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**Derivation of general Doppler effect equations**

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# Abstract

In this paper we will derive the general equations for Doppler effect. It will be proved that regardless of the nature of the emitted waves and the medium through which the waves propagate the formula always has the same form and is identical to the general Doppler effect formula for sound. We will also show that Doppler effect can be used to establish a relationship between the local time of the source and the local time of the receiver. In addition, some new features of the Doppler effect have been presented that have not been discussed in the literature so far.

**Keywords:** *Doppler effect equations*

**Introduction**

In the general formula for Doppler effect for sound waves, a reference frame is the body of air through which these waves travel. This means that we measure the speed of a source of sound waves and the speed of the observer of those waves relative to the body of air. The emitted frequency $f$ and the detected frequency $f'$ , are related by following formula [1].

$$f' = f\left(\frac{v \pm v_o}{v \pm v_s}\right) \tag{1}$$

where

$f'$ - the frequency observed by the observer
$f$ - the frequency of sound which the observer emits
$v$ - the speed of sound in the medium (air)
$v_o$ - the velocity of the observer
$v_s$ - the velocity of the sender

The positive direction is directed from the observer toward the sender.

For the electromagnetic waves there is no medium that we can use as a reference frame to measure velocities. We use a frame of reference in which is the observer at the rest. The relativistic Doppler effect for electromagnetic waves depends only on the the relative velocity $v$ between source and observer [1], [2].

$$f' = f\sqrt{\frac{c-v}{c+v}} \tag{2}$$

From (2) it follows that relative velocity $v$ is equal to:

$$v = \frac{f^2 - f'^2}{f^2 + f'^2} * c \tag{3}$$

**Derivation of general Doppler effect equations**

Suppose that the source $(Z)$ and the receiver $(S)$ of the signal move uniformly at velocities $\mathbf{u}$ and $\mathbf{v}$ respectively with respect to the coordinate system $(K)$ Figure (1). The coordinate system $(K)$ is not defined but we will assume that there is such a coordinate system. Let a signal be emitted in all directions at some point, which we will denote by $\tau_1$. The speed of the signal is denoted by $c$. The time when this signal is registered by the receiver $(S)$ is denoted by $t_1$. In a similar way we can assume that a second signal was emitted from $(Z)$ at some point $\tau_2$, and that this signal was received by the receiver $(S)$ at time $t_2$.

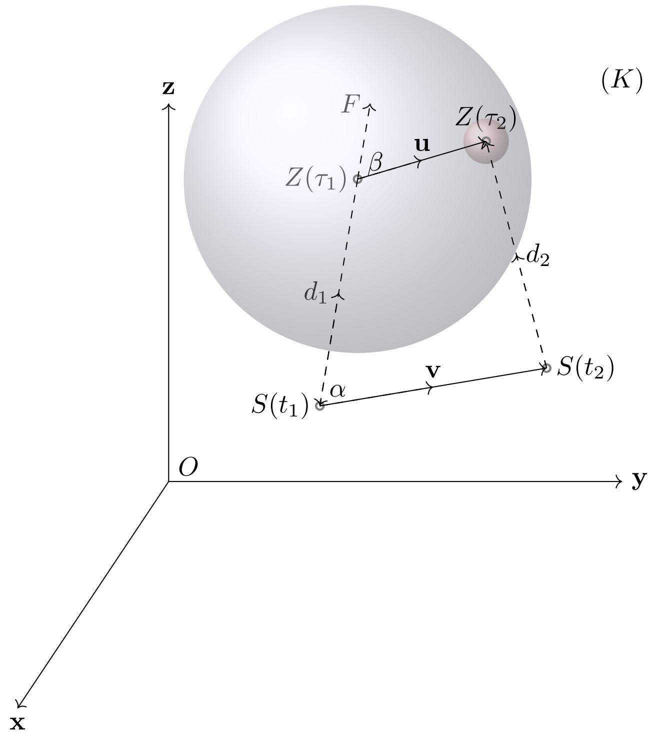


Figure 1: The sender $Z$ and the observer $S$ move uniformly with respect to the coordinate system $(K)$

Referring to Figure (1) we have that

$$\Delta t = t_2 - t_1 \tag{4}$$
$$\Delta\tau = \tau_2 - \tau_1 \tag{5}$$
$$Z_1 \equiv Z(\tau_1) \equiv [Z_x(\tau_1), Z_y(\tau_1), Z_z(\tau_1)] \tag{6}$$
$$Z_2 \equiv Z(\tau_2) \equiv [Z_x(\tau_2), Z_y(\tau_2), Z_z(\tau_2)] \tag{7}$$
$$S_1 \equiv S(t_1) \equiv [S_x(t_1), S_y(t_1), S_z(t_1)] \tag{8}$$
$$S_2 \equiv S(t_2) \equiv [S_x(t_2), S_y(t_2), S_z(t_2)] \tag{9}$$
$$\mathbf{S}_1\mathbf{S}_2 = \mathbf{v}\Delta t \tag{10}$$
$$\mathbf{Z}_1\mathbf{Z}_2 = \mathbf{u}\Delta\tau \tag{11}$$
$$\alpha = \angle(S_1S_2, S_1Z_1) \tag{12}$$
$$\beta = \angle(Z_1Z_2, Z_1F) \tag{13}$$

$$\mathbf{d}_1 = \mathbf{S}_1\mathbf{Z}_1 \tag{14}$$
$$\mathbf{d}_2 = \mathbf{S}_2\mathbf{Z}_2 \tag{15}$$
$$\mathbf{v} = [v_x, v_y, v_z] \tag{16}$$
$$\mathbf{u} = [u_x, u_y, u_z] \tag{17}$$
$$v = ||\mathbf{v}|| \tag{18}$$
$$u = ||\mathbf{u}|| \tag{19}$$
$$\mathbf{u}\cdot\mathbf{v} = u_x * v_x + u_y * v_y + u_z * v_z \tag{20}$$

The velocity $\mathbf{u}$ can be decomposed into two components, radial and tangential, which we usually denote by $\mathbf{u}_r$ and $\mathbf{u}_t$ respectively. In the same way the velocity $\mathbf{v}$ can be decomposed into components $\mathbf{v}_r$ and $\mathbf{v}_t$.

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_t \tag{21}$$
$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t \tag{22}$$
$$v_r = \frac{\mathbf{d}_1\cdot\mathbf{v}}{||\mathbf{d}_1||} = v\cos(\alpha) \tag{23}$$
$$u_r = \frac{\mathbf{d}_1\cdot\mathbf{u}}{||\mathbf{d}_1||} = u\cos(\beta) \tag{24}$$

Now we have that:

$$\frac{d_1}{c} + \Delta t = \Delta\tau + \frac{d_2}{c} \tag{25}$$
$$d_2 = d_1 + c\ \Delta t - c\Delta\tau \tag{26}$$
$$d_2^2 = d_1^2 + (c\Delta t)^2 + (c\Delta\tau)^2 + 2d_1c\Delta t - 2d_1c\Delta\tau - 2c^2\Delta\tau\Delta t \tag{27}$$
$$\mathbf{d}_2 = \mathbf{d}_1 - \mathbf{v}\Delta t + \mathbf{u}\Delta\tau \tag{28}$$
$$\mathbf{d}_2\cdot\mathbf{d}_2 = \mathbf{d}_1\cdot\mathbf{d}_1 + \mathbf{v}\cdot\mathbf{v}\Delta t^2 + \mathbf{u}\cdot\mathbf{u}\Delta\tau^2 + 2\Delta\tau\mathbf{d}_1\cdot\mathbf{u} - 2\Delta t\mathbf{d}_1\cdot\mathbf{v} - 2\Delta\tau\Delta t\mathbf{v}\cdot\mathbf{u} \tag{29}$$
$$d_2^2 = d_1^2 + (v\ \Delta t)^2 + (u\ \Delta\tau)^2 + 2\Delta\tau\ d_1\ u\ \cos(\beta) - 2\Delta t\ d_1\ v\ \cos(\alpha) - 2\ \Delta\tau\ \Delta t\ v\ u\ \cos(\beta-\alpha) \tag{30}$$
$$d_2^2 = d_1^2 + (v\ \Delta t)^2 + (u\ \Delta\tau)^2 + 2\Delta\tau\ d_1\ u_r - 2\Delta t\ d_1\ v_r - 2\ \Delta\tau\ \Delta t\ v\ u\ \cos(\beta-\alpha) \tag{31}$$

From (27) and (31) it follows that

$$(c^2 - v^2)\Delta t^2 + 2\left(d_1(v_r + c) + \Delta\tau(\mathbf{u}\cdot\mathbf{v} - c^2)\right)\Delta t + (c^2 - u^2)\Delta\tau^2 - 2\Delta\tau d_1(u_r + c) = 0 \tag{32}$$

Suppose that we have a quadratic equation :

$$A\Delta t^2 + 2B\Delta t + C = 0 \tag{33}$$

Equation (33) has two solutions $\Delta t_1$ and $\Delta t_2$.

$$\Delta t_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A} = -\frac{B}{A} \pm \frac{B\sqrt{1 - \frac{AC}{B^2}}}{A} \tag{34}$$

If $\sqrt{B^2 - AC} \geq 0$ then solutions $\Delta t_1$ and $\Delta t_2$ have real values. We choose the solution for which the condition ( $d_2 > 0$) (Equation (26)) is fulfilled.

Generally we have that:

$$\left\{\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2*4}x^2 + ...\right\} \implies \left\{\sqrt{1+x} \approx 1 + \frac{1}{2}x \ \ (where\ |x| << 1)\right\} \tag{35}$$

Suppose that $B^2 >> |AC|$. We can write that

$$\left\{B^2 >> |AC|\right\} \implies \left\{4B^2 >> |AC|\right\} \implies \left\{\frac{|2B|}{|A|} >> \frac{|C|}{|2B|}\right\} \tag{36}$$

From (34) and (36) it follows that

$$\Delta t_{1,2} \approx -\frac{B}{A} \pm \frac{B\left(1 - \frac{AC}{2B^2}\right)}{A} = -\frac{B}{A} \pm \left(\frac{B}{A} - \frac{C}{2B}\right) \tag{37}$$

$$\Delta t_1 \approx -\frac{C}{2B} \tag{38}$$

$$\Delta t_2 \approx -\frac{2B}{A} + \frac{C}{2B} \overset{(36)}{\implies} \Delta t_2 \approx -\frac{2B}{A} \tag{39}$$

Let make substitutions as it follows:

$$A = (c^2 - v^2) \tag{40}$$

$$B = d_1(v_r + c) + \Delta\tau(\mathbf{u}\cdot\mathbf{v} - c^2) \tag{41}$$

$$C = (c^2 - u^2)\Delta\tau^2 - 2\Delta\tau d_1(u_r + c) \tag{42}$$

Let define wavelength $\lambda$ as follows $\lambda = \Delta\tau * c$

We will assume that the inequality $\lambda << d_1$ holds.

Now we have that:

$$A = (c^2 - v^2) = c^2 \left(1 - \frac{v^2}{c^2}\right) = c^2 * \varepsilon_0 \tag{43}$$

$$B = d_1(v_r + c) + \Delta\tau(\mathbf{u}\cdot\mathbf{v} - c^2) = d_1 c\left(1 + \frac{v_r}{c}\right) - \Delta\tau c^2\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) = d_1 c * \varepsilon_1 - \lambda c * \varepsilon_2 \approx d_1 c * \varepsilon_1 \tag{44}$$

$$B^2 = d_1^2 c^2 * \varepsilon_1^2 - 2d_1 c^2 \lambda * \varepsilon_1\varepsilon_2 + \lambda^2 c^2 * \varepsilon_2^2 \tag{45}$$

$$C = (c^2 - u^2)\Delta\tau^2 - 2\Delta\tau d_1(u_r + c) = c^2\Delta\tau^2\left(1 - \frac{u^2}{c^2}\right) - 2c\Delta\tau d_1\left(1 + \frac{u_r}{c}\right) \tag{46}$$

$$C = \lambda^2 * \epsilon_3 - 2\lambda d_1 * \epsilon_4 \approx -2\lambda d_1 * \epsilon_4 \tag{47}$$

*where* $\left\{\varepsilon_0 = \left(1 - \frac{v^2}{c^2}\right),\ \varepsilon_1 = \left(1 + \frac{v_r}{c}\right),\ \varepsilon_2 = \left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right),\ \varepsilon_3 = \left(1 - \frac{u^2}{c^2}\right),\ \varepsilon_4 = \left(1 + \frac{u_r}{c}\right)\right\}$

$$\frac{B^2}{c^2 d_1} = \frac{d_1^2 c^2 * \varepsilon_1^2 - 2d_1 c^2\lambda * \varepsilon_1\varepsilon_2 + \lambda^2 c^2 * \varepsilon_2^2}{c^2 d_1} = d_1 * \varepsilon_1^2 - 2\lambda * \varepsilon_1\varepsilon_2 + \frac{\lambda^2}{d_1} * \varepsilon_2^2 \approx d_1 * \varepsilon_1^2 - 2\lambda * \varepsilon_1\varepsilon_2 \tag{48}$$

$$\frac{A * C}{c^2 d_1} = \frac{c^2\lambda^2\varepsilon_0\varepsilon_3 - 2\lambda d_1 c^2\varepsilon_0\varepsilon_4}{c^2 d_1} = \frac{\lambda^2}{d_1}\varepsilon_0\varepsilon_3 - 2\lambda\varepsilon_0\varepsilon_4 \approx -2\lambda\varepsilon_0\varepsilon_4 \tag{49}$$

$$\frac{B^2}{AC} \approx \frac{d_1 * \varepsilon_1^2 - 2\lambda * \varepsilon_1\varepsilon_2}{-2\lambda\varepsilon_0\varepsilon_4} \approx \frac{d_1 * \varepsilon_1^2}{-2\lambda * \varepsilon_0\varepsilon_4} = -\frac{d_1}{2\lambda} * \frac{(c + v_r)^2 * c}{(c^2 - v^2)(c + u_r)} \tag{50}$$

$$\tag{51}$$

And finally we have :

$$\Delta t_1 \approx -\frac{C}{2B} \approx \frac{2\lambda d_1 * \epsilon_4}{2d_1 c * \varepsilon_1} = \frac{1 + \frac{u_r}{c}}{1 + \frac{v_r}{c}} * \frac{\lambda}{c} = \frac{u_r + c}{v_r + c} * \Delta\tau \tag{52}$$

$$\Delta t_2 \approx -\frac{2B}{A} = -\frac{2d_1 c * \varepsilon_1}{c^2 - v^2} = -\frac{2d_1 c}{c^2 - v^2} * \left(1 + \frac{v_r}{c}\right) = -\frac{2d_1(c + v_r)}{(c - v)(c + v)} \tag{53}$$

$$\left\{\frac{\Delta t_2}{\Delta t_1} = -\frac{2d_1(c + v_r)}{(c - v)(c + v)} * \frac{c + v_r}{(c + u_r) * \Delta\tau} = -\frac{2d_1}{\lambda} * \frac{c(c + v_r)^2}{(c - v)(c + v)(c + u_r)}\right\} \implies |\Delta t_2| >> |\Delta t_1| \tag{54}$$

We will consider following cases:

1° $\{ (c + v_r) > 0 \,,\ (c + u_r) > 0 \,,\ (c - v) > 0 \} \implies \{ \Delta t_1 > 0 \,,\ \Delta t_2 < 0 \}$
There is only one solution $\Delta t_1$.

2° $\{ (c + v_r) > 0 \,,\ (c + u_r) > 0 \,,\ (c - v) < 0 \} \implies \{ \Delta t_1 > 0 \,,\ \Delta t_2 > 0 \}$
There are two solutions. This means that the signal sent from point $Z(\tau_2)$ will be registered twice by the receiver, at time $t_1 + \Delta t_1$ and at time $t_1 + \Delta t_2$.

3° $\{ (c + v_r) > 0 \,,\ (c + u_r) < 0 \,,\ (c - v) > 0 \} \implies \{ \Delta t_1 < 0 \,,\ \Delta t_2 < 0 \}$
There are no solutions.

4° $\{ (c + v_r) > 0 \,,\ (c + u_r) < 0 \,,\ (c - v) < 0 \} \implies \{ \Delta t_1 < 0 \,,\ \Delta t_2 > 0 \}$
There is only one solution $\Delta t_2$.

5° $\{ (c + v_r) < 0 \,,\ (c + u_r) > 0 \,,\ (c - v) > 0 \} \implies \{ (v + v_r) < 0 \}$
This case is not possible.

6° $\{ (c + v_r) < 0 \,,\ (c + u_r) > 0 \,,\ (c - v) < 0 \} \implies \{ \Delta t_1 < 0 \,,\ \Delta t_2 < 0 \}$
There are no solutions.

7° $\{ (c + v_r) < 0 \,,\ (c + u_r) < 0 \,,\ (c - v) > 0 \} \implies \{(v + v_r) < 0 \}$

This case is not possible.

$8^\circ$ $\{ \ (c+v_r)<0\ ,\ (c+u_r)<0\ ,\ (c-v)<0 \ \} \implies \{ \ \Delta t_1 > 0\ ,\ \Delta t_2 > 0 \ \}$

There are two solutions. This means that the signal sent from point $Z(\tau_2)$ will be registered twice by the receiver, at time $t_1 + \Delta t_1$ and at time $t_1 + \Delta t_2$.

Let us consider the case when $u < c$ and $v < c$. We have the following implication:

$\{u < c \bigwedge v < c\} \implies \{\Delta t_1 > 0 \bigwedge \Delta t_2 < 0\}$

This means that $\Delta t_1$ is the only solution of Equation (33).

Let define the frequencies $f = \frac{1}{\Delta\tau}$ and the $f' = \frac{1}{\Delta t}$.

Now we can write that:

$$\Delta t = \Delta\tau \frac{c+u_r}{c+v_r} \tag{55}$$

$$f' = \frac{c+v_r}{c+u_r} * f = \frac{c+v*cos(\alpha)}{c+u*cos(\beta)} * f \tag{56}$$

In that way we prove that formula (56) is identical to formula (1).

After multiplying both sides of (55) by $c$ we get the following expression

$$\lambda' = \frac{c+u_r}{c+v_r} * \lambda = \frac{c+u*cos(\alpha)}{c+v*cos(\beta)} * \lambda \tag{57}$$

where $\lambda$ is wavelength of the emitted light and $\lambda'$ the wavelength we observe.

Doppler shift $z_f$ in frequency and Doppler shift $z_\lambda$ in wavelength are defined by the following formulas respectively:

$$z_f = \frac{f'-f}{f} = -\frac{u_r - v_r}{c+u_r} = -\frac{\Delta u_r}{c+u_r} - \frac{\Delta u_r}{c}\left(1 - \frac{u_r}{c} + \frac{{u_r}^2}{c^2} - ...\right) \tag{58}$$

$$z_\lambda = \frac{\lambda'-\lambda}{\lambda} = \frac{u_r - v_r}{c+v_r} = \frac{\Delta u_r}{c+v_r} = \frac{\Delta u_r}{c}\left(1 - \frac{v_r}{c} + \frac{{v_r}^2}{c^2} - ...\right) \tag{59}$$

If $\mid u_r \mid \ << \ c$ then we have

$$z_f \approx -\frac{\Delta u_r}{c} \tag{60}$$

And if $\mid v_r \mid \ << \ c$ it follows that

$$z_\lambda \approx \frac{\Delta u_r}{c} \tag{61}$$

It is important to note that as long as at least one of the velocities $\mathbf{u}_t$ or $\mathbf{v}_t$ is different from zero, the observer's line of sight changes, and therefore the angles $\alpha$ and $\beta$ change. This means that over time the frequency $f'$ changes even if the velocities $\mathbf{u}$ and $\mathbf{v}$ are constant.

There is no need to distinguish between the Transverse Doppler Effect (transverse velocity) and the Longitudinal Doppler effect (radial velocity) as two different types of Doppler effect, as is usually done. Both cases are covered by a single formula given by (56).

# 1 Comparison between the local times of the sender and receiver

Suppose that $(n+1)$ signals have been emitted. We denote by $\{\tau_0, \tau_1, ..., \tau_n\}$ the times when the signals were sent and by $\{t_0, t_1, ..., t_n\}$ the times when the signals were received. Let define $\Delta\tau_i$ $(\Delta\tau_i = \tau_i - \tau_{i-1})$ , $\Delta t_i$ $(\Delta t_i = t_i - t_{i-1})$ and the frequencies $f_i = \frac{1}{\Delta\tau_i}$ for the emitted signals and the $f'_i = \frac{1}{\Delta t_i}$ for the received signals, where $i \in \{1, 2, ...n\}$. We have the following equations.

$$f_i * \Delta\tau_i = f'_i * \Delta t_i \tag{62}$$

$$\Delta\tau_i = \frac{f'_i}{f_i}\Delta t_i \tag{63}$$

Suppose that the signals are emitted at equal time intervals $\Delta\tau$ and received at approximately the equal time intervals $\Delta t$. It follows that:

$$f = \frac{1}{\Delta\tau} \tag{64}$$

$$f' = \frac{1}{\Delta t} \tag{65}$$

$$\{\Delta\tau_i = \Delta\tau\} \quad \Longrightarrow \{f_i = f\} \tag{66}$$

$$\{\Delta t_i \approx \Delta t\} \quad \Longrightarrow \{f'_i \approx f'\} \tag{67}$$

$$\Delta\tilde{\boldsymbol{\tau}} = \tau_n - \tau_0 = \sum_{n=1}^{n} \Delta\tau_i \approx \sum_{n=1}^{n} \frac{f' * \Delta t_i}{f} \approx \frac{f'}{f}(t_n - t_0) = \frac{f'}{f}(\Delta\tilde{\mathbf{t}}) \tag{68}$$

$$\Delta\tilde{\boldsymbol{\tau}} \approx \frac{f'}{f}\Delta\tilde{\mathbf{t}} \tag{69}$$

The relationship between the local time $\Delta\tilde{\boldsymbol{\tau}}$ of the sender and the local time $\Delta\tilde{\mathbf{t}}$ of the receiver is given by formula (69). The difference in times is not because the duration of the clock cycles in the two frame of reference are different but because the speed of the signal is finite and because the sender and receiver change position in relation to the direction determined by the position of the sender at the time when the signal is emitted and the position of the receiver when that signal is detected. If the clock cycles were different then the frequency $(f)$ corresponding to a certain electromagnetic signal would not be constant for each sender but would change. We can conclude that time is universal except that there is local time in each reference frame.

## 2 Discussion and Conclusion

The frequency $f'$ can be determined in two ways, by direct measurement or by applying formula (56). While in the first case we have exactly one value in the second case that value depends on what is chosen as the reference for measuring the velocities **u** and **v**. Since $f'$ must have exactly one value it means that there is exactly one reference frame. The question is whether this frame is fixed or depends on how the sender and receiver of the signal are selected (for example the recipient is on Earth and the sender is somewhere in the Earth's orbit, the solar system, the Galaxy ...). At this point, we cannot give a precise answer to that question. If we accept the principle that measurements are made in relation to the receiver then it is obvious that this rule does not apply to sound waves. We can go so far as to claim that this principle has not been proven even in the case of electromagnetic waves, because there is no experiment to determine that the relative velocity at which the sender moves in relation to the observer is exactly that obtained by formula (3).

One of the ways in which it would be possible to choose a coordinate system $(K)$ is already described in the paper in which we dealt with the problem of stellar aberration [3]. The ray connecting the sender and the receiver (telescope) represents the $z - axis$ and the plane in which the objective lens lies is chosen as the $(xy)$ plane.

The discussion on how to choose the reference coordinate systems $(K)$ is not over yet and we will deal with this problem in the future.

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