Application of The Magnus and Fer Expansions to A Perturbative Study of The Time Dependent Jaynes Cummings Model

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Abstract

We apply the Magnus and Fer perturbation methods to Jaynes Cummings Model (JCM) with linear time dependence and show that the perturbation methods yield results which are appreciably close to the results obtained by Wei-Norman method, leading to exact solution when the Hamiltonian under consideration is an element of su(2), thus making the perturbation methods important for Hamiltonians for which a method of exact solution is not available.

Keywords: JCM, Matrix, Evolution, Operator, Perturbation, Ramp, Sweep, Wei-Norman, Feynman-Dyson, Magnus, Fer.

Subject Classication: Quantum optics.

Introduction

Study of Jaynes Cummings Model[2] by several authors explored interesting behaviours of both the two-level atom and the radiation field of a coupled radiation-matter system [6,7,8,9,10,11,12,13]. The underlying SU(2) structure of the model[16] made it possible for application of the Wei-Norman method[3] for an exact solution. Dasgupta showed that the model is exactly solvable for a sech pulse[17] and for linear time dependence[19]. Limitations of exact solvability of the model with arbitrary time dependence lead us to follow perturbation methods as an alternative approach. The Hamiltonian of the model in the interaction picture is

$$H(t) = 2\delta(\Delta)J_3 + 2\Lambda(t)\sqrt{\chi(\Delta)}J_1 \quad (1)$$

where is a constant operator

$$\Delta = a^{\dagger}a + \frac{1+\sigma_3}{2} \quad (2)$$

and c = $a^{\dagger}a$, $a^{\dagger}a$ being the number operator describing the strength of the field, ($a^{\dagger}$, a) being the creation and annihilation operator, respectively, describing the dynamics of the field and Ji are the generators of the SU(2) inherent in the Hamiltonian. We study the Hamiltonian (1) by assuming only one part of it to be linearly time dependent. The particular form of the Hamiltonian allows us to apply the methods of perturbation, assuming the time dependent part to be exactly solvable and the rest as a small perturbation. we expect that the procedures shall generate results with close proximity with those given by the exact solution, thereby, making it an alternative trick to approach time dependent Hamiltonians for which a method of exact solution is unavailable.

1 Perturbation Methods

When a time dependent Hamiltonian H(t) does not commute with itself at dierent times, the usual approach for solution is the Feynman-Dyson method.

1.1 Feynman-Dyson Formula

The evolution operator is given by[21]

$$U(t) = 1 + \sum_{k=1}^{\infty}(-i)^k \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{k-1}} dt_k H(t_1)H(t_2)\cdots H(t_k) \quad (3)$$

Apart from the increasing difficulties in calculations in higher order, the formula suffers from a major drawback - the time evolution operator is not unitary at each order, which can be avoided by applying Magnus and Fer perturbation methods.

1.2 Magnus Formula

The evolution operator is expressed as exponential of an innite sum of anti-hermitian operators[1,4,5,15,22]

$$U(t,t_0) = \exp[\sum_n A_n(t,t_0)] \quad (4)$$

where the expansion terms An are given by

$$A_1(t) = -i\int_{t_0}^{t} dt_1 H(t_1)$$

$$A_2(t) = \frac{1}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 [H_2, H_1]$$

$$A_3(t) = \frac{i}{3!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 [[H_3,[H_2,H_1]] + [[H_3,H_2],H_1]]$$

and so on to the higer orders.

1.3 Fer Formula

According to Fer the time evolution operator is written as an innite product of exponential operators[5,17,23]

$$U(t,t_0) = \prod_{j=1}^{\infty} \exp[S_j(t)] \quad (5)$$

where

$$S_1(t) = -i\int_{t_0}^{t} H(t_1)dt_1$$

$$S_2(t) = \frac{1}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 [H_2, H_1]$$

$$S_3(t) = \frac{i}{3!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 [[H_3,[H_2,H_1]] + [H_2,[H_3,H_1]]]$$

and so on to the higer orders.

2 Perturbation method applied to JCM with linear time dependence

2.1 Interaction picture Hamiltonian

We rename the picture in which the Hamiltonian (1) is written, to be 'Schrödinger Picture'. When the time dependence of the Hamiltonian is carried by only one term, it has the general form

$$H(t) = H_0(t) + V \quad (6)$$

The time dependent part of the Hamiltonian is assumed to have an exact solution and the other time independent part is treated as a small perturbation to H0(t). With the initial time set at zero, the exact solution of the time dependent part is given by

$$U_0(t) = \exp\left[i\int_0^t dt' H_0(t')\right] \quad (7)$$

We rid of the time dependent part by transforming (6) to 'Interaction Picture'

$$H_I(t) = U_0 V U_0^\dagger \quad (8)$$

The form of this interaction picture Hamiltonian depends on the time dependence of L(t) and involves the generators Ji of the SU(2), as evident from (1). This determines the evolution operator in interaction picture, at each order of a perturbation scheme.

2.2 The evolution matrix

We evaluate the time evolution operator at k^{th} order of a perturbation scheme as a 2x2 matrix V(t) = $v_{ij}^{(k)}$ using the algebra followed by Ji and its connection to the Pauli matrices through

$$J_i = \sigma_i/2 \quad (9)$$

For Feynman-Dyson formula, straightforward substitution of (9) gives the evolution operator in matrix form.

For Magnus formula the sum inside the exponential of (4) is expressed in terms of the generators Ji

$$A(t) = -i[g_1(t)J_1 + g_2(t)J_2 + g_3(t)J_3] \quad (10)$$

Using (9) the evolution operator is given in terms of the Pauli matrices

$$U(t) = \cos f\,\mathbf{1} - i\frac{\sin f}{f}[f_1(t)\sigma_1 + f_2(t)\sigma_2 + f_3(t)\sigma_3] \quad (11)$$

where fi(t) = gi(t)/2 and f = $(f1^2 + f2^2 + f3^2)^{1/2}$. With $f_\pm$ = f1 ±if2, the evolution matrix becomes

$$U(t) = \begin{pmatrix} \cos f - i f_3 \frac{\sin f}{f} & -i f_- \frac{\sin f}{f} \\ -i f_+ \frac{\sin f}{f} & \cos f + i f_3 \frac{\sin f}{f} \end{pmatrix} \quad (12)$$

Thus at the first order the evolution matrix is exp[A1(t)], at the second order, it is exp[A1(t) + A2(t)] and so on.

The argument Sj(t) of the exponential in Fer formula (5), at each order, is given by

$$S_j(t) = -i[\alpha_j(t)\sigma_1 + \beta_j(t)\sigma_2 + \gamma_j(t)\sigma_3] \quad (13)$$

by using (9). Thus, the evolution matrix at the kth order is given by

$$U_k(t) = \begin{pmatrix} \cos f_k - i\gamma_k \frac{\sin f_k}{f_k} & -if_k^{(-)} \frac{\sin f_k}{f_k} \\ -if_k^{(+)} \frac{\sin f_k}{f_k} & \cos f_k + i\gamma_k \frac{\sin f_k}{f_k} \end{pmatrix} \quad (14)$$

with

$$f_k^2 = \alpha_k^2 + \beta_k^2 + \gamma_k^2, \quad \text{and} \quad f_k^{\pm} = \alpha_k \pm i\beta_k \quad (15)$$

At the first order, the evolution operator is U(t) = U1(t), to the second order it is U(t) = U1(t)U2(t) and so on. Thus, one may calculate the evolution operator in explicit analytic form to any desired order.

2.3 Linear Ramp

We call the linear time dependence a linear ramp when the detuning parameter is linearly time dependent, d(D)=bt and the interaction parameter is time independent, L(t)=l0 , so that the Hamiltonian (1) becomes

$$H(t) = 2\beta t J_3 + 2\lambda_0 \sqrt{\chi} J_1 \quad (16)$$

Evolution operator for the time dependent part of (16) is

$$U_0(t) = \exp[i\beta t^2 J_3] \quad (17)$$

The 'Interaction Picture' Hamiltonian is given by

$$H_I(t) = 2\delta[\cos(\beta t^2)J_1 - \sin(\beta t^2)J_2] \quad (18)$$

2.3.1 Feynman-Dyson

Up to first order, the evolution operator is given by

$$V^{(1)} = \begin{pmatrix} 1 & \delta(I_2 - iI_1) \\ -\delta(I_2 + iI_1) & 1 \end{pmatrix} \quad (19)$$

Up to second order,

$$V^{(2)} = \begin{pmatrix} 1 - \delta^2[(I_3 + I_6) + i(I_5 - I_4)] & \delta(I_2 - iI_1) \\ -\delta(I_2 + iI_1) & 1 - \delta^2[(I_3 + I_6) - i(I_5 - I_4)] \end{pmatrix} \quad (20)$$

Up to third order,

$$V^{(3)} = \begin{pmatrix} 1 - \delta^2[(I_3 + I_6) + i(I_5 - I_4)] & \delta(I_2 - iI_1) + i\delta^3(k_1 + ik_2) \\ -\delta(I_2 + iI_1) + i\delta^3(k_1 - ik_2) & 1 - \delta^2[(I_3 + I_6) - i(I_5 - I_4)] \end{pmatrix} \quad (21)$$

with

$$k_1 = I_7 + I_{10} - I_{12} + I_{13}$$
$$k_2 = I_8 - I_9 + I_{11} + I_{14} \quad (22)$$

2.3.2 Magnus

As mentioned in subsection 2.2, the functions fk(t) occurring in (4) determines the matrix elements of the evolution operator at each order.
At the first order,

$$f_1(t) = 2\delta I_1, \qquad f_2(t) = -2\delta I_2, \qquad f_3(t) = 0 \quad (23)$$

At the second order,

$$f_1 = 2\delta I_1, \qquad f_2 = -2\delta I_2, \qquad f_3 = 2\delta^2(I_5 - I_4) \quad (24)$$

At the third order,

$$f_1 = 2\delta I_1 - \frac{4}{3}\delta^3(I_{10} - 2I_{12} + I_{13}), \qquad f_2 = -2\delta I_2 + \frac{4}{3}\delta^3(I_8 - 2I_9 + I_{11}), \qquad f_3 = 2\delta^2(I_5 - I_4) \quad (25)$$

2.3.3 Fer

As mentioned in subsection 2.3, the functions fk(t) occurring in (13) determines the matrix elements of the evolution operator at each order.
At the first order,

$$f_1(t) = 2\delta I_1, \qquad f_2(t) = -2\delta I_2, \qquad f_3(t) = 0 \quad (26)$$

At the second order,

$$f_1 = 0, \qquad f_2 = 0, \qquad f_3 = 2\delta^2(I_5 - I_4) \quad (27)$$

At the third order,

$$f_1(t) = \frac{4}{3}\delta^3(-2I_{10} + I_{12} + I_{13}), \qquad f_2(t) = \frac{4}{3}\delta^3(-I_8 - I_9 + 2I_{11}), \qquad f_3(t) = 0 \quad (28)$$

2.4 Linear sweep

Calling the case with time independent detuning d(D)=d and linearly time dependent interaction parameter L(t)=l0t a linear sweep, the Hamiltonian (1) becomes

$$H(t) = 2\delta J_3 + 2\lambda_0 t\sqrt{\chi} J_1 \quad (29)$$

Evolution operator for the time dependent part of (29) is

$$U_0(t) = \exp[i\beta t^2 J_1] \quad (30)$$

The 'Interaction Picture' Hamiltonian is given by

$$H_I(t) = 2\delta[\sin(\beta t^2)J_2 + \cos(\beta t^2)J_3] \quad (31)$$

2.4.1 Feynman-Dyson

Up to first order,

$$V^{(1)} = \begin{pmatrix} 1 - i\delta I_1 & -\delta I_2 \\ \delta I_2 & 1 + i\delta I_1 \end{pmatrix} \quad (32)$$

Up to second order,

$$V^{(2)} = \begin{pmatrix} 1 - i\delta I_1 - \delta^2 (I_3 + I_6) & -\delta I_2 + i\delta^2 (I_4 - I_5) \\ \delta I_2 + i\delta^2 (I_4 - I_5) & 1 + i\delta I_1 - \delta^2 (I_3 + I_6) \end{pmatrix} \quad (33)$$

Up to third order,

$$V^{(3)} = \begin{pmatrix} 1 - i\delta I_1 - \delta^2 (I_3 + I_6) + i\delta^3 k_1 & -\delta I_2 + i\delta^2 (I_4 - I_5) + \delta^3 k_2 \\ \delta I_2 + i\delta^2 (I_4 - I_5) - \delta^3 k_2 & 1 + i\delta I_1 - \delta^2 (I_3 + I_6) - i\delta^3 k_1 \end{pmatrix} \quad (34)$$

with

$$k_1 = I_7 + I_{10} - I_{12} + I_{13}$$
$$k_2 = I_8 - I_9 + I_{11} + I_{14}$$

2.4.2 Magnus

As mentioned in subsection 2.2, the functions fk(t) occurring in (10) determines the matrix elements of the evolution operator at each order.
First order,

$$f_1(t) = 0, \qquad f_2(t) = 2\delta I_2, \qquad f_3(t) = 2\delta I_1 \quad (35)$$

Second order,

$$f_1 = 2\delta^2 (I_5 - I_4), \qquad f_2 = 2\delta I_2, \qquad f_3 = 2\delta I_1 \quad (36)$$

Third order,

$$f_1 = 2\delta^2 (I_5 - I_4), \qquad f_2 = 2\delta I_2 + \frac{4}{3}\delta^3 (I_8 - 2I_9 + I_{11}), \qquad f_3 = 2\delta I_1 - \frac{4}{3}\delta^3 (I_{10} - 2I_{12} + I_{13}) \quad (37)$$

2.4.3 Fer

As mentioned in subsection 2.3, the functions fk(t) occurring in (13) determines the matrix elements of the evolution operator at each order.
First order,

$$f_1(t) = 0, \qquad f_2(t) = 2\delta I_2, \qquad f_3(t) = 2\delta I_1 \quad (38)$$

Second order,

$$f_1 = 2\delta^2(I_5 - I_4), \qquad f_2 = 2\delta I_2, \qquad f_3 = 2\delta I_1 \quad (39)$$

Third order,

$$f_1 = 0, \qquad f_2 = \frac{4}{3}\delta^3(I_8 + I_9 - 2I_{11}), \qquad f_3 = \frac{4}{3}\delta^3(-2I_{10} + I_{12} + I_{13}) \quad (40)$$

The time dependent integrals Ik(t) are presented in the appendix and are computed using python scripts.

3 Results and Discussion

3.1 Linear ramp

Fig.1 and Fig.2 show, respectively, that the time evolution of inversion and Mandel Q parameter determined by Feynman-Dyson(FD) formula at second and third orders, are much better than those at the first order. Similar feature is evident for squeezing - FD formula, even at first order, gives results which are appreciably close to the result of Wei-Norman(WN) method(Fig.2). Also, the figures indicate an order-by-order improvement of the results. This order-by-order improvement of the results are studied with a greater detail by computing the logarithmic difference of inversions obtained from WN method and FD formula(Fig.4). As shown, a larger negative value of the logarithmic difference shows an improvement of the result at higher order of the perturbation method. Study of the physical quantities with Magnus and Fer perturbation formulae show similar features(Fig.7,8,9 and Fig.13,14,15), along with a clear indication that Magnus and Fer methods yield even better results than FD formula(Fig.10,11,12 and Fig.16,17,18).

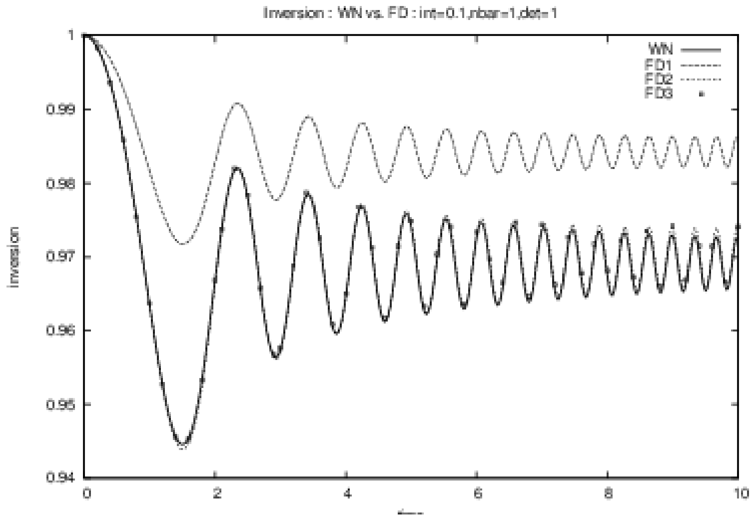

Figure 1: Ramp : Comparing WN and FD : Variation of inversion.

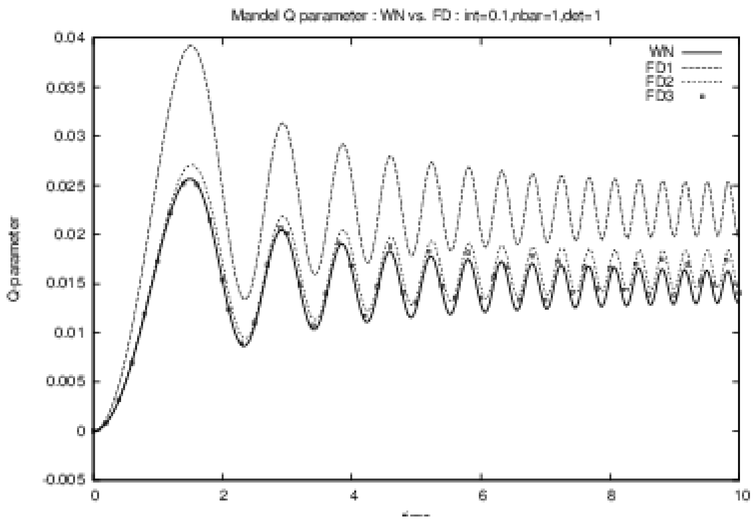

Figure 2: Ramp : Comparing WN and FD : Variation of Mandel Q parameter.

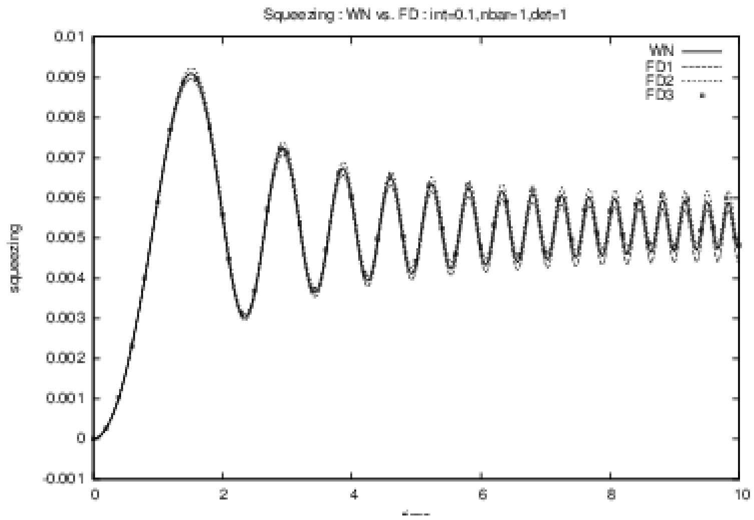

Figure 3: Ramp : Comparing WN and FD : Variation of squeezing.

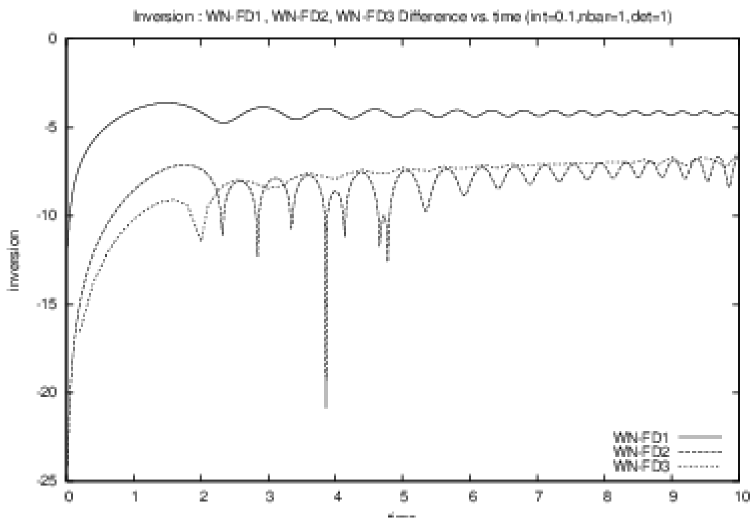

Figure 4: Ramp : Comparing WN and FD : Variation of inversion : Lorgarithmic Difference.

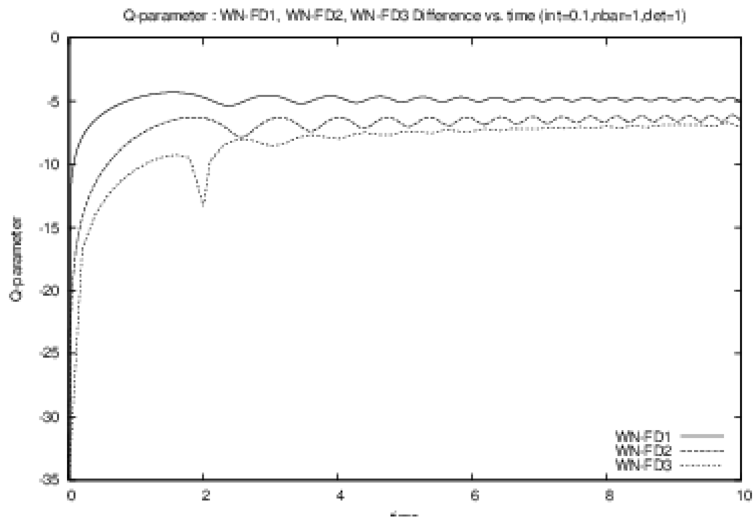

Figure 5: Ramp : Comparing WN and FD : Variation of Mandel Q parameter : Lorgarithmic Difference.

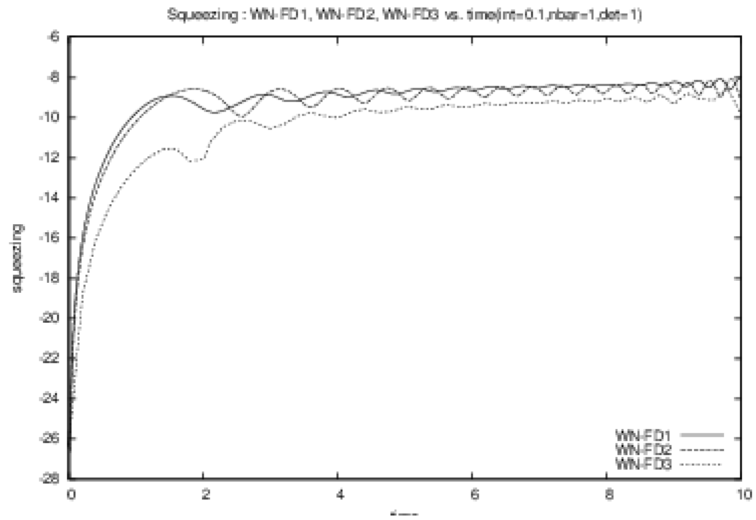

Figure 6: Ramp : Comparing WN and FD : Variation of squeezing : Lorgarithmic Difference.

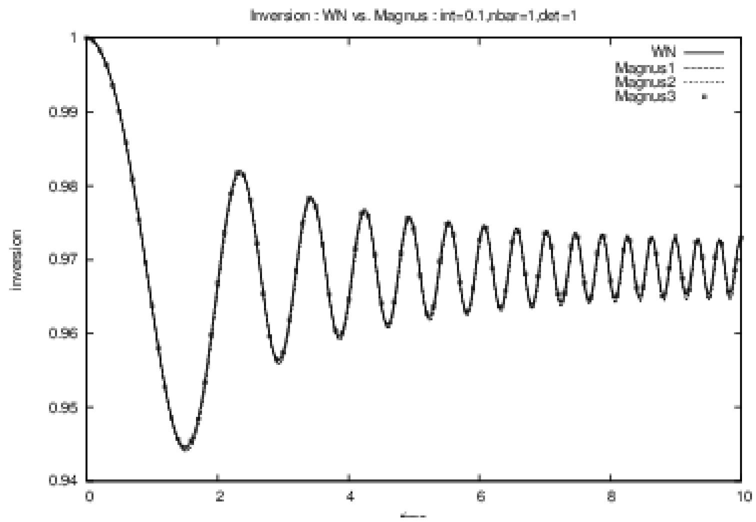

Figure 7: Ramp : Comparing WN and Magnus : Variation of inversion.

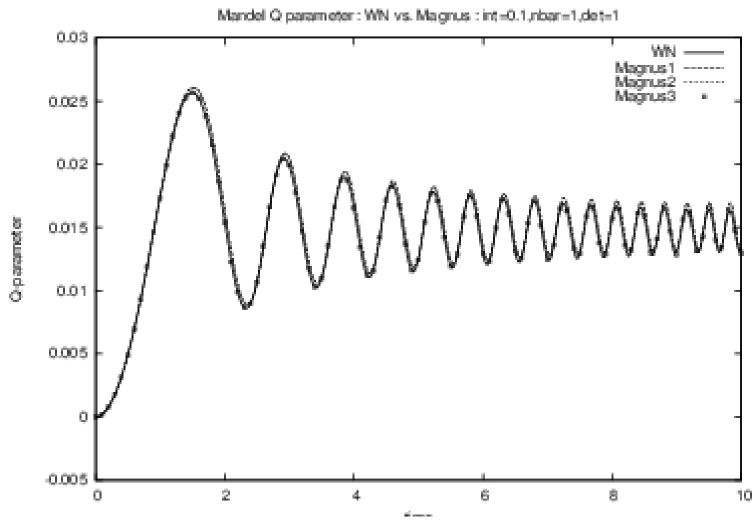

Figure 8: Ramp : Comparing WN and Magnus : Variation of Mandel Q parameter.

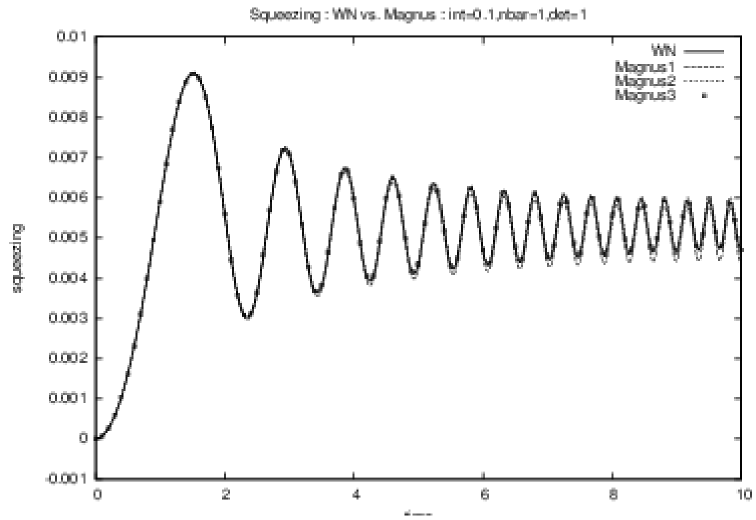

Figure 9: Ramp : Comparing WN and Magnus : Variation of squeezing.

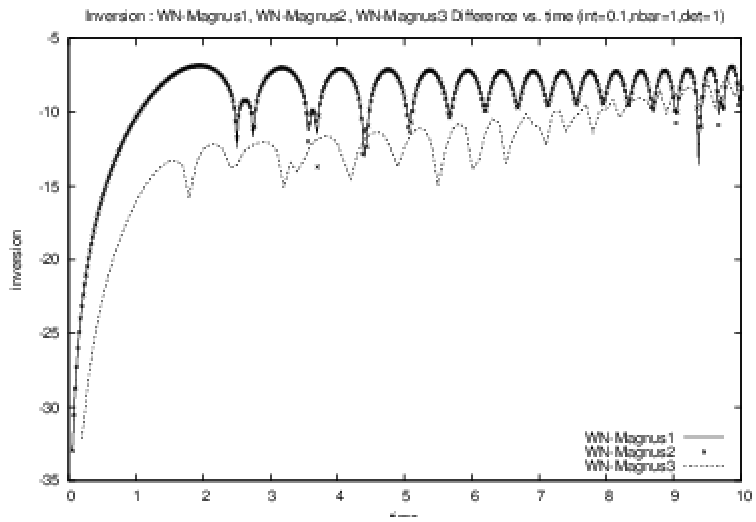

Figure 10: Ramp : Comparing WN and Magnus : Variation of inversion : Lorgarithmic Difference.

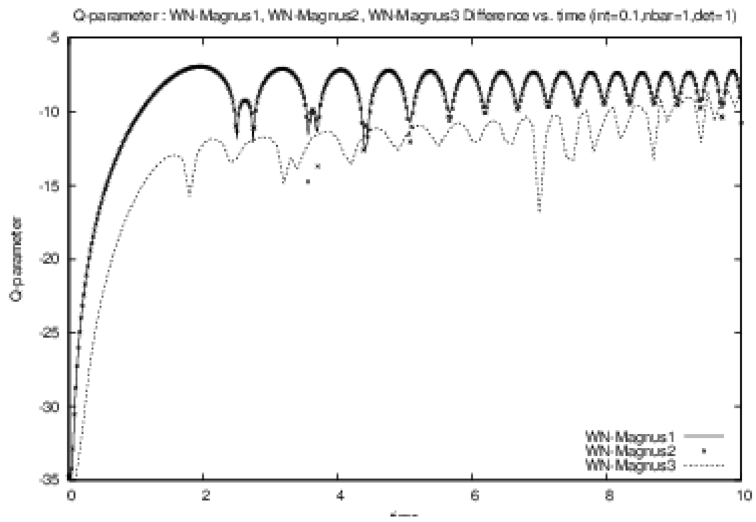

Figure 11: Ramp : Comparing WN and Magnus : Variation of Mandel Q parameter : Lorgarithmic Difference.

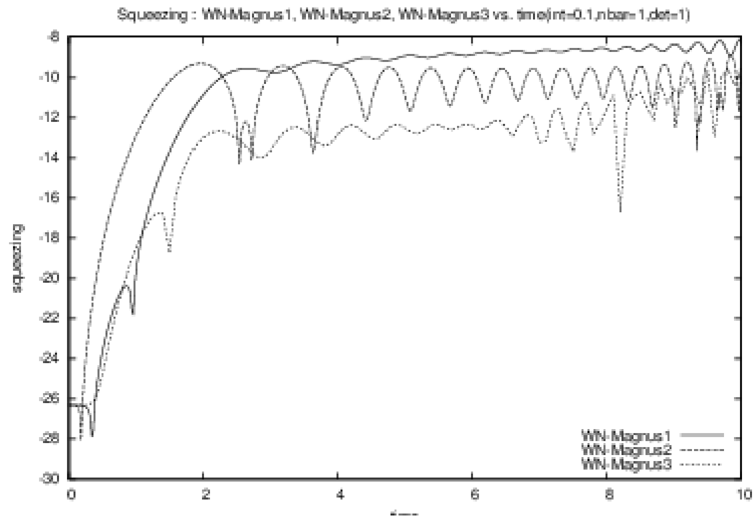

Figure 12: Ramp : Comparing WN and Magnus : Variation of squeezing : Lorgarithmic Difference.

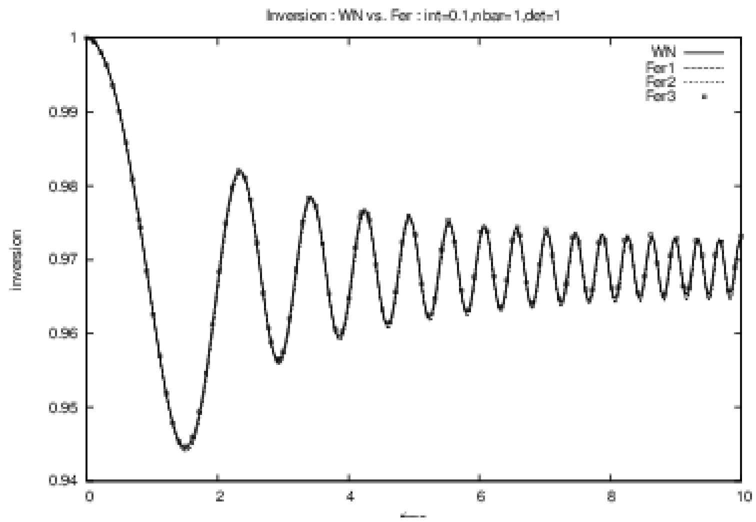

Figure 13: Ramp : Comparing WN and Fer : Variation of inversion.

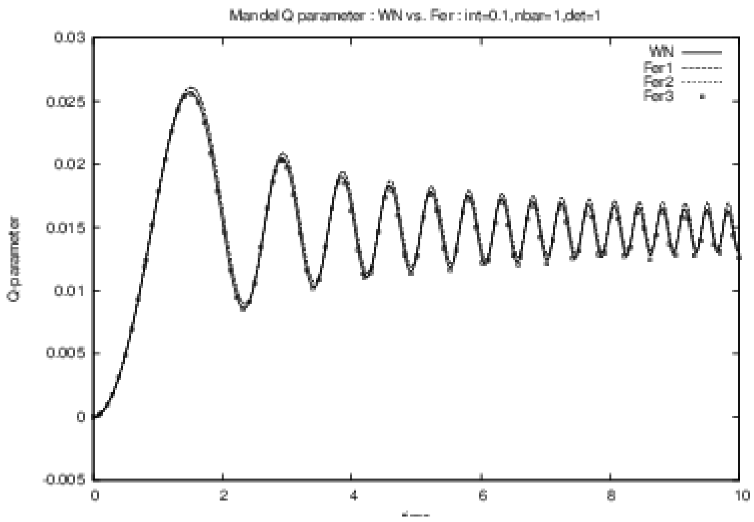

Figure 14: Ramp : Comparing WN and Fer : Variation of Mandel Q parameter.

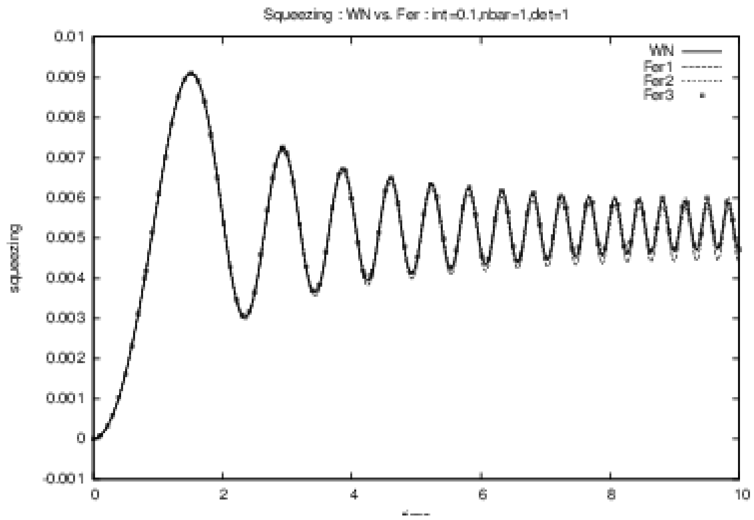

Figure 15: Ramp : Comparing WN and Fer : Variation of squeezing.

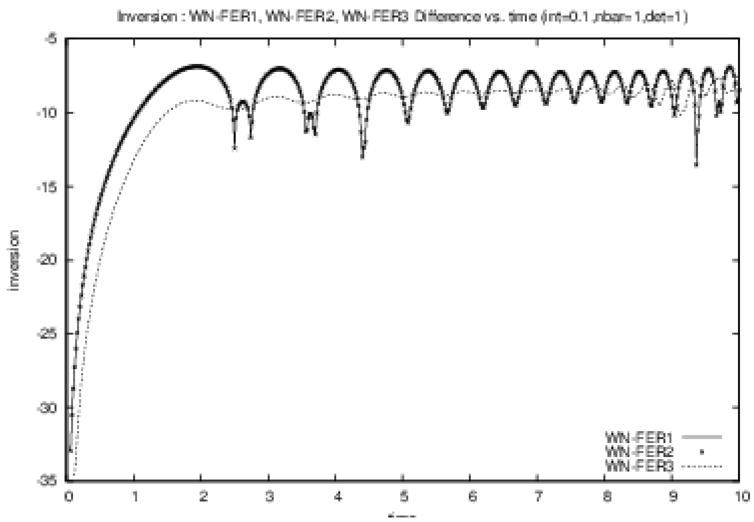

Figure 16: Ramp : Comparing WN and Fer : Variation of inversion : Lorgarithmic Difference.

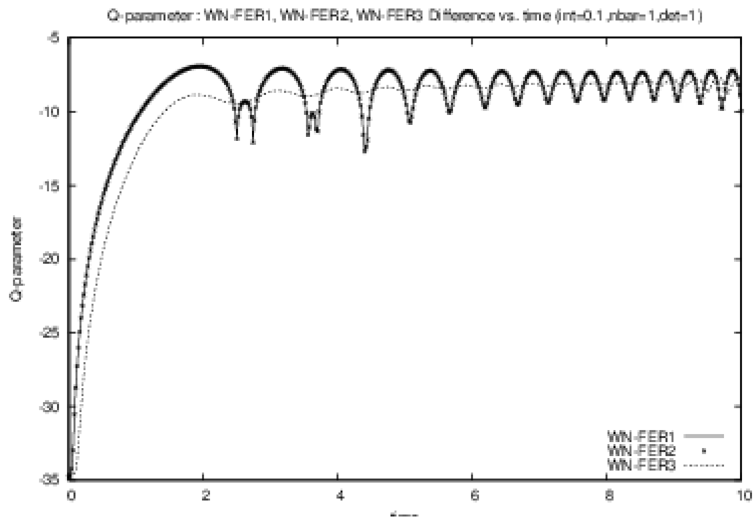

Figure 17: Ramp : Comparing WN and Fer : Variation of Mandel Q parameter : Lorgarithmic Difference.

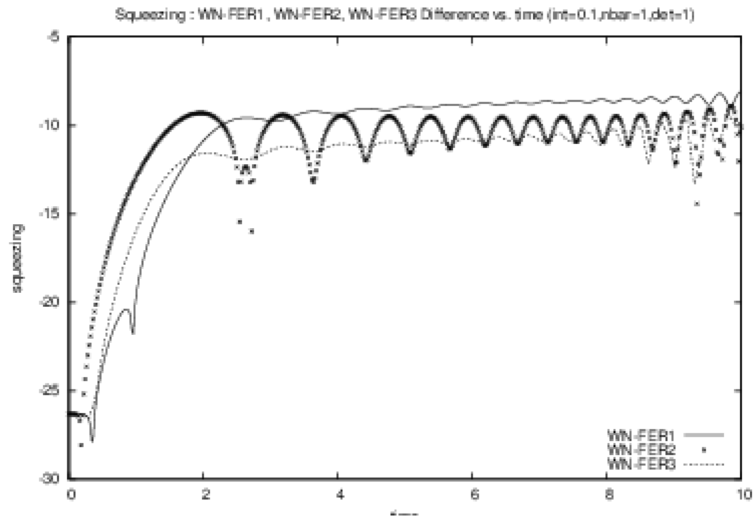

Figure 18: Ramp : Comparing WN and Fer : Variation of squeezing : Lorgarithmic Difference.

3.2 Linear sweep

As for the case of linear ramp, the closeness of the results yielded by the perturbation methods to those generated by Wei-Norman method is evident from the figures 19,20,21,25,26,27,31,32 and 33. The gures 22,23,24,28,29,30,34,35 and 36 show the order-by-order improvement of a perturbation scheme.

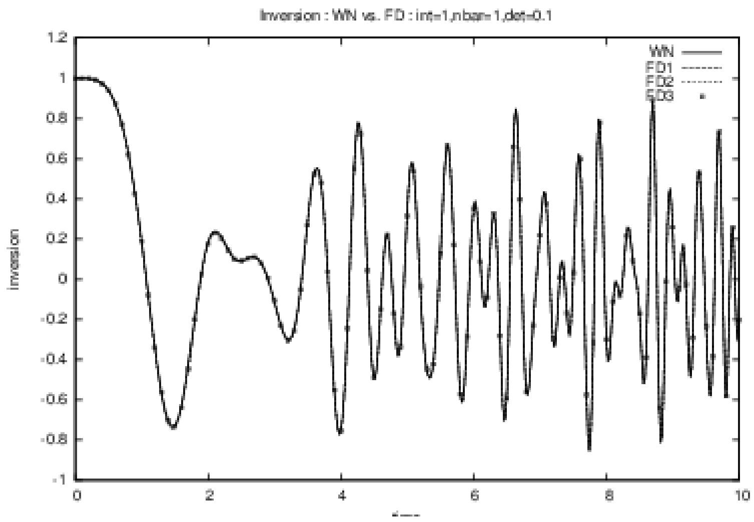

Figure 19: Sweep : Comparing WN and FD : Variation of inversion.

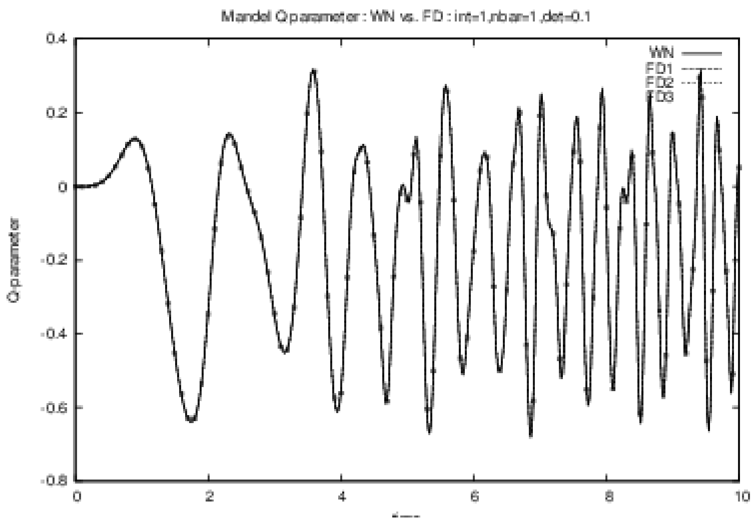

Figure 20: Sweep : Comparing WN and FD : Variation of Mandel Q parameter.

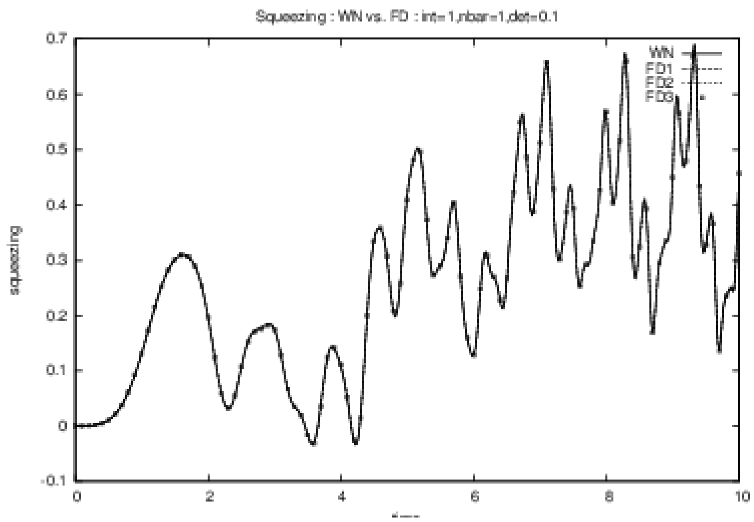

Figure 21: Sweep : Comparing WN and FD : Variation of squeezing.

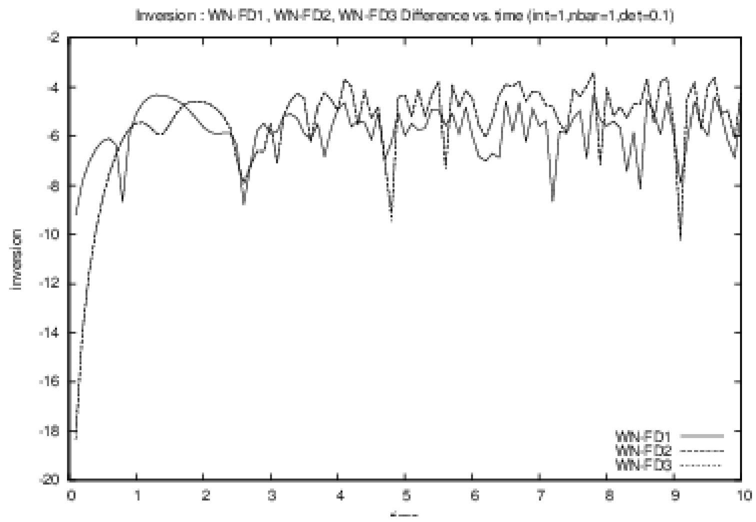

Figure 22: Sweep : Comparing WN and FD : Variation of inversion : Lorgarithmic Difference.

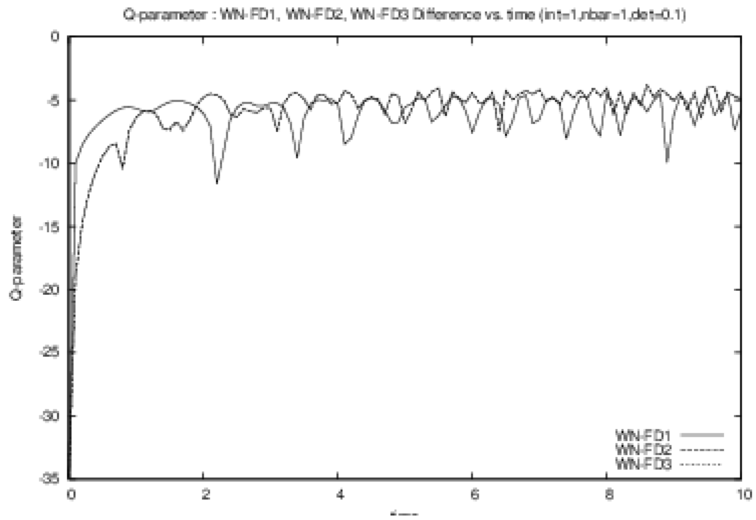

Figure 23: Sweep : Comparing WN and FD : Variation of Mandel Q parameter : Lorgarithmic Difference.

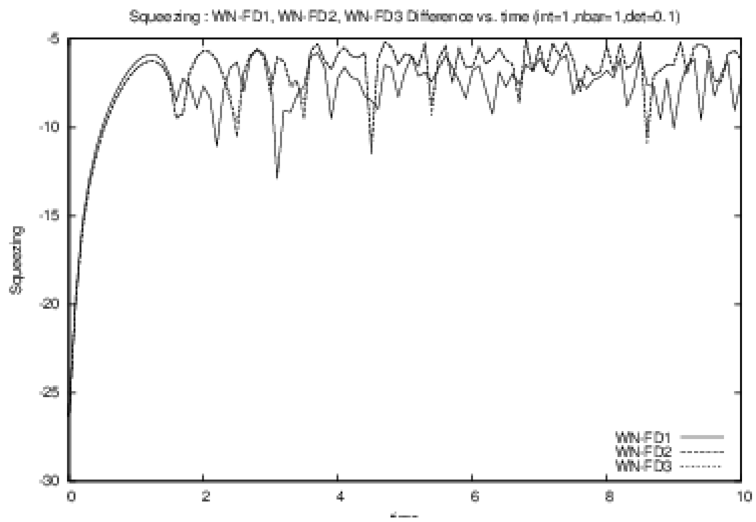

Figure 24: Sweep : Comparing WN and FD : Variation of squeezing : Lorgarithmic Dffierence.

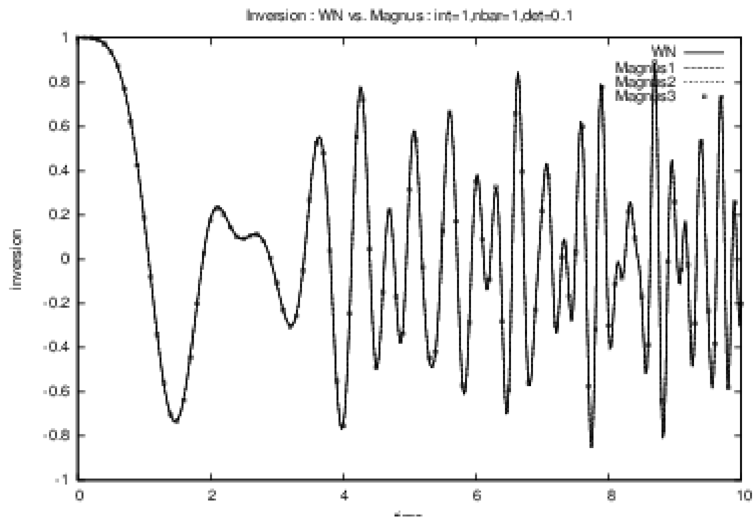

Figure 25: Sweep : Comparing WN and Magnus : Variation of inversion.

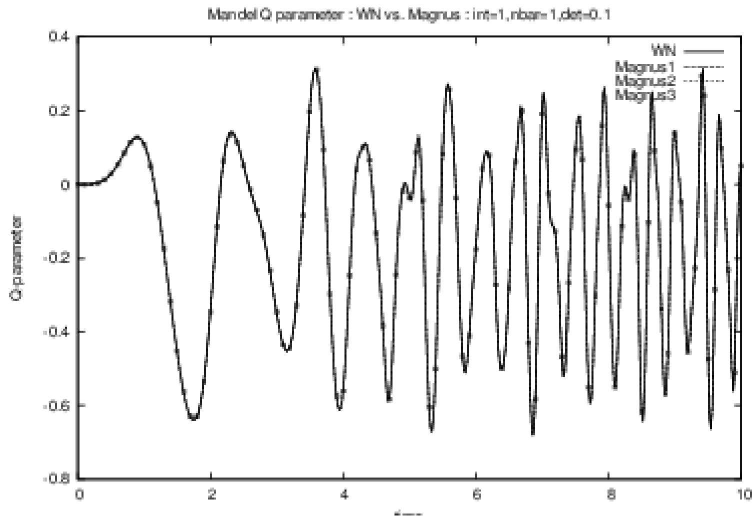

Figure 26: Sweep : Comparing WN and Magnus : Variation of Mandel Q parameter.

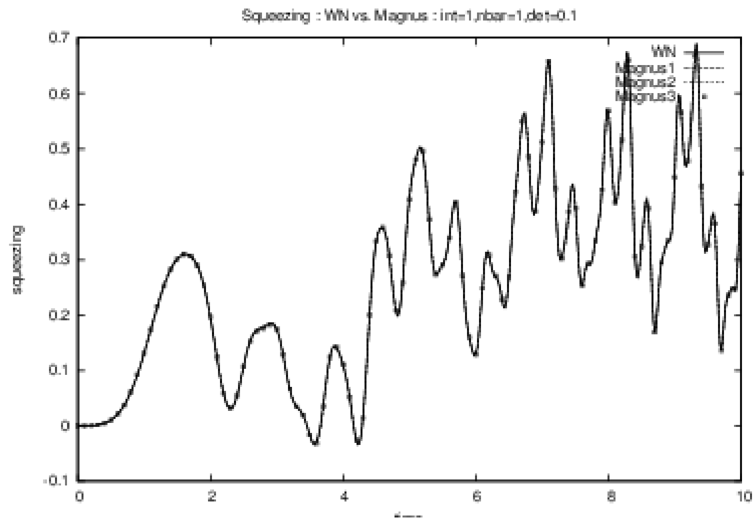

Figure 27: Sweep : Comparing WN and Magnus : Variation of squeezing.

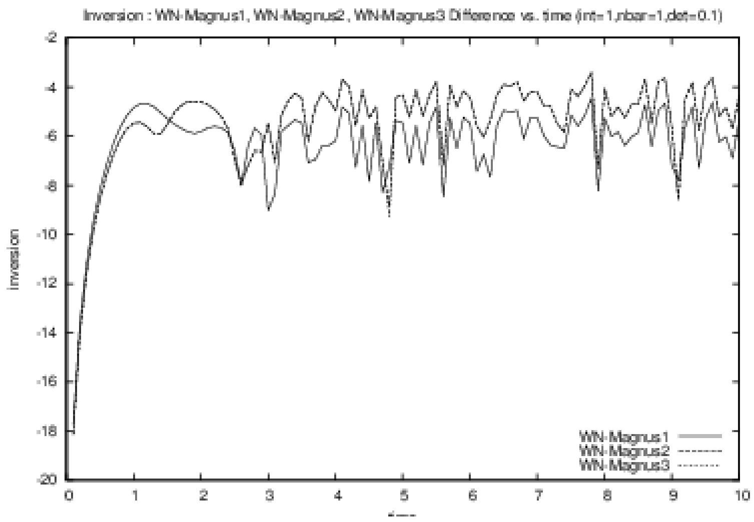

Figure 28: Sweep : Comparing WN and Magnus : Variation of inversion : Lorgarithmic Difference.

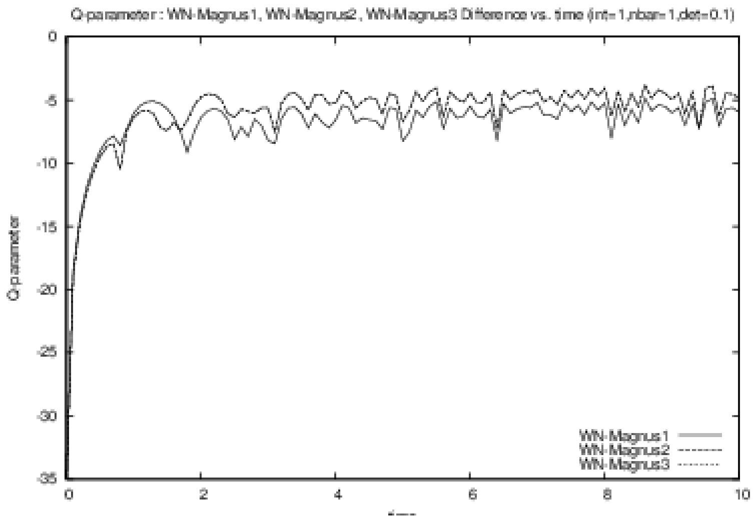

Figure 29: Sweep : Comparing WN and Magnus : Variation of Mandel Q parameter : Lorgarithmic Difference.

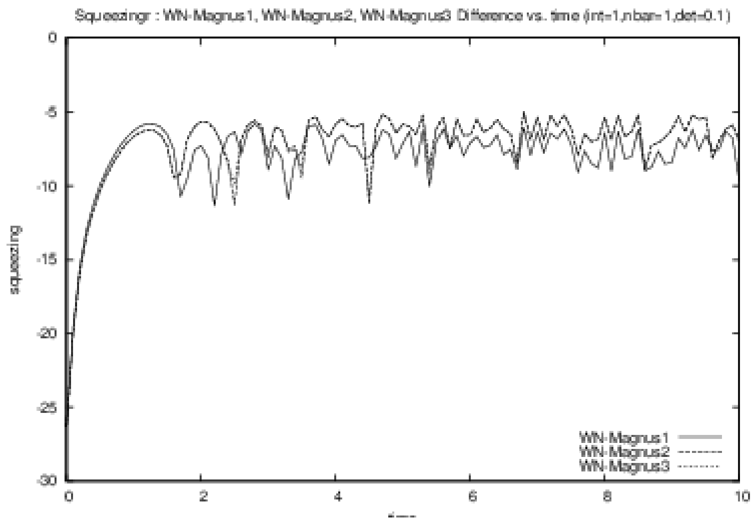

Figure 30: Sweep : Comparing WN and Magnus : Variation of squeezing : Lorgarithmic Difference.

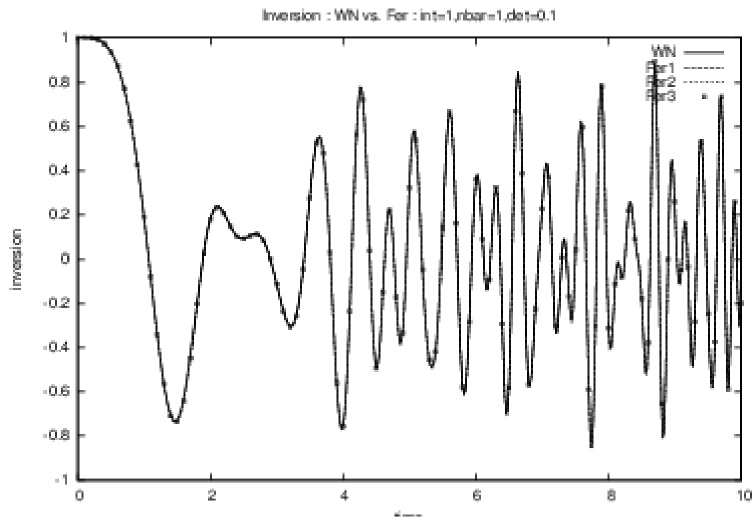

Figure 31: Sweep : Comparing WN and Fer : Variation of inversion.

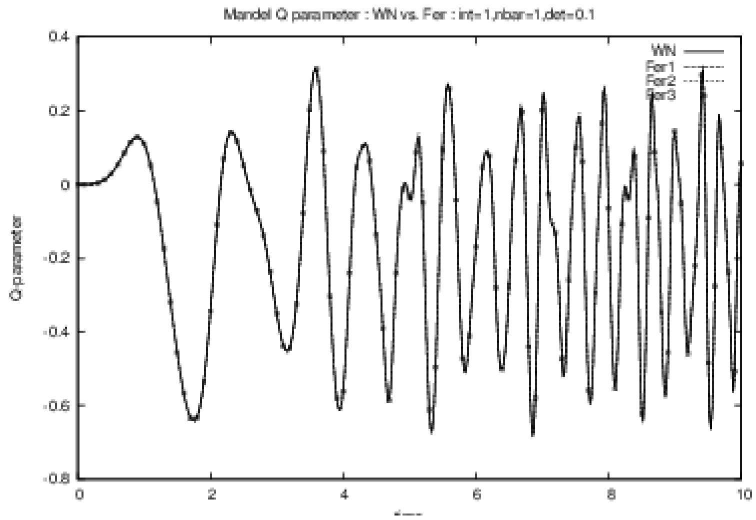

Figure 32: Sweep : Comparing WN and Fer : Variation of Mandel Q parameter.

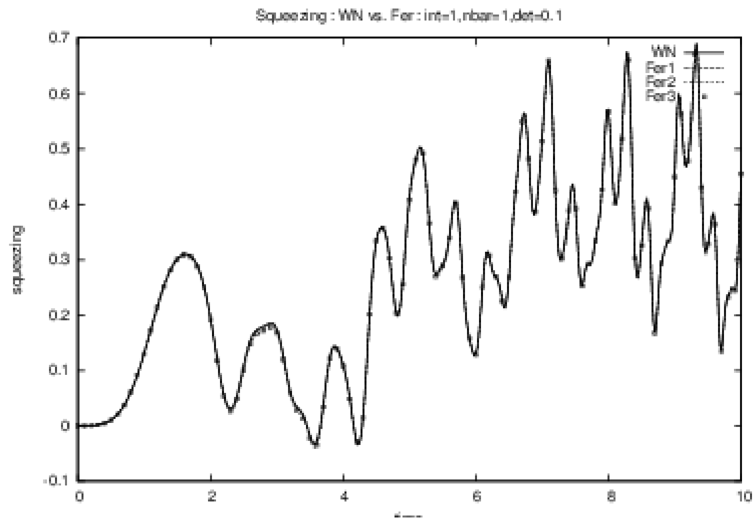

Figure 33: Sweep : Comparing WN and Fer : Variation of squeezing.

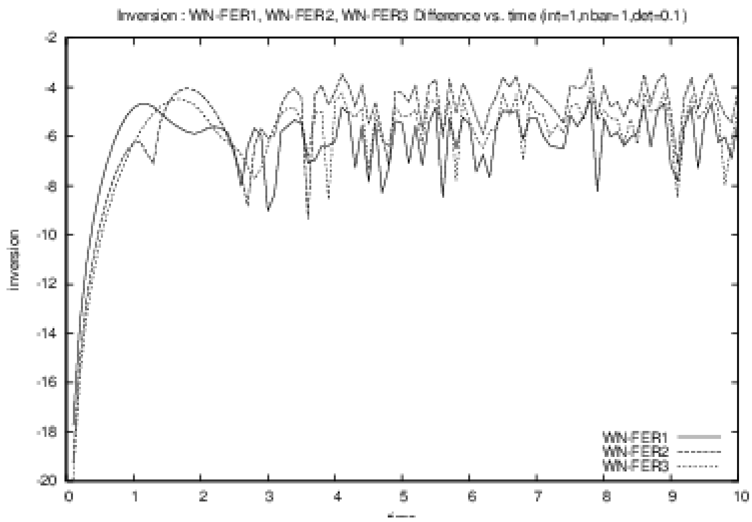

Figure 34: Sweep : Comparing WN and Fer : Variation of inversion : Lorgarithmic Difference.

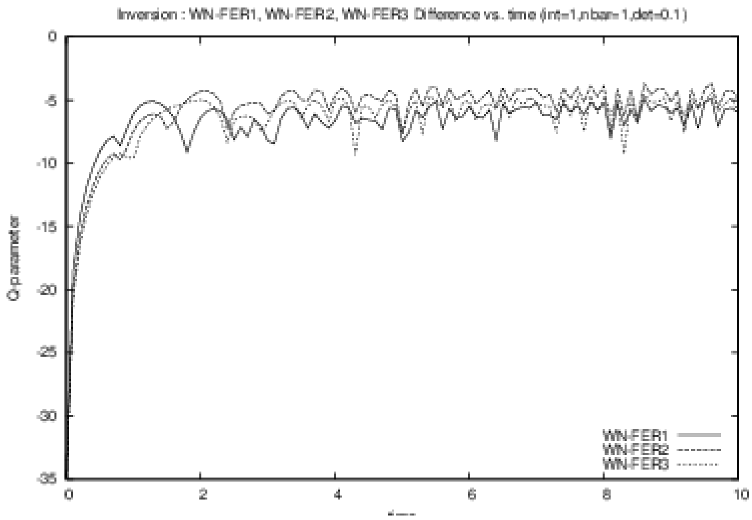

Figure 35: Sweep : Comparing WN and Fer : Variation of Mandel Q parameter : Lorgarithmic Difference.

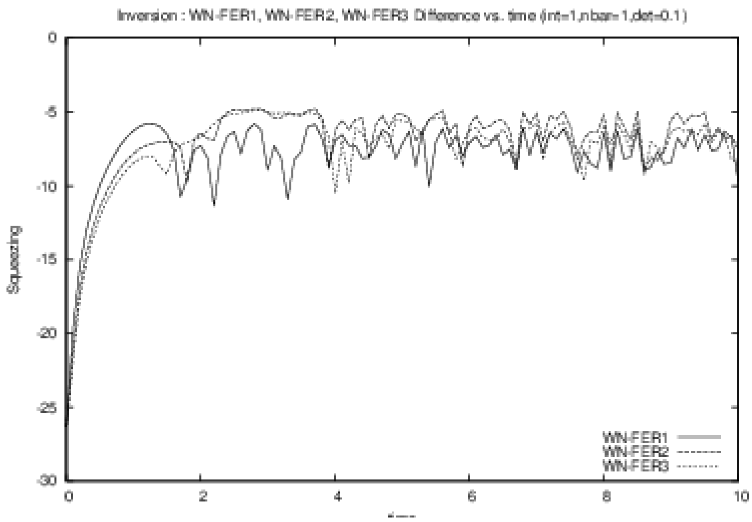

Figure 36: Sweep : Comparing WN and Fer : Variation of squeezing : Lorgarithmic Difference.

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Appendix : List of integrals

$$I_1(t) = \int_0^t dt_1 \cos(\beta t_1^2, \qquad I_2(t) = \int_0^t dt_1 \sin(\beta t_1^2),$$
$$I_3(t) = \int_0^t dt_1 \cos(\beta t_1^2) I_1(t_1), \qquad I_4(t) = \int_0^t dt_1 \cos(\beta t_1^2) I_2(t_1),$$
$$I_5(t) = \int_0^t dt_1 \sin(\beta t_1^2) I_1(t_1), \qquad I_6(t) = \int_0^t dt_1 \sin(\beta t_1^2) I_2(t_1),$$
$$I_7(t) = \int_0^t dt_1 \cos(\beta t_1^2) I_3(t_1), \qquad I_8(t) = \int_0^t dt_1 \cos(\beta t_1^2) I_4(t_1),$$
$$I_9(t) = \int_0^t dt_1 \cos(\beta t_1^2) I_5(t_1), \qquad I_{10}(t) = \int_0^t dt_1 \cos(\beta t_1^2) I_6(t_1),$$
$$I_{11}(t) = \int_0^t dt_1 \sin(\beta t_1^2) I_3(t_1), \qquad I_{12}(t) = \int_0^t dt_1 \sin(\beta t_1^2) I_4(t_1),$$
$$I_{13}(t) = \int_0^t dt_1 \sin(\beta t_1^2) I_5(t_1), \qquad I_{14}(t) = \int_0^t dt_1 \sin(\beta t_1^2) I_6(t_1)$$

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