4^2\Bigg)\partial_\mu U_\nu+\\
&+\Bigg(\big(a_{331}+a_{313}\big)k_1k_3+\big(a_{341}+a_{314}\big)k_1k_4+\big(a_{332}-a_{323}\big)k_2k_3+\big(a_{342}-a_{324}\big)k_2k_4\Bigg)\partial_\mu G_{3\nu}+\\
&+\Bigg(\big(a_{331}+a_{313}\big)k_1k_4-\big(a_{314}+a_{341}\big)k_1k_3-\big(a_{324}+a_{342}\big)k_2k_3+\big(a_{323}+a_{332}\big)k_2k_4\Bigg)\partial_\mu G_{4\nu}\Bigg]\boldsymbol{\partial^\mu\alpha\partial^\nu\alpha}+
\end{aligned}
\tag{H.3}
$$

$$
\begin{aligned}
&+\Bigg[\big(a_{111}k_1+a_{121}k_2\big)\partial_\mu(A_\nu)A^\mu+\big(a_{121}k_1+a_{122}k_2\big)\partial_\mu(A_\nu)U^\mu+\\
&+\big(a_{211}k_1+a_{212}k_2\big)\partial_\mu(U_\nu)A^\mu+\big(a_{221}k_1+a_{222}k_2\big)\partial_\mu(U_\nu)U^\mu+\\
&\left[(a_{331}k_1+a_{332}k_2)\,\partial_\mu(G_{3\nu})G_3^\mu+(a_{341}k_1+a_{342}k_2)\,\partial_\mu(G_{3\nu})G_4^\mu+\right.\\
&\left.(a_{313}k_3+a_{314}k_4)\,\partial_\mu(G_{3\nu})A^\mu+(a_{323}k_3+a_{324}k_4)\,\partial_\mu(G_{3\nu})U^\mu\right]+\\
&+(a_{133}k_3+a_{134}k_4)\,\partial_\mu(A_\nu)G_3^\mu+(a_{233}k_3+a_{234}k_4)\,\partial_\mu(U_\nu)G_3^\mu+\\
&+(a_{331}k_1+a_{332}k_2)\,\partial_\mu(G_{4\nu})G_4^\mu-(a_{341}k_1+a_{342}k_2)\,\partial_\mu(G_{4\nu})G_3^\mu+\\
&+(a_{313}k_3-a_{314}k_4)\,\partial_\mu(G_{4\nu})A^\mu+(a_{323}k_3-a_{324}k_4)\,\partial_\mu(G_{4\nu})U^\mu+\\
&+(a_{133}k_4-a_{134}k_3)\,\partial_\mu(A_\nu)G_4^\mu+(a_{233}k_4-a_{234}k_3)\,\partial_\mu(U_\nu)G_4^\mu\Bigg]\boldsymbol{\partial^\nu\alpha}+
\end{aligned}
\tag{H.4}
$$

$$
\begin{aligned}
&+\Big[\big(a_{111}k_1+a_{121}k_2\big)\partial_\mu(A_\nu)A^\nu+\big(a_{112}k_1+a_{122}k_2\big)\partial_\mu(A_\nu)U^\nu+\\
&+\big(a_{211}k_1+a_{221}k_2\big)\partial_\mu(U_\nu)A^\nu+\big(a_{212}k_1+a_{222}k_2\big)\partial_\mu(U_\nu)U^\nu+\\
&\big[\left(a_{331}k_3+a_{341}k_4\right)\partial_\mu(G_{3\nu})A^\nu+\left(a_{332}k_3+a_{342}k_4\right)\partial_\mu(G_{3\nu})U^\nu+\\
&\left(a_{313}k_1+a_{323}k_2\right)\partial_\mu(G_{3\nu})G_3^\nu+\left(a_{314}k_1+a_{324}k_2\right)\partial_\mu(G_{3\nu})G_4^\nu\big]+\\
&+\left(a_{133}k_3+a_{134}k_4\right)\partial_\mu(A_\nu)G_3^\nu+\left(a_{233}k_3+a_{234}k_4\right)\partial_\mu(U_\nu)G_3^\nu+\\
&+\left(a_{331}k_4-a_{341}k_3\right)\partial_\mu(G_{4\nu})A^\nu+\left(a_{332}k_4-a_{342}k_3\right)\partial_\mu(G_{4\nu})U^\nu+\\
&+\left(a_{313}k_1+a_{323}k_2\right)\partial_\mu(G_{4\nu})G_4^\nu-\left(a_{314}k_1+a_{324}k_2\right)\partial_\mu(G_{4\nu})G_3^\nu+\\
&+\left(a_{133}k_3-a_{134}k_4\right)\partial_\mu(A_\nu)G_4^\nu+\left(a_{233}k_3-a_{234}k_4\right)\partial_\mu(U_\nu)G_4^\nu\Big]\boldsymbol{\partial^\mu\alpha}+
\end{aligned}
\tag{H.5}
$$

$$
\begin{aligned}
&+\Big[\big(a_{111}k_1+a_{211}k_2\big)A_\mu A^\nu+\big((a_{112}+a_{121})k_1+a_{212}k_2\big)A_\mu U^\nu+\big(a_{122}k_1+a_{222}k_2\big)U_\mu U^\nu+\\
&+\big[(a_{331}+a_{313})k_3+(a_{341}+a_{314})k_4\big]A^\mu G_3^\nu+\big[(a_{332}+a_{323})k_3+(a_{342}-a_{324})k_4\big]U^\mu G_3^\nu+\\
&+\big[(a_{331}+a_{313})k_4+(a_{341}-a_{341})k_3\big]A^\mu G_4^\nu+\big[(a_{332}+a_{323})k_4+(a_{324}-a_{342})k_3\big]U^\mu G_4^\nu+\\
&+\big[(a_{133}+a_{134})k_1+(a_{233}+a_{234})k_2\big]G_3^\mu G_3^\nu+\big[(a_{133}-a_{134})k_1+(a_{233}-a_{234})k_2\big]G_4^\mu G_4^\nu\Big]\boldsymbol{\partial^\mu\partial^\nu\alpha}
\end{aligned}
\tag{H.6}
$$

$$\mathcal{L}_1 = \Bigg(\beta_{111}k_1^3 + 2\beta_{112}k_1^2k_2 + \beta_{122}k_1k_2^2 + \beta_{211}k_1^2k_2 + 2\beta_{212}k_1k_2^2 + 2\beta_{222}k_2^3 +$$
$$+ \left(2\left(\beta_{313} + \beta_{133}\right)k_1 + 2\left(\beta_{323} + \beta_{233}\right)k_2\right)\left(k_3^2 + k_4^2\right)\Bigg)\Box\left(\alpha\right)\partial_\nu\alpha\partial^\nu\alpha +$$
$$+\Bigg[(\beta_{111}k_1^2 + 2\beta_{112}k_1k_2 + \beta_{122}k_2^2 + \beta_{133}\left(k_3^2 + k_4^2\right))\partial_\mu(A^\mu) +$$
$$+(\beta_{211}k_1^2 + 2\beta_{212}k_1k_2 + 2\beta_{222}k_2^2 + \beta_{233}\left(k_3^2 + k_4^2\right))\partial_\mu(U^\mu)\Big) +$$
$$+\Big(2\left(\beta_{313}k_3 + \beta_{314}k_4\right)k_1 + 2\left(\beta_{323}k_3 + \beta_{324}k_4\right)k_2\Big)\partial_\mu G_3^\mu +$$
$$+\Big(2\left(\beta_{313}k_4 - \beta_{314}k_3\right)k_1 + 2\left(\beta_{323}k_4 - \beta_{324}k_3\right)k_2\Big)\partial_\mu G_4^\mu\Bigg]\partial_\nu(\alpha)\partial^\nu(\alpha) +$$
$$+\Bigg((2\beta_{111}k_1^2 + 2\beta_{112}k_1k_2 + 4\beta_{211}k_1k_2 + 2\beta_{212}k_2^2 + 2\beta_{313}\left(k_3^2 + k_4^2\right))A^\nu +$$
$$+(2\beta_{112}k_1^2 + 2\beta_{122}k_1k_2 + 2\beta_{212}k_1k_2 + 4\beta_{222}k_2^2 + 2\beta_{313}\left(k_3^2 + k_4^2\right))U^\nu +$$
$$+\Big[2k_1\left((\beta_{313} + \beta_{133})k_3 - \beta_{314}k_4\right) + 2k_2\left((\beta_{323} + \beta_{233})k_3 - \beta_{324}k_4\right)\Big]G_{3\nu} +$$
$$+\Big[2k_1\left((\beta_{313} + \beta_{133})k_4 - \beta_{314}k_3\right) + 2k_2\left((\beta_{323} + \beta_{233})k_4 - \beta_{324}k_3\right)\Big]G_{4\nu}\Bigg)\Box(\alpha)\partial_\nu(\alpha)$$
$$+\Bigg[\Big((2\beta_{111}k_1 + 2\beta_{112}k_2)A^\nu + (2\beta_{112}k_1 + 2\beta_{122}k_2)U^\nu + (2\beta_{133}k_3)\,G_3^\nu + (2\beta_{133}k_4)\,G_4^\nu\Big)\partial_\mu(A^\mu) +$$
$$+\Big((4\beta_{211}k_1 + 2\beta_{212}k_2)A^\nu + (2\beta_{212}k_1 + 4\beta_{222}k_2)U^\nu + (2\beta_{233}k_3)\,G_3^\nu + (2\beta_{233}k_4)\,G_4^\nu\Big)\partial_\mu(U^\mu) +$$
$$+2\Big[\left(k_3\beta_{313} + k_4\beta_{314}\right)A_\nu + \left(k_3\beta_{323} + k_4\beta_{324}\right)U_\nu +$$
$$+\left(k_1\beta_{313} + k_2\beta_{323}\right)G_{3\nu} + \left(k_1\beta_{314} + k_2\beta_{324}\right)G_{4\nu}\Big]\partial_\mu G_3^\mu +$$
$$+2\Big[\left(k_4\beta_{313} - k_3\beta_{314}\right)A_\nu + \left(k_4\beta_{323} - k_3\beta_{324}\right)U_\nu +$$
$$-\left(k_1\beta_{314} + k_2\beta_{324}\right)G_{3\nu} + \left(k_1\beta_{313} + k_2\beta_{323}\right)G_{4\nu}\Big]\partial_\mu G_4^\mu\Bigg]\partial_\nu\alpha +$$
$$+\Bigg[(\beta_{111}k_1 + 2\beta_{211}k_2)A_\nu A^\nu + (\beta_{122}k_1 + \beta_{222}k_2)U_\nu U^\nu + 2(\beta_{112}k_1 + \beta_{212}k_2)A_\nu U^\nu +$$
$$+\left[2\left(\beta_{313}k_3 - \beta_{314}k_4\right)G_{3\nu} + 2\left(\beta_{313}k_4 + \beta_{314}k_3\right)G_{4\nu}\right]A^\nu +$$
$$+\left[2\left(\beta_{323}k_3 - \beta_{324}k_4\right)G_{3\nu} + 2\left(\beta_{323}k_4 + \beta_{324}k_3\right)G_{4\nu}\right]U^\nu +$$
$$+\left(\beta_{133}k_1 + \beta_{233}k_2\right)\left(G_{3\nu}G_3^\nu + G_{4\nu}G_4^\nu\right)\Bigg]\Box\alpha \tag{H.7}$$

Appendix I. $U(1)\times SO(2)_{global}$ invariance: L_4

$$\mathcal{L}_1 = \Big(a_{1111}k_1^4 + a_{2222}k_2^4 + 4a_{1112}k_1^3k_2 + 4a_{1222}k_2^3k_1 + 2a_{1122}k_1^2k_2^2 + 2a_{1212}k_1^2k_2^2 + $$
$$+2a_{1221}k_1^2k_2^2 + 2\left(a_{1133}+a_{1331}+a_{1313}\right)\left(k_1^2k_3^2+k_1^2k_4^2\right) + $$
$$+2\left(a_{2233}+a_{2332}+a_{2323}\right)\left(k_2^2k_3^2+k_2^2k_4^2\right) + 4\left(a_{1233}+a_{1332}+a_{1323}\right)\left(k_1k_2k_3^2+k_1k_2k_4^2\right) + $$
$$+4\left(a_{3344}+a_{3443}+a_{3434}\right)\left(k_3^4+k_4^4\right) + 2a_{3434}k_3^2k_4^2 \Big)\partial_\mu\alpha\partial^\mu\alpha\partial_\nu\alpha\partial^\nu\alpha \tag{I.1}$$

$$\mathcal{L}_2 = \Big[\Big(4a_{1111}k_1^3 + 12a_{1112}k_1^2k_2 + 4a_{1222}k_2^3 + 4a_{1122}k_1k_2^2 + 4a_{1212}k_1k_2^2 + 4a_{1221}k_1k_2^2 + $$
$$+4\left(a_{1133}+a_{1331}+a_{1313}\right)\left(k_3^2k_1+k_4^2k_1\right) + 8\left(a_{1233}+a_{1332}+a_{1323}\right)\left(k_3^2k_2+k_4^2k_2\right)\Big)\mathbf{A}_\mu + $$
$$+\Big(4a_{2222}k_2^3 + 12a_{1222}k_2^2k_1 + 4a_{1122}k_1^2k_2 + 4a_{1212}k_1^2k_2 + 4a_{1221}k_1^2k_2 + $$
$$+4\left(a_{2233}+a_{2332}+a_{2323}\right)\left(k_3^2k_2+k_4^2k_2\right) + 8\left(a_{1233}+a_{1332}+a_{1323}\right)\left(k_3^2k_1+k_4^2k_1\right)\Big)\mathbf{U}_\mu + $$
$$+\Big(4\left(a_{1133}+a_{1331}+a_{1313}\right)k_1^2k_3 + 4\left(a_{2233}+a_{2332}+a_{2323}\right)k_2^2k_3 + 8\left(a_{1233}+a_{1332}+a_{1323}\right)k_1k_2k_3 + $$
$$+4\left(a_{3344}+a_{3443}+a_{3434}\right)k_3^3 + 4a_{3434}k_3^2k_4\Big)\mathbf{G}_{3\mu} + $$
$$+\Big(4\left(a_{1133}+a_{1331}+a_{1313}\right)k_1^2k_4 + 4\left(a_{2233}+a_{2332}+a_{2323}\right)k_2^2k_4 + 8\left(a_{1233}+a_{1332}+a_{1323}\right)k_1k_2k_4 + $$
$$+4\left(a_{3344}+a_{3443}+a_{3434}\right)k_4^3 + 4a_{3434}k_4^2k_3\Big)\mathbf{G}_{4\mu}\Big)\Big]\partial^\mu\alpha\partial_\nu\alpha\partial^\nu\alpha \tag{I.2}$$

$$\mathcal{L}_3 = \Big(\left(4a_{1111}k_1^2 + 4a_{1112}k_1k_2 + 2a_{1212}k_2^2 + 2a_{1313}k_3^2\right)\mathbf{A}_\mu\mathbf{A}^\mu + $$
$$+\left(4a_{2222}k_2^2 + 4a_{1222}k_1k_2 + 2a_{1212}k_1^2 + 2a_{2323}k_3^2\right)\mathbf{U}_\mu\mathbf{U}^\mu + $$
$$+\left(4(a_{1112}+a_{1222})k_1^2 + 4(a_{1122}+a_{1221})k_1k_2\right)\mathbf{A}_\mu\mathbf{U}^\mu + $$
$$+\left(4(a_{1133}+a_{1331})k_1k_3 + 2(a_{1234}-a_{1342})k_2k_4\right)\mathbf{A}_\mu\mathbf{G}_3^\mu + $$
$$+\left(4(a_{1133}+a_{1331})k_1k_4 + 2(a_{1342}-a_{1234})k_2k_3\right)\mathbf{A}_\mu\mathbf{G}_4^\mu + $$
$$+\left(4(a_{2233}+a_{2323})k_2k_3 + 2(a_{1342}-a_{1234})k_1k_4\right)\mathbf{U}_\mu\mathbf{G}_3^\mu + $$
$$+\left(4(a_{2233}+a_{2323})k_2k_4 + 2(a_{1234}-a_{1342})k_1k_3\right)\mathbf{U}_\mu\mathbf{G}_4^\mu + $$
$$+\left(2a_{1313}k_1^2 + 2a_{2323}k_2^2 + 2(a_{3344}+a_{3443}+a_{3434})(k_3^2+k_4^2)\right)\mathbf{G}_{3\mu}\mathbf{G}_3^\mu + $$
$$+\left(2a_{1313}k_1^2 + 2a_{2323}k_2^2 + 2(a_{3344}+a_{3443}+a_{3434})(k_3^2+k_4^2)\right)\mathbf{G}_{4\mu}\mathbf{G}_4^\mu \Big)\partial_\nu\alpha\partial^\nu\alpha \tag{I.3}$$

$$\mathcal{L}_4 = \Bigg(\Bigg(4a_{1111}k_1^2 + 8a_{1112}k_1k_2 + 2(a_{1122} + a_{1221})k_2^2 + 2(a_{1133} + a_{1331})(k_3^2 + k_4^2)\Bigg)\mathbf{A}_\mu\mathbf{A}_\nu +$$
$$+\Bigg(4a_{2222}k_2^2 + 8a_{1222}k_1k_2 + 2(a_{1122} + a_{1221})k_1^2 + 2(a_{2233} + a_{2332})(k_3^2 + k_4^2)\Bigg)\mathbf{U}_\mu\mathbf{U}_\nu +$$
$$+\Bigg(8a_{1112}k_1^2 + 4(a_{1122} + 2a_{1212} + a_{1221})k_1k_2)\Bigg)\mathbf{A}_\mu\mathbf{U}_\nu +$$
$$+\Bigg(4(a_{1133} + a_{1331} + 2a_{1313})k_1k_3 + 4(a_{1233} + 2a_{1323} + a_{1331})k_2k_3 +$$
$$+2(a_{1342} - a_{1234})k_2k_4\Bigg)\mathbf{A}_\mu\mathbf{G}_{3\nu} +$$
$$+\Bigg(4(a_{1133} + a_{1331} + 2a_{1313})k_1k_3 + 4(a_{1233} + 2a_{1332} + a_{1332})k_1k_4 +$$
$$+2(a_{1234} - a_{1342})k_2k_3\Bigg)\mathbf{A}_\mu\mathbf{G}_{4\nu} +$$
$$+\Bigg(4(a_{2233} + a_{2332} + 2a_{2323})k_2k_3 + 4(a_{1233} + 2a_{1332} + a_{1323})k_1k_3 +$$
$$+2(a_{1234} - a_{1342})k_1k_4\Bigg)\mathbf{U}_\mu\mathbf{G}_{3\nu} +$$
$$+\Bigg(4(a_{2233} + a_{2323} + 2a_{2323})k_2k_4 + 4(a_{1233} + 2a_{1332} + a_{1323})k_2k_3 +$$
$$+2(a_{1342} - a_{1234})k_1k_3\Bigg)\mathbf{U}_\mu\mathbf{G}_{4\nu} +$$
$$+\Bigg(2(a_{1133} + a_{1331})k_1^2 + 2(a_{2233} + 2a_{2332})k_2^2 + 4(a_{1233} + a_{13322}) +$$
$$+4(a_{3344} + a_{3443} + a_{3434})(k_3^2 + k_4^2) + \Bigg)\mathbf{G}_{3\mu}\mathbf{G}_{3\nu} +$$
$$+\Bigg(2(a_{1133} + a_{1331})k_1^2 + 2(a_{2233} + 2a_{2332})k_2^2 + 4(a_{1233} + a_{13322}) +$$
$$+4(a_{3344} + a_{3443} + a_{3434})(k_3^2 + k_4^2) + \Bigg)\mathbf{G}_{4\mu}\mathbf{G}_{4\nu} + 8a_{3434}k_3k_4\mathbf{G}_{3\mu}\mathbf{G}_{4\nu}\Bigg)\partial^\mu\alpha\partial^\nu\alpha \tag{I.4}$$

$$
\begin{aligned}
\mathcal{L}_5 = \Bigg(& 4(a_{1111}k_1 + a_{1112}k_2)\mathbf{A}_\mu\mathbf{A}^\mu\mathbf{A}_\nu + 4(a_{2222}k_2 + a_{1222}k_1)\mathbf{U}_\mu\mathbf{U}^\mu\mathbf{U}_\nu + \\
& +4(a_{1112}k_1 + a_{1212}k_2)\mathbf{A}_\mu\mathbf{A}^\mu\mathbf{U}_\nu + 4(2a_{1222}k_2 + (a_{1122} + a_{1221})k_1)\mathbf{A}_\mu\mathbf{U}^\mu\mathbf{U}_\nu + \\
& +4(2a_{1112}k_1 + 2(a_{1112} + a_{1221})k_2)\mathbf{A}_\mu\mathbf{U}^\mu\mathbf{A}_\nu + 4(a_{1222}k_2 + a_{1212}k_1)\mathbf{U}_\mu\mathbf{U}^\mu\mathbf{A}_\nu + \\
& +\Big(4(a_{1133} + a_{1331})(k_1 + k_3) + 4(a_{1233} + a_{1332} + a_{1342})k_2\Big)\mathbf{A}_\mu\mathbf{G}_3^\mu\mathbf{G}_{3\nu} + \\
& +\Big(4a_{1313}k_1 + a_{1323}k_3\Big)\mathbf{G}_{3\mu}\mathbf{G}_3^\mu\mathbf{A}_\nu + \Big(4a_{1313}k_3\Big)\mathbf{A}_\mu\mathbf{A}^\mu\mathbf{G}_{3\nu} + \Big(4a_{1313}k_4\Big)\mathbf{A}_\mu\mathbf{A}^\mu\mathbf{G}_{4\nu} + \\
& +\Big(4a_{1313}k_1 + a_{1323}k_4\Big)\mathbf{G}_{4\mu}\mathbf{G}_4^\mu\mathbf{A}_\nu + \Big(4(a_{1133} + a_{1323})k_4\Big)\mathbf{A}_\mu\mathbf{G}_4^\mu\mathbf{A}_\nu + \\
& +\Big(4(a_{1133} + a_{1331}k_1 + 4(a_{1233} + a_{1331})k_2\Big)\mathbf{A}_\mu\mathbf{G}_4^\mu\mathbf{G}_{4\nu} + \\
& +4a_{1323}k_3\mathbf{A}_\mu\mathbf{U}^\mu\mathbf{G}_{3\nu} + 4a_{1323}k_4\mathbf{A}_\mu\mathbf{U}^\mu\mathbf{G}_{4\nu} + 4a_{1334}k_1\mathbf{U}_\mu\mathbf{G}_4^\mu\mathbf{G}_{3\nu} + \\
& +2(a_{1334} - a_{1342})k_2\mathbf{A}_\mu\mathbf{G}_3^\mu\mathbf{G}_{4\nu} + 2a_{1334}k_2\mathbf{A}_\mu\mathbf{G}_4^\mu\mathbf{G}_{3\nu} + 2(a_{2323})k_3\mathbf{U}_\mu\mathbf{U}^\mu\mathbf{G}_{3\nu} \\
& +\Big(4a_{2323}k_2 + 4a_{1323}k_1\Big)\mathbf{G}_{4\mu}\mathbf{G}_4^\mu\mathbf{U}_\nu + \Big(4a_{2323}k_4\Big)\mathbf{A}_\mu\mathbf{G}_3^\mu\mathbf{U}_\nu + \\
& +\Big(4(a_{1233} + a_{1332})k_3 + 2(a_{1331} - a_{1342})k_4\Big)\mathbf{A}_\mu\mathbf{G}_3^\mu\mathbf{U}_\nu + \\
& +\Big(4(a_{1233} + a_{1332})k_3 + 2(a_{1342} - a_{1334})k_4\Big)\mathbf{U}_\mu\mathbf{G}_3^\mu\mathbf{A}_\nu + \\
& +\Big(4(a_{2233} + a_{2332})k_2 + 4(a_{1233} + a_{1332})k_1\Big)\mathbf{G}_{3\mu}\mathbf{U}^\mu\mathbf{G}_{3\nu} + \\
& +\Big(4(a_{2233} + a_{2332})k_3\Big)\mathbf{U}_\mu\mathbf{G}_3^\mu\mathbf{U}_\nu + \Big(4(a_{2233} + a_{2332})k_4\Big)\mathbf{U}_\mu\mathbf{G}_4^\mu\mathbf{U}_\nu + \\
& +\Big(4(a_{2233} + a_{2332})k_2 + 4(a_{1233} + a_{1332})k_1\Big)\mathbf{U}_\mu\mathbf{G}_4^\mu\mathbf{G}_{4\nu} \\
& +\Big(4(a_{2323}k_2 + a_{1323}k_1)\Big)\mathbf{G}_{3\mu}\mathbf{G}_3^\mu\mathbf{U}_\nu + \Big(4(a_{1233} + a_{1332})k_2\Big)\mathbf{G}_{4\mu}\mathbf{A}^\mu\mathbf{G}_{4\nu} + \\
& +\Big(4(a_{1233} + a_{1332})k_4 + 2(a_{1342} - a_{1334})k_3\Big)\mathbf{A}_\mu\mathbf{G}_4^\mu\mathbf{U}_\nu + \\
& +\Big(4(a_{1233} + a_{1332})k_4 + 2(a_{1334} - a_{1312})k_3\Big)\mathbf{U}_\mu\mathbf{G}_4^\mu\mathbf{A}_\nu + \\
& +\Big(2(a_{1342} - 2a_{1334})k_1\Big)\mathbf{G}_{3\mu}\mathbf{U}^\mu\mathbf{G}_{4\nu} - \Big(2a_{1342}k_1\Big)\mathbf{G}_{4\mu}\mathbf{U}^\mu\mathbf{G}_{3\nu} + \\
& +4\Big(a_{3344} + a_{3443} + a_{3434}\Big)k_3\mathbf{G}_{3\mu}\mathbf{G}_3^\mu\mathbf{G}_{3\nu} + 4\Big(a_{3344} + a_{3443} + a_{3434}\Big)k_4\mathbf{G}_{4\mu}\mathbf{G}_4^\mu\mathbf{G}_{4\nu} + \\
& +4a_{3434}k_3\mathbf{G}_{4\mu}\mathbf{G}_4^\mu\mathbf{G}_{3\nu} + 4a_{3434}k_4\mathbf{G}_{3\mu}\mathbf{G}_3^\mu\mathbf{G}_{4\nu}\Bigg)\partial^\nu\alpha
\end{aligned} \tag{I.5}
$$