



## Rosen-Chambers variation theory of linearly-damped classic and quantum oscillator

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### ABSTRACT

Phenomena of damped harmonic oscillator is important in the description of the elementary dissipative processes of linear responses in our physical world. Its classical description is clear and understood, however it is not so in the quantum physics, where it also has a basic role. Starting from the Rosen-Chambers restricted variation principle a Hamilton like variation approach to the damped harmonic oscillator will be given. The usual formalisms of classical mechanics, as Lagrangian, Hamiltonian, Poisson brackets, will be covered too. We shall introduce two Poisson brackets. The first one has only mathematical meaning and for the second, the so-called constitutive Poisson brackets, a physical interpretation will be presented. We shall show that only the fundamental constitutive Poisson brackets are not invariant throughout the motion of the damped oscillator, but these show a kind of universal time dependence in the universal time scale of the damped oscillator. The quantum mechanical Poisson brackets and commutation relations belonging to these fundamental time dependent classical brackets will be described. Our objective in this work is giving clearer view to the challenge of the dissipative quantum oscillator.

### Keywords

variation-theory, damped-oscillator, quantum-oscillator

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## INTRODUCTION

There has been an old challenge for a long time that the dissipation causes difficulties in the standard quantization of quantum-mechanical description of the damping harmonic oscillators, [1], [2], [3]. This problem had remained under intensive investigation [4], [5]. There are some widely accepted Hamilton like variation theories about the treatment of the linearly-dependent classic or quantum damped oscillator. One of these theories is the Bateman's mirror-image model [1], which consists of two different damped oscillators, where one of them represents the main one-dimensional linearly damped harmonic oscillator. The energy dissipated by the main oscillator will be absorbed by the other oscillator and thus the energy of the total system will be conserved. The commutation relations of this model in quantum theory are time independent, however, the time-dependent uncertainty products, obtained this way, vanish as time tends to infinity [6]. The Caldirola-Kanai theory with an explicit time dependent Hamiltonian is another kind of variation theory [7], [8], [9]. In the quantum version of this theory both the canonical commutation rules and the uncertainty products tend to zero as time tends to infinity. The system-plus-reservoir model ([10], [11]) is another standard model to investigate dissipation of the damped oscillator. In this model the main simple harmonic oscillator is coupled linearly to a fluctuating bath. If the bath is weakly perturbed by the system then it can be modelled with a continuous bath of the harmonic oscillator. A quantum Langevin equation in the Heisenberg picture can be deduced in this model. However, this equation in general does not obey [12], the Onsager's regression hypothesis, [13], but only in case when  $\hbar \rightarrow 0$ . A direct consequence of this fact is that the expected value of the fundamental observable does not satisfy the equation of the classic linearly-damped oscillator. Another consequence is that no spontaneous dissipative process exists.

Starting from the Rosen-Chambers restricted variation principle [14], [15], [16] a Hamilton like variation approach to the damped harmonic oscillator will be given. The usual formalisms of the classical mechanics, as the Lagrangian, the Hamiltonian and the Poisson brackets, will be covered too. We shall introduce two Poisson brackets. The first one has only mathematical meaning and for the second, a physical interpretation could be provided for the so-called constitutive Poisson brackets. We shall show that only the fundamental constitutive Poisson brackets are not invariant throughout the motion of the damped oscillator, but these show a kind of universal time dependence in the universal time scale of the damped oscillator. We will derive the quantum mechanical Poisson brackets and commutation relations belong to these fundamental time dependent classical brackets.

The commutation relations and the time-dependent uncertainty products obtained this way are evidently time dependent and vanish as time tends to infinity.

By means of canonical quantization we shall give the quantum canonical equations of the linearly-damped oscillator. According to the commutation relations we shall evaluate these canonical equations by algebraic methods. The resulted Heisenberg operator differential equation of the damped oscillator, which is consistent with the classical equation, will be solved by using ladder operators, which, in this theory, are time dependent. We shall give the actual form of the fundamental observables of the damped oscillator and the results will cause some irreversible problems, for example the natural width of the spectral line.

## ROSEN-CHAMBERS VARIATION PRINCIPLE

Rosen and Chambers formulated their variation principle for the deduction of the transport equations of the continuous non-equilibrium thermodynamic system [15], [14], [16]. We are going to adapt this so called *restricted variation principle* to the classical and quantum damped oscillator. The Rosen-Chambers variation principle is significantly different from the Lagrangian one since the classical Lagrange formalism cannot be applied for dissipative systems. So, we are going to use a mechanical model instead of a non-equilibrium thermodynamic model to describe the damped quantum oscillator.

### Lagrange formalism

In the next section we are going to give a restricted variation principle for a linear damped oscillator. Also, we are going to consider the equation of motion of

$$m \frac{d^2 u}{dt^2} + c \frac{du}{dt} + ku = 0, \quad (1)$$

where  $u$  is the displacement of the mass of oscillator from its equilibrium position, and the  $m$ ,  $c$ ,  $k$  physical parameters of the oscillator are positive constants. The equation (1) can be deduced from the following form of the Rosen-Chambers's variation principle:

$$\delta \int_0^t L \left( u, \frac{du}{dt}, \lambda \right) dt \Big|_{\lambda=u} = \delta \int_0^t \left[ -cu \frac{du}{dt} + \frac{1}{2} m \left( \frac{du}{dt} \right)^2 - \frac{1}{2} ku^2 + \lambda c \frac{du}{dt} \right] dt \quad (2)$$

By assuming the restrictions of  $\delta u|_{t=0} = \delta u|_{t=t} = 0$  and integrating them part by part, we get a simpler form of the restricted variation principle, namely



$$\delta \int_0^t L^* dt \Big|_{\lambda=u} = \delta \int_0^t \left( \frac{1}{2} m \left( \frac{du}{dt} \right)^2 - \frac{1}{2} k u^2 + \lambda c \frac{du}{dt} \right) dt \Big|_{\lambda=u} = 0. \quad (3)$$

We might express the equation (1) as an Euler-Lagrange equation, belonging to this restricted variation principle:

$$\frac{\partial L^*}{\partial u} \Big|_{\lambda=u} - \frac{d}{dt} \frac{\partial L^*}{\partial \frac{du}{dt}} \Big|_{\lambda=u} = - \left( m \frac{d^2 u}{dt^2} + c \frac{du}{dt} + k u \right) = 0. \quad (4)$$

## Canonical formalism

We have seen that it is possible to give a Hamilton type restricted variation principle that is equivalent to the equations of motions of the damped oscillator.

Next, we are going to give the canonical formalism of this restricted variation principle. To do it, let us start from the Lagrange density of variation principle (3) and introduce the

$$\pi \left( u, \frac{du}{dt}, \lambda \right) := \frac{\partial L^*}{\partial \frac{du}{dt}} = m \frac{du}{dt} + \lambda c \quad (5)$$

generalised momentum. Now, we can see that

$$\pi \Big|_{\lambda=u} = m \frac{du}{dt} + c u. \quad (6)$$

The Hamiltonian density defined by the Legendre transformation is

$$H(u, \pi, \lambda) := \pi \frac{du}{dt} - L^* = \frac{1}{2m} (\pi - \lambda c)^2 + \frac{1}{2} k u^2. \quad (7)$$

It is easy to see that  $H \Big|_{\lambda=u}$  is equal to the instantaneous free energy of the oscillator.

With this Hamiltonian density, the variation principle (4) may be written in another form:

$$\begin{aligned} \delta \int_0^t L^* \left( u, \frac{du}{dt}, \lambda \right) dt \Big|_{\lambda=u} &= \delta \int_0^t \left( \pi \frac{du}{dt} - H \right) dt \Big|_{\lambda=u} = \\ &= \int_0^t \left[ \left( \frac{du}{dt} - \frac{\partial H}{\partial \pi} \right) \Big|_{\lambda=u} \delta \pi - \left( \frac{d\pi}{dt} + \frac{\partial H}{\partial \pi} \right) \Big|_{\lambda=u} \delta u \right] dt = 0 \end{aligned} \quad (8)$$

from which we get the canonical equations of damped oscillator:

$$\frac{d\pi}{dt} \Big|_{\lambda=u} = - \frac{\partial H}{\partial u} \Big|_{\lambda=u} \quad (9)$$

and

$$\frac{du}{dt} \Big|_{\lambda=u} = \frac{\partial H}{\partial \pi} \Big|_{\lambda=u}. \quad (10)$$

Applying these equations to the Hamiltonian (7), we obtain

$$\frac{d\pi}{dt} = -k u, \quad (11)$$



$$\frac{du}{dt} = \frac{\pi - uc}{m}, \tag{12}$$

which are equivalent to equation (1).

### Poisson formulation

To proceed towards the quantum theory of damped oscillator we must introduce the Poisson brackets. Let  $f$  and  $g$  be arbitrary differentiable functions of the variables  $u, \pi, \lambda$ . The restricted Poisson bracket of two observables  $f$  and  $g$  are defined as

$$(f, g) = \left( \frac{\partial f}{\partial u} \frac{\partial g}{\partial \pi} - \frac{\partial f}{\partial \pi} \frac{\partial g}{\partial u} \right)_{\lambda=u}, \tag{13}$$

which is a minor generalization of the form well-known from classical mechanics [17]. Now, the (9) and (10) and canonical equations of damped oscillator can be formulated by the Poisson brackets as

$$\frac{du}{dt} = (u, H), \quad \frac{d\pi}{dt} = (\pi, H) \tag{14}$$

replacing the derivatives of the Hamiltonian by the following Poisson brackets

$$\frac{\partial H}{\partial \pi} = (u, H), \quad \frac{\partial H}{\partial u} = -(\pi, H). \tag{15}$$

It is easy to see that the restricted Poisson bracket defined this way has the same algebraic properties as the classical one [17], i.e. it is asymmetric and linear in both variables, besides, they are equal to zero in case of a constant coefficient and fulfil the Jacobi identity. From this fact it follows that the infinite-dimensional vector space of functions (observables) on phase space of the damped oscillator, equipped with the Poisson bracket as a generalized „product” is the Lie algebra [18]. Because of this feature of Poisson formulation, the above equation can be evaluated by the Lie algebraic method, if we take into account the following fundamental brackets

$$(u, u) = (\pi, \pi) = (u, \lambda) = (\pi, \lambda) = 0, \quad (\pi, u) = 1, \tag{16}$$

which can be easily derived by Poisson bracket defined in equation (13) and the equations of motion (11) and (12).

This way only algebraic operations take the place of derivatives and the canonical equations (14) can be evaluated by algebraic method taking into account the fundamental brackets (16). Beside the restricted Poisson bracket, defined in equation (13), the damped oscillator has another, so called constitutive Poisson bracket to. The relation between these Poisson brackets, as it can be shown in 6.1, has the form:

$$(u, u)_K = (\pi, \pi)_K = (u, \lambda)_K = (\pi, \lambda)_K = 0, \quad (\pi, u)_K = (\pi, u) e^{-2\beta t} = e^{-2\beta t}, \quad \beta = \frac{c}{2m}, \tag{17}$$

Additionally, the constitutive Poisson bracket unlike the Poisson bracket (13) is not a remaining constant throughout the motion of the damped oscillator, but it is a universal time function in the natural time scale  $\beta t$  of the oscillator (for proof see A.1 Using the Poisson bracket, the time derivative of an observable  $f(u, \pi, \lambda, t)$  can be expressed as

$$\left. \frac{df}{dt} \right|_{\lambda=u} = \frac{\partial f}{\partial t} + (f, H) + \left( \frac{\partial f}{\partial \lambda} \right)_{\lambda=u} \frac{du}{dt}. \tag{18}$$

By applying this expression to the Hamilton function – not including the time explicitly – we get the energy law of the oscillator, from which follows the expression of the rate of energy dissipation of the damped oscillator follows, i. e.

$$D = -\left. \frac{dH}{dt} \right|_{\lambda=u} = c \left( \frac{du}{dt} \right)^2. \tag{19}$$

An observable – which does not include the  $\lambda$  variable explicitly – is the first integral of the damped oscillator if

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (f, H) = 0. \tag{20}$$

An observable – including neither the time  $t$  nor the  $\lambda$  variables explicitly – is a constant of motion if





$$\frac{df}{dt} = (f, H) = 0. \quad (21)$$

Now let us give the first integral for the damped harmonic oscillator. It was Bohlin, who first dealt with the problem of the first integral of the damped linear oscillator [19], [20]. Detailed description on this subject can be found in [21]. It is easy to prove that the Bohlin's observable, defined as

$$F = \frac{m}{2} e^{2\beta t} \left( \frac{du}{dt} - \gamma u \right) \left( \frac{du}{dt} - \gamma^* u \right), \quad (22)$$

where

$$\gamma = -\beta + i\omega, \quad \gamma^* = -\beta - i\omega, \quad \omega = \sqrt{\omega_0^2 - \beta^2}, \quad \omega_0 = \sqrt{\frac{k}{m}} \quad (23)$$

is the first integral of the damped oscillator. Now, it may be seen that the Bohlin's observable is equal to the maxima of the Hamiltonian (free energy) of the damped oscillator and in case of the undamped oscillator it is equal to the Hamiltonian.

## CANONICAL QUANTIZATION OF DAMPED HARMONIC OSCILLATOR

Next, we are going to follow the Dirac's method. In Dirac's quantum mechanics, a physical state of a damped oscillator is represented by a vector in an abstract vector space (in the so-called ket space), which is known as the Hilbert space of quantum states. The dynamical variables (observables), i.e.,  $u, \pi, H, D, F$ , etc., are represented by operators that act linearly on the Hilbert space of quantum states. The procedure of canonical quantization of damped oscillator consists that to the dynamical variables  $u, \pi, H$  of the canonical equations of classical damped oscillator (14), we associate the operators  $\mathbf{u}, \boldsymbol{\pi}, \mathbf{H}$ , which act on the Hilbert space of quantum states, and we replace the classical Poisson brackets by quantum Poisson brackets. In case of damped harmonic oscillator, the main problem of the classical Poisson brackets is that they are not invariant throughout the motion. Fortunately, they show universal time dependence (in the natural time scale of the damped oscillator) throughout the motion of the damped oscillator.

Next, we shall construct a time dependent quantum Poisson bracket, in which universal time dependence will be included.

### Quantum Poisson bracket (QPB)

To give the quantum Poisson bracket, we proceed from the main properties of the classical restricted Poisson brackets (RPB) which are as follows:

$$\begin{aligned} \text{Antisymmetry} \quad & (f, g)_K = -(g, f)_K, \\ (f, c)_K &= 0, \\ \text{Bilinearity} \quad & (af_1 + bf_2, g)_K = a(f_1, g)_K + b(f_2, g)_K, \quad (f, ag_1 + bg_2)_K = a(f, g_1)_K + b(f, g_2)_K, \\ \text{Product rules} \quad & (f_1 f_2, g)_K = (f_1, g)_K f_2 + f_1 (f_2, g)_K, \quad (f, g_1 g_2)_K = (f, g_1)_K g_2 + g_1 (f, g_2)_K, \\ \text{Jacobi identity} \quad & (f, (g, h))_K + (g, (h, f))_K + (h, (f, g))_K = 0, \end{aligned} \quad (24)$$

where  $a, b$  and  $c$  are numbers or pure time functions. The product rules of restricted Poisson brackets are simple consequences of the Leibnitz's product rule of derivation. In case of RPB the product of observables is commutative. We shall formulate the equations of quantum damped oscillator in terms of the Poisson brackets as Dirac did.. To do so, we accept the above mentioned main relations of brackets, but we use another definition of brackets, since in quantum mechanics the observables are non-commuting linear Hermitian operators acting in a ket space which represent all the possible states of the damped oscillator. The above mentioned "number like" behavior of pure time function in quantum mechanics can be based on the following conclusion of W. Pauli, to which J. Hilgevoord called the attention of Physicists [22]: "...the introduction of an operator  $t$  is basically forbidden and the time must necessarily be considered as an ordinary number ("c-number")..." [23], [24], [25]. In the following we are going to show that some combinations of non-commuting operators exist which satisfy all the above relations (28). To give the actual form of this quantum Poisson bracket, we shall evaluate the quantum bracket  $(f_1 f_2, g_1 g_2)_Q$  in two different ways using either of the first or second product rule of (20) first, then we obtain



$$\begin{aligned}
 (f_1 f_2, g_1 g_2)_Q &= (f_1, g_1 g_2)_Q f_2 + f_1 (f_2, g_1 g_2)_Q = \\
 &= [(f_1, g_1)_Q g_2 + g_1 (f_1, g_2)_Q] f_2 + f_1 [(f_2, g_1)_Q g_2 + g_1 (f_2, g_2)_Q] = \\
 &= (f_1, g_1)_Q g_2 f_2 + g_1 (f_1, g_2)_Q f_2 + f_1 (f_2, g_1)_Q g_2 + f_1 g_1 (f_2, g_2)_Q \quad , \\
 (f_1 f_2, g_1 g_2)_Q &= (f_1 f_2, g_1)_Q g_2 + g_1 (f_1 f_2, g_2)_Q = \\
 &= (f_1, g_1)_Q f_2 g_2 + g_1 (f_1, g_2)_Q f_2 + f_1 (f_2, g_1)_Q g_2 + g_1 f_1 (f_2, g_2)_Q
 \end{aligned}
 \tag{25}$$

where the order of various observables has been preserved, since they are now non-commuting operators. Equating the above two expressions, we obtain [26], [27].

$$(f_1, g_1)_Q (f_2 g_2 - g_2 f_2) = (f_1 g_1 - g_1 f_1) (f_2 g_2)_Q \tag{26}$$

Next, we shall give another solution of this function equation then the standard one.

Since this relation must hold for every  $f_1$  and  $g_1$  quite independent of  $f_2$  and  $g_2$ , it follows that

$$(f_1 g_1 - g_1 f_1) = \frac{i\hbar}{g(t)} (f_1, g_1)_Q, \quad (f_2 g_2 - g_2 f_2) = \frac{i\hbar}{g(t)} (f_2, g_2)_Q, \tag{27}$$

where the function  $\frac{\hbar}{g(t)}$  does not depend on the operators  $f_1, f_2, g_1, g_2$  and also commute with  $(f_1 g_1 - g_1 f_1)$ .

Since  $f_1, f_2, g_1, g_2$  are arbitrary it follows that  $\frac{\hbar}{g(t)}$  is a pure time function. If we assume that the QPB of two Hermitian operators are Hermitian operators themselves, then  $\hbar$  must be a real number and  $g(t)$  a real function. Here  $\hbar$  is Dirac's notation for  $\frac{h}{2\pi}$ , where  $h$  is the Planck's constant. The actual form of the function  $g(t)$  could be originated from the Bolin's first integral of the quantum damped oscillator. If introduce the Dirac's notation  $[f, g]$  for the commutator  $fg - gf$ , then the QPB is as follows

$$(f, g)_Q = \frac{g(t)}{i\hbar} [f, g]. \tag{28}$$

It is easily shown that this new form of Poisson brackets satisfies all the previous operational rules (28) of the Poisson brackets including the Jacobi identity. If  $g(t) = 1$ , then we get the standard QPB introduced by Dirac in quantum theory of reversible system [28]. In our case the actual form of the QPB is determined by the constitution of the system, therefore the QPB (28) is rather a constitutive law than a universal one. In the classical theory the fundamental bracket equations (16) and (17) are simple consequences of the classical RPBs, defined in equation (13). In Dirac's theory these are absolutely essential, because they express the commutation rules for the quantum mechanical observables  $\mathbf{u}$  and  $\boldsymbol{\pi}$ . By the above found strong analogy between classical and quantum mechanical Poisson brackets, we may make the assumption that in the case of fundamental brackets the QPBs have the same values or pure time function as the corresponding classical PBs (17). From this assumption it follows that the fundamental quantum mechanical commutation relations are

$$[\mathbf{u}, \mathbf{u}] = [\boldsymbol{\pi}, \boldsymbol{\pi}] = [\mathbf{u}, \boldsymbol{\lambda}] = [\boldsymbol{\pi}, \boldsymbol{\lambda}] = 0, \quad [\boldsymbol{\pi}, \mathbf{u}] = \frac{i\hbar}{g(t)e^{2\beta t}} \boldsymbol{\delta}. \tag{29}$$

where from here the bold letter refers to the operator and  $\boldsymbol{\delta}$  is the unit operator. If there were  $g(t)e^{2\beta t} = 1$ , then the last expression of (29) would transform to the Heisenberg's fundamental commutation relation.

### Canonical equations of quantum damped oscillator

As it has already been mentioned above, in Quantum Mechanics, a physical state of a damped oscillator is represented by a vector in an abstract vector space (ket space), which is called the Hilbert space of quantum states. The dynamical variables (observables),  $\mathbf{u}, \boldsymbol{\pi}, \mathbf{H}, \mathbf{F}$ , etc., are represented by the Hermitian operators that act linearly on the Hilbert space of quantum states. According to Dirac formulation, the procedure of canonical quantization of damped oscillator consists of the procedure of canonical quantization dynamical variables  $\mathbf{u}, \boldsymbol{\pi}, \boldsymbol{\lambda}, \mathbf{H}$  of canonical equations of the classical damped oscillator (14). We associate the Hermitian operators  $\mathbf{u}, \boldsymbol{\pi}, \boldsymbol{\lambda}, \mathbf{H}$  be acting in the Hilbert space on quantum



states; and we replace the classical Poisson brackets by quantum Poisson brackets. Formally, this procedure can be written

$$\left. \begin{aligned} \frac{du}{dt} &= e^{2\beta t} (u, H)_K \\ \frac{d\pi}{dt} &= e^{2\beta t} (\pi, H)_K \end{aligned} \right\} \xrightarrow{\text{Canonical quantization}} \left\{ \begin{aligned} \frac{d\mathbf{u}}{dt} &= e^{2\beta t} (\mathbf{u}, \mathbf{H})_Q \\ \frac{d\boldsymbol{\pi}}{dt} &= e^{2\beta t} (\boldsymbol{\pi}, \mathbf{H})_Q \end{aligned} \right. \quad (30)$$

In terms of fundamental brackets (16), (17), and (29) this procedure means that with the classical canonical pairs  $(u, u), (u, \pi), (\pi, \pi), \text{etc.}$ , that satisfy the fundamental brackets (17) we associate pairs of operators  $\mathbf{u}, \boldsymbol{\pi}, \text{etc.}$ , respectively, which act on the ket space of quantum states, and are required to obey the canonical commutation relations (29).

In terms of commutation relations the canonical equations of the quantum oscillator have the form

$$\frac{d\mathbf{u}}{dt} = \frac{1}{i\hbar} e^{2\beta t} g(t) [\mathbf{u}, \mathbf{H}], \quad \frac{d\boldsymbol{\pi}}{dt} = \frac{1}{i\hbar} e^{2\beta t} g(t) [\boldsymbol{\pi}, \mathbf{H}], \quad (31)$$

where from (7) the Hamiltonian is

$$\mathbf{H}(\mathbf{u}, \boldsymbol{\pi}, \lambda) := \frac{1}{2m} (\boldsymbol{\pi} - \lambda c)^2 + \frac{1}{2} k \mathbf{u}^2 \quad (32)$$

and the commutator takes the form of

$$[\mathbf{f}, \mathbf{H}] := \mathbf{fH} - \mathbf{Hf} \Big|_{\lambda=\mathbf{u}} \quad (33)$$

It is easy to show that from the canonical equations we get the

$$\frac{d\boldsymbol{\pi}}{dt} = -k \mathbf{u} \quad (34)$$

and the

$$\frac{d\mathbf{u}}{dt} = \frac{\boldsymbol{\pi} - \mathbf{u}c}{m} \quad (35)$$

equations by using (33) and the fundamental commutation relations (29).

From (18) and the quantum theory from the above mentioned we found that any observable  $\mathbf{G}(\mathbf{u}, \boldsymbol{\pi})$  of damped oscillator obeys

$$\frac{d\mathbf{G}}{dt} = \frac{1}{i\hbar} e^{2\beta t} g(t) [\mathbf{G}, \mathbf{H}]. \quad (36)$$

## EVALUATION OF THE EQUATION OF MOTION OF THE QUANTUM DAMPED OSCILLATOR

From the canonical equations (34) and (35) we get that

$$m \frac{d^2 \mathbf{u}}{dt^2} + c \frac{d\mathbf{u}}{dt} + k \mathbf{u} = 0. \quad (37)$$

This is an operator differential equation which corresponds formally to the equation of classic damped oscillator. The solution of equation is formally identical with that of the classical one, namely

$$\mathbf{u} = \mathbf{a} e^{-\frac{c}{2m}t} e^{i\omega t} + \mathbf{b} e^{-\frac{c}{2m}t} e^{-i\omega t}. \quad (38)$$

If we require that the physical quantities of oscillator shall be hermetic then we get from the  $\mathbf{u} = \mathbf{u}^\dagger$  condition that



$$\mathbf{u} = \mathbf{a} e^{-\frac{c}{2m}t} e^{i\omega t} + \mathbf{a}^+ e^{-\frac{c}{2m}t} e^{-i\omega t}, \tag{39}$$

where the cross symbol denotes the simultaneous transposing and conjugation. In the interest of the better similarity to the simple harmonic oscillator, we are going to use the above equation in the

$$\mathbf{u} = \sqrt{\frac{\hbar}{2m\omega_0}} (\mathbf{A} e^{i\omega t} + \mathbf{A}^+ e^{-i\omega t}), \quad \mathbf{A} = \mathbf{a} e^{-\frac{c}{2m}t} \tag{40}$$

equivalent form. Using (6) the generalised momentum operator can be expressed as

$$\boldsymbol{\pi} = \sqrt{\frac{\hbar m}{2\omega_0}} \left( \left( \gamma + \frac{c}{m} \right) \mathbf{A} e^{i\omega t} + \left( \gamma^* + \frac{c}{m} \right) \mathbf{A}^+ e^{-i\omega t} \right). \tag{41}$$

By substituting the above two expressions into the commutation relations (29), we get the following

$$[\mathbf{A}, \mathbf{A}^+] = \frac{1}{g(t)e^{2\beta t}} \delta \tag{42}$$

commutation relation for the operator  $\mathbf{A}$ .

To solve the damped oscillator problem we have to determine the operator  $\mathbf{A}$ , because this should be known for the specification of displacement impulse and the energy of the oscillator. The non-damped  $\mathbf{A}$  can be determined from the Hamiltonian of the oscillator, which is a constant of the motion. This is, however, not true for our case, thus we are going to use the Bohlin's first integral introduced earlier. For this, we have to rewrite it into the language of quantum mechanics. The operator belonging to the Bohlin's first integral shall be hermetic, because in this case the eigenvalues are real. It is to be expected that the Bohlin's first integral operator shall be free of dispersion. From this it follows that the matrix belonging to Bohlin's first integral operator is diagonal. From this diagonal condition we are able to specify the matrix belonging to  $\mathbf{A}$ .

By rewriting the Bohlin's operator (further on: exergy operator) into the language of quantum mechanics, we get

$$\begin{aligned} \mathbf{F} &= \frac{m}{4} e^{2\beta t} \left[ \frac{\left( \frac{d\mathbf{u}}{dt} - \gamma \mathbf{u} \right) \left( \frac{d\mathbf{u}}{dt} - \gamma^* \mathbf{u} \right)}{2} + \frac{\left( \frac{d\mathbf{u}}{dt} - \gamma^* \mathbf{u} \right) \left( \frac{d\mathbf{u}}{dt} - \gamma \mathbf{u} \right)}{2} \right] = \tag{43} \\ &= \frac{\hbar \omega_m}{2} e^{2\beta t} (\mathbf{A} \mathbf{A}^+ + \mathbf{A}^+ \mathbf{A}) = \hbar \omega_m e^{2\beta t} \left( \mathbf{A}^+ \mathbf{A} + \frac{1}{2e^{2\beta t} g(t)} \right), \quad \omega_m = \omega^2 \omega_0^{-1} \end{aligned}$$

where we took into account the (42) commutation relation. This operator must be independent of time. From this we get the pure time function of the QPB (28) and commutation relation (29) that  $g(t) = 1$ , and by this the exergy operator of quantum oscillator is

$$\mathbf{F} = \hbar \omega_m \left( \mathbf{a}^+ \mathbf{a} + \frac{1}{2} \right), \tag{44}$$

where  $\mathbf{a} = \mathbf{A}(t = 0)$  is the initial amplitude operator which according to (42) it satisfies the

$$[\mathbf{a}^+, \mathbf{a}] = 1 \tag{45}$$

commutation relation. Pursuant to the two relations above it is easy to show that

$$[\mathbf{F}, \mathbf{a}^+] = \hbar \omega_m \mathbf{a}^+, \quad [\mathbf{a}, \mathbf{F}] = \hbar \omega_m \mathbf{a}. \tag{46}$$

Now, we see that if we replace the operator  $\mathbf{F}$  by the Hamiltonian  $\mathbf{H}$  of simple oscillator, these equations are identical with the corresponding equations of the simple quantum oscillator [29], [30]. According to this strong analogy, we are able to determine the amplitude matrix and the matrices of the exergy operator  $\mathbf{F}$ , the displacement operator, and the momentum operator. The results are as follows:





- Since  $\mathbf{a}^+ \mathbf{a}$  is positive definite, therefore  $\mathbf{F}$  can possess no negative eigenvalue. The lowest  $F_0$  exergy eigenvalue belongs to the eigenket  $|0\rangle$  of the operator  $\mathbf{a}$  for which the relation  $\mathbf{a}|0\rangle = 0$  holds. From (44) to this so-called vacuum state belonging to  $\frac{1}{2}\hbar\omega_m$  zero-point exergy, and the  $F_n$  exergy eigenvalue belonging to the eigenket  $|n\rangle$ , can be calculated in the form of  $|n\rangle = \frac{\mathbf{a}^{+n}}{\sqrt{n!}}|0\rangle$ , is  $\left(\frac{1}{2} + n\right)\hbar\omega_m$ . For the actions of the energy eigenket  $|n\rangle$  of the ladder operators  $\mathbf{a}^+$  and  $\mathbf{a}$  can be written as

$$\mathbf{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \mathbf{a}|n\rangle = \sqrt{n-1}|n-1\rangle. \tag{47}$$

Also  $\mathbf{a}^+$  is a creation and  $\mathbf{a}$  is an annihilation operator.

From these properties of ladder operators it follows that the energy eigenkets are eigenkets of the occupation number operator  $\mathbf{N} = \mathbf{a}^+ \mathbf{a}$ , which is a positive definite operator having positive eigenvalues, i.e.

$$\mathbf{N}|n\rangle = n|n\rangle. \tag{48}$$

- The matrices belonging to the above mentioned operators are

$$\mathbf{a} = \begin{bmatrix} 0 & \sqrt{1} & 0 & \dots \\ 0 & 0 & \sqrt{2} & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}, \tag{49}$$

$$\mathbf{N} = \mathbf{a} \mathbf{a}^+ = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 2 & 0 & \dots \\ 0 & 0 & 3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{50}$$

The exergy energy eigenkets are

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \dots \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \dots \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \dots \end{bmatrix}, \text{etc.,.} \tag{51}$$

It is easy to see that the  $\mathbf{a} \mathbf{a}^+$  and the  $\mathbf{a}^+ \mathbf{a}$  matrices are not diagonal. From the above equations it follows that the rules for time dependent ladder and occupation number operators are

$$\begin{aligned} \mathbf{A}^+(t)|n\rangle &= e^{-\beta t} \sqrt{n+1}|n+1\rangle, & \mathbf{A}(t)|n\rangle &= e^{-\beta t} \sqrt{n-1}|n-1\rangle, \\ \mathbf{N}(t)|n\rangle &= e^{-2\beta t} n|n\rangle. \end{aligned} \tag{52}$$

Also, the eigenvalue of these operators are time dependent. From the theory of quantum-mechanical measurement upon a measurement of the observable  $\mathbf{G}$  of the system in a state  $|\Psi\rangle$  the outcome can only be one of the eigenvalues  $g_n, (i = 1, 2, 3, \dots)$  of the observable  $\mathbf{G}$  and the probability for that event to occur is  $\langle G_n | \Psi \rangle^2$ . Here  $|G_n\rangle$  is the n-th eigenket of  $\mathbf{G}$  and  $\langle G_n | \Psi \rangle$  is the probability amplitude of that event. Consequently, in the case of damped oscillator the measuring values are time dependent.



### Expected value of the main operators of damped oscillator

Moreover, the expected value of occupation number in  $n$ -th energy eigenstate at time  $t$  is

$$\overline{N(t)} := \langle n | N(t) | n \rangle = N_0 e^{-2\beta t}, \tag{53}$$

where  $N_0$  is the occupation number at  $t = 0$ . This result agrees well with the corresponding result derived from the system-reservoir models [31]. Because in the energy representation, the matrices of the operators  $\mathbf{a}, \mathbf{a}^+, \mathbf{a}^2, \mathbf{a}^{+2}$  have zero diagonal elements, the expected values of the operators  $\mathbf{A}, \mathbf{A}^+, \mathbf{A}^2, \mathbf{A}^{+2}$  are zero in every energy eigenstate, i.e.

$$\overline{\mathbf{A}} = 0, \overline{\mathbf{A}^+} = 0, \overline{\mathbf{A}^2} = 0, \overline{\mathbf{A}^{+2}} = 0. \tag{54}$$

According to these equations the expected values  $\overline{\mathbf{u}} =: \langle n | \mathbf{u} | n \rangle, \overline{\mathbf{p}} =: \langle n | \mathbf{p} | n \rangle$  of the displacement and the Newtonian momentum operator

$$\mathbf{p} = \sqrt{\frac{\hbar m}{2\omega_0}} (\gamma \mathbf{A} e^{-i\omega t} + \gamma^* \mathbf{A}^+ e^{i\omega t}) \tag{55}$$

in the  $n$ -th energy eigenstate are zero.

The variance  $\overline{\mathbf{u}^2} = \langle n | \mathbf{u}^2 | n \rangle, \overline{\mathbf{p}^2} = \langle n | \mathbf{p}^2 | n \rangle$  of the displacement and momentum operator in the  $n$ -th energy eigenstate can be evaluated as

$$\begin{aligned} \overline{\mathbf{u}^2} &= \frac{\hbar}{2m\omega_0} (\overline{\mathbf{A}\mathbf{A}^+ + \mathbf{A}^+\mathbf{A}}) = \frac{\hbar}{2m\omega_0} (\overline{\delta e^{-2\beta t} + 2\mathbf{A}\mathbf{A}^+}) = \\ &= \frac{\hbar}{2m\omega_0} e^{-2\beta t} (1 + 2n) \end{aligned} \tag{56}$$

$$\begin{aligned} \overline{\mathbf{p}^2} &= \frac{\hbar m}{2\omega_0} (\overline{\gamma\gamma^* \mathbf{A}\mathbf{A}^+ + \gamma\gamma^* \mathbf{A}^+\mathbf{A}}) = \frac{m\hbar\omega_0}{2} (\overline{\delta e^{-2\beta t} + 2\mathbf{A}\mathbf{A}^+}) = \\ &= m\hbar\omega_0 e^{-2\beta t} \left( \frac{1}{2} + n \right), \end{aligned} \tag{57}$$

where we considered the commutation relation (42).

According to these results we obtain the expected value of energy

$$\begin{aligned} \overline{H} &= \frac{\overline{\mathbf{p}^2}}{2m} + m\omega_0^2 \frac{\overline{\mathbf{u}^2}}{2} = \hbar\omega_0 (\overline{\delta e^{-2\beta t} + 2\mathbf{A}\mathbf{A}^+}) = \\ &= \hbar\omega_0 e^{-2\beta t} \left( n + \frac{1}{2} \right) \end{aligned} \tag{58}$$

and the uncertainty relation

$$\begin{aligned} \sqrt{\overline{\mathbf{u}^2} \overline{\mathbf{p}^2}} &= \frac{\hbar}{2} e^{-2\beta t} (2n + 1) \rightarrow \Delta\pi\Delta u \geq \frac{\hbar}{2} e^{-2\beta t}, \\ (\Delta u)^2 &:= \overline{(\mathbf{u} - \overline{\mathbf{u}})^2} = \overline{\mathbf{u}^2} - 2\overline{\mathbf{u}\mathbf{u}} + \overline{\mathbf{u}}^2 = \overline{\mathbf{u}^2}, \\ (\Delta p)^2 &:= \overline{(\mathbf{p} - \overline{\mathbf{p}})^2} = \overline{\mathbf{p}^2} - 2\overline{\mathbf{p}\mathbf{p}} + \overline{\mathbf{p}}^2 = \overline{\mathbf{p}^2}, \end{aligned} \tag{59}$$



where we considered that  $\bar{\mathbf{u}} = 0, \bar{\mathbf{p}} = 0$ . In case of the simple oscillator and also when  $c = 0$ , this relation transforms into the Heisenberg's one.

### The coherent state of the damped oscillator

The coherent state is a specific kind of state of the quantum damped oscillator the dynamics of which most closely resembles the oscillating behavior of a classical damped oscillator. The coherent state  $|a\rangle$  is defined to be the eigenstate of the annihilation operator  $\mathbf{A}(t)$ . Mathematically, this reads:

$$\mathbf{A}(t)|a\rangle = \lambda(t)|a\rangle \tag{60}$$

Since  $\mathbf{A}(t)$  is not Hermitian,  $\lambda(t)$  is a complex function. In the exponential form it can be represented as  $|\lambda(t)|e^{\varepsilon(t)}$ , where the amplitude  $|\lambda(t)|$  is a real function and the phase  $\varepsilon(t)$  of state  $|a\rangle$  is a real function of time to. Since  $\mathbf{A}(t)$  can be represented in the form

$$\mathbf{A}(t) = \mathbf{A}(0)e^{-\beta t} = \mathbf{a}e^{-\beta t}, \quad \mathbf{a} = \mathbf{A}(0) \tag{61}$$

where the annihilation operator  $\mathbf{a}$  is shown a strong analogy with the annihilation operator of the simple harmonic oscillator, as we can see in section 3. From this analogy, it follows that the amplitude and phase of the state  $|a\rangle$  satisfy the same eigenvalue problem as in the case of the simple harmonic oscillator, i.e.

$$\mathbf{a}|a\rangle = \lambda_0 e^{i\varepsilon_0}|a\rangle \tag{62}$$

The above two formulas mean that a coherent state is left unchanged by the annihilation or the creation of a particle. From the equations (60) and (61) it follows that

$$\lambda(t) = (\lambda_0 e^{i\varepsilon_0})e^{-\beta t} \tag{63}$$

Thus the phase of state  $|a\rangle$  is time independent. Since the Bohlin's exergy operator has the same eigenkets as the Hamiltonian in case of the simple oscillator, thus the representation of the coherent state of the quantum damped oscillator - on the basis of exergy operator - shall be identical. Also, the representation of the coherent state on the basis of the exergy operator eigenkets, the so-called Fock states, is:

$$|a\rangle = e^{-\frac{\lambda_0^2}{2}} \sum_{n=0}^{\infty} \frac{\lambda_0^n}{\sqrt{n!}} |n\rangle \tag{64}$$

where  $|n\rangle$  are eigenkets of the exergy operator. This is a Poissonian distribution, which is a necessary and sufficient condition that all annihilations are statistically independent. On the other hand, from the equation (53) and (64) the expected value of the occupation number operator in coherent state at time  $t = 0$  is

$$\overline{n(t=0)} = N_0 = \langle a | \mathbf{a}^+ \mathbf{a} | a \rangle = e^{-\frac{\lambda_0^2}{2}} \sum_{n=0}^{\infty} n \frac{\lambda_0^{2n}}{\sqrt{n!}} = \lambda_0^2 \tag{65}$$

Thus the coherent state (64) can be written by means of the expected photon number at  $t = 0$  as

$$|a\rangle = e^{-\frac{N_0}{2}} \sum_{n=0}^{\infty} \frac{N_0^{\frac{n}{2}}}{\sqrt{n!}} |n\rangle \tag{66}$$

Consequently, the probability of detecting  $n$  photons in coherent state is:

$$\langle n | e \rangle^2 = \frac{N_0^n e^{-N_0}}{n!} \tag{67}$$

Since  $N_0 = \overline{n(t)}e^{2\beta t}$ , where  $\overline{n(t)}$  is the expected value of photons at time  $t$  if the expected photon number were  $N_0$  at time  $t = 0$ , the above detecting probability may be written as



$$\langle n|e \rangle^2 = \frac{\overline{n(t)}^n e^{-2n\beta t} e^{-\overline{n(t)} e^{2n\beta t}}}{n!} \tag{68}$$

This is the probability of detecting  $n$  photons at time  $t$ , if the expected photon number is  $\overline{n(t)}$  at the same time. Since in case of the Poisson distribution the variance is equal with the expected value, we obtain

$$\overline{\Delta n^2} = N_0 = \overline{n(t)} e^{2\beta t} \tag{69}$$

Also the probability of detecting  $n$  photons and the variance of this detection is determined by the expected initial number of photons only.

### Probability description of the wave packet motion belonging to the damped oscillator

To learn something about the time dependence of our system in a certain state  $| \rangle$ , we shall calculate

$$\langle u(t) | \rangle \tag{70}$$

and

$$\langle u(t) | \rangle^2, \tag{71}$$

which represent the probability amplitude and probability of finding the damped oscillator at  $u$  at the time  $t$ . In particular, it is useful to study the coherent state  $| a \rangle$ , which is an eigenstate of the non-hermitian time dependent operator  $\mathbf{A}$ , i.e.

$$\mathbf{A} e^{i\omega t} | a \rangle = \sqrt{\frac{2m\omega^2}{\hbar\omega_0}} d e^{-\beta t} e^{i\omega t} | a \rangle. \tag{72}$$

We shall also calculate the probability amplitude  $\Psi_d(u) = \langle u | a \rangle$  of the wave packet  $| a \rangle$  at  $u$ .

To do it, we shall start the following fact of bracket calculus

$$\begin{aligned} \langle u | \mathbf{A} e^{i\omega t} | u \rangle &= \int \langle u | \mathbf{A} e^{i\omega t} | u' \rangle \langle u' | d \rangle du' = \int \langle u | \mathbf{A} e^{i\omega t} | u' \rangle \Psi_d(u') du' = \\ &= \sqrt{\frac{2m\omega^2}{\hbar\omega_0}} d e^{-\beta t} e^{i\omega t} \Psi_d(u) \end{aligned} \tag{73}$$

Now, we are going to express the operator  $\mathbf{A}$  by using the displacement (40) and the generalised momentum operator (41), then, we get

$$\mathbf{A} e^{i\omega t} = \sqrt{\frac{2}{\hbar\omega_0 m}} (i\boldsymbol{\pi} + m\omega \mathbf{u}). \tag{74}$$

Take this expression into (73), we obtain

$$\int \langle u | (i\boldsymbol{\pi} + m\omega \mathbf{u}) | u' \rangle \Psi_d(u') du' = \omega d e^{-\beta t} e^{i\omega t} \Psi_d(u). \tag{75}$$

According to the following

$$\langle u | \boldsymbol{\pi} | u' \rangle = e^{-2\beta t} \hbar \delta'(u - u') \quad \langle u | \mathbf{u} | u' \rangle = u \delta(u - u') \tag{76}$$

coordinate representation of the operators originated from commutation relation (29) and addendum 3., we get the

$$\left( \omega \mathbf{u} + \frac{\hbar}{m} \frac{d}{du} e^{-2\beta t} \right) \Psi_d(u) = \omega d e^{-\beta t} e^{i\omega t} \Psi_d(u) \tag{77}$$

ordinary differential equation. The solution of this equation in normalized form is



$$\Psi_d(q) = \left( \frac{\hbar \omega}{\pi} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar} \frac{(u - d e^{-\beta t} e^{i\omega t})^2}{e^{-2\beta t}}} \quad (78)$$

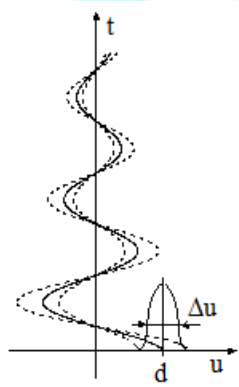
from which the probability will be

$$|\Psi_d(u)|^2 = \left( \frac{m\omega}{2\hbar\pi} \right)^{\frac{1}{2}} e^{\beta t} e^{-\frac{m\omega}{\hbar} e^{2\beta t} (u - d e^{-\beta t} \cos(\omega t))^2} \quad (79)$$

Now, we might see that this is the Gaussian distribution with the

$$|\Psi_d(u)|^2 = \frac{1}{\sqrt{2\pi} \left( \sqrt{\frac{\hbar}{m\omega}} e^{-\beta t} \right)} e^{-\frac{(u - d e^{-\beta t} \cos(\omega t))^2}{2 \left( \sqrt{\frac{\hbar}{m\omega}} e^{-\beta t} \right)^2}} \quad (80)$$

probability density function. Therefore, the motion of the centre of the wave packet  $|d\rangle$  is a damped oscillation, and its width decreases exponentially from the initial value of  $\sqrt{\frac{\hbar}{m\omega}}$  to zero (see Figure 1.).



**Fig 1: Representation of the evolution of the wave packet  $|a\rangle$ . The motion of the centre of packet and its uncertainty  $\Delta u$  are represented**

Now, we see that the initial uncertainty of the packet  $|a\rangle$  keeps getting smaller with the progress of time and becomes negligible as  $t \rightarrow \infty$ . Also, the evolution of the wave packet continually proceeds to the motion of classic damped oscillator with the progress of time.

### Calculation of wave function by matrix calculus

As resulted from (40) using (49), the matrix of displacement operator in energy representation has the form

$$\mathbf{u} = \sqrt{\frac{\hbar}{2m\omega_0}} \begin{bmatrix} 0 & \bar{z} & 0 & 0 & 0 \\ 1\bar{z}^* & 0 & \sqrt{2}\bar{z} & 0 & 0 \\ 0 & \sqrt{2}\bar{z}^* & 0 & \sqrt{3}\bar{z} & 0 \\ 0 & 0 & \sqrt{3}\bar{z}^* & 0 & \sqrt{4}\bar{z} \\ 0 & 0 & 0 & \sqrt{4}\bar{z}^* & 0 \\ & & & etc. & \end{bmatrix} e^{-\beta t}, \quad \bar{z} = e^{i\omega t}, \quad (81)$$

where the asterisk denotes the conjugation. Next, we are going to solve the



$$\mathbf{u}|d\rangle = u|d\rangle \tag{82}$$

eigenvalue problem belonging to this and it looks like in matrix form as

$$\sqrt{\frac{\hbar}{2m\omega_0}} \begin{bmatrix} 0 & 1\bar{z} & 0 & 0 & 0 \\ 1\bar{z}^* & 0 & \sqrt{2}\bar{z} & 0 & 0 \\ 0 & \sqrt{2}\bar{z}^* & 0 & \sqrt{3}\bar{z} & 0 \\ 0 & 0 & \sqrt{3}\bar{z}^* & 0 & \sqrt{4}\bar{z} \\ 0 & 0 & 0 & \sqrt{4}\bar{z}^* & 0 \\ & & & etc. & \end{bmatrix} e^{-\beta t} \begin{bmatrix} d_0 \\ d_1 \\ d_3 \\ \vdots \\ \cdot \\ \cdot \end{bmatrix} = u(t) \begin{bmatrix} d_0 \\ d_1 \\ d_3 \\ \vdots \\ \cdot \\ \cdot \end{bmatrix} \tag{83}$$

From this we get the difference schema

$$d_1 = \frac{q^\circ}{z\sqrt{1}}d_0, \quad d_2 = \frac{q^\circ}{z\sqrt{2}}d_1 - \frac{\bar{z}^*}{z\sqrt{2}}d_0, \dots, \tag{84}$$

$$d_n = \frac{q^\circ}{z\sqrt{n}}d_{n-1} - \frac{\bar{z}^*\sqrt{n-1}}{z\sqrt{n}}d_{n-2}$$

$$q^\circ = \sqrt{\frac{2m\omega_0}{\hbar}} e^{\beta t} u$$

After some algebra, we obtain another form

$$d'_n = 2\frac{q^\circ}{\sqrt{2}}d'_{n-1} - 2(n-1)d'_{n-2} \tag{85}$$

$$d'_n = \sqrt{2^n n!} \frac{d_n}{\left(\frac{\bar{z}^*}{z}\right)^n}$$

By introducing a new coordinate variable, we have the difference equation

$$\hat{q} d'_n = 2q d'_{n-1} - 2(n-1)d'_{n-2} \tag{86}$$

$$\hat{q} = \frac{q^\circ}{\sqrt{2}}$$

This difference equation is satisfied by the Hermite polynomials. Thus, we obtain

$$\langle n|u\rangle = d_n = c_0 \left( e^{\beta t} \sqrt{\frac{m\omega_0}{\hbar}} u \right) \frac{\left(\frac{\bar{z}^*}{z}\right)^n}{\sqrt{2^n n!}} H_n \left( e^{\beta t} \sqrt{\frac{m\omega_0}{\hbar}} u \right), \tag{87}$$

where  $c_0 \left( e^{\beta t} \sqrt{\frac{m\omega_0}{\hbar}} u \right)$  is a function to be determined. Starting from the fact  $\langle u'|u''\rangle = \delta(u'-u'')$ , we get that



$$\begin{aligned} \langle u' | u' \rangle &= \sum_n \langle u' | n \rangle \langle n | u' \rangle = \sum_n c_0 \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' \right) \frac{\left( \frac{-}{z} \right)^n}{\sqrt{2^n n!}} H_n \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' \right) \times \\ &\times c_0^* \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u'' \right) \frac{\left( \frac{-}{z} \right)^n}{\sqrt{2^n n!}} H_n \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u'' \right) \\ &= \sum_n c_0 \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' \right) c_0^* \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u'' \right) \frac{1}{2^n n!} H_n \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' \right) \times \\ &H_n \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u'' \right) = \delta(u' - u'') \end{aligned}$$

By using the

$$\begin{aligned} \sum_n \frac{1}{2^n n!} H_n \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' \right) \times H_n \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u'' \right) &= \tag{88} \\ &= \sqrt{\pi} e^{\left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' \right)^2} \delta \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' - e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u'' \right) = \\ &= \sqrt{\frac{\pi m \omega_0}{\hbar}} e^{\left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u' \right)^2} e^{\beta t} \delta(u'' - u') \end{aligned}$$

relationship, the final form of (87) is given by

$$\Psi_n(u) = \langle n | u \rangle = d_n = e^{\frac{1}{2} \beta t} \left( \frac{m \omega_0}{\hbar \pi} \right)^{\frac{1}{4}} e^{-\frac{1}{2} e^{2\beta t} \frac{m \omega_0}{\hbar} u^2} \frac{\left( \frac{-}{z} \right)^n}{\sqrt{2^n n!}} H_n \left( e^{\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u \right). \tag{89}$$

According to this result, the probability density of the  $n$ -th energy state when the displacement  $u$  is

$$\begin{aligned} |\Psi_n(u)|^2 = |\langle n | u \rangle|^2 &= \frac{1}{2^n n!} e^{\beta t} \left( \frac{m \omega_0}{\hbar \pi} \right)^{\frac{1}{2}} e^{-\frac{m \omega_0}{\hbar} \frac{u^2}{e^{-2\beta t}}} H_n^2 \left( e^{-\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u \right) \tag{90} \\ &= \frac{1}{\sqrt{2\pi} \left( \sqrt{\frac{\hbar}{2m \omega_0}} e^{-\beta t} \right)} e^{-\frac{u^2}{2 \left( \sqrt{\frac{\hbar}{2m \omega_0}} e^{-\beta t} \right)^2}} \left[ \frac{1}{2^n n!} H_n^2 \left( e^{-\beta t} \sqrt{\frac{m \omega_0}{\hbar}} u \right) \right]. \end{aligned}$$

Now, we might see that this is the density function of a modulated Gaussian distribution, where the modulating term has finite amplitude which runs over in time, while the Gaussian distribution sharpens towards to a Dirac delta distribution. This means that the particle will get closer and closer to the equilibrium point as  $t \rightarrow \infty$ . From last result, we can conclude that in the case of  $\beta \rightarrow 0$  we get back to the well-known wave function of the simple oscillator.

## SPECTRUM OF THE RATE OF ENERGY DISSIPATION OF THE DAMPED OSCILLATOR

In the next section, we are going to explain the finite width of spectrum line and give the intensity frequency spectrum of radiation. As the atom emits photons, the energy drops and the amplitude of transition decreases in time. Therefore, the



emission is not harmonic, thus spectrum occurs. We shall see that the width of spectral line can be connected to the attenuation coefficient of the damped oscillator. Inversely, from the width of the spectral line we might determine the attenuation coefficient of the oscillator.

### Spectral density of the energy dissipation

In section 1.3 we already introduced the rate of energy dissipation for a damped oscillator. The rate of energy dissipation operator can be originated from equation (19) as

$$\mathbf{D} = c \frac{d\mathbf{u}}{dt} \frac{d\mathbf{u}}{dt} \tag{91}$$

From this it follows that the expected value of the operator of energy dissipation is

$$\overline{\mathbf{W}}_{dissz} = \int_0^{\infty} \overline{\mathbf{D}} dt \tag{92}$$

In the next section we shall evaluate this energy dissipation formula. To do it, let us introduce the velocity operator, which comes from (40) as

$$\frac{d\mathbf{u}}{dt} = \sqrt{\frac{\hbar}{2m\omega_0}} (\gamma \mathbf{a} e^{\gamma t} + \gamma^* \mathbf{a}^+ e^{\gamma^* t}) \tag{93}$$

The elements of this operator in energy (exergy) representation are:

$$\left( \frac{d\mathbf{u}}{dt} \right)_{ik} = \sqrt{\frac{\hbar}{2m\omega_0}} (\gamma \langle i | \mathbf{a} | k \rangle e^{\gamma t} + \gamma^* \langle i | \mathbf{a}^+ | k \rangle e^{\gamma^* t}) \tag{94}$$

According to (47) we get the final form of matrix elements

$$\left( \frac{d\mathbf{u}}{dt} \right)_{ik} = \sqrt{\frac{\hbar}{2m\omega_0}} (\gamma \sqrt{k} \delta_{ik-1} e^{\gamma t} + \gamma^* \sqrt{k+1} \delta_{ik+1} e^{\gamma^* t}) \tag{95}$$

Substitute this into (91) then the rate of energy dissipation is given by

$$\overline{\mathbf{D}} = \hbar \omega_0 \beta n (e^{\gamma t} e^{\gamma^* t}), (n = 1, 2, 3, \dots) \tag{96}$$

and the energy dissipation is given by

$$\overline{\mathbf{W}}_{dissz} = \hbar \omega_0 \beta n \int_0^{\infty} (e^{\gamma t} e^{\gamma^* t}) dt, (n = 1, 2, 3, \dots) \tag{97}$$

According to the Parseval theorem of the Fourier transformation theory the energy dissipation may be written as

$$\begin{aligned} \overline{\mathbf{W}}_{dissz} &= \hbar \omega_0 \beta n \int_0^{\infty} (e^{\gamma t})^2 dt = \hbar \omega_0 \beta n \int_0^{\infty} \{F[e^{\gamma t}]\} \{F[e^{\gamma t}]\}^* d\omega' \\ &= \hbar \omega_0 \beta n \int_0^{\infty} \frac{d\omega'}{(\omega - \omega')^2 + \beta^2}, (n = 1, 2, 3, \dots) \end{aligned} \tag{98}$$

where  $F[e^{\gamma t}]$  is the Fourier transform of  $e^{\gamma t}$ .

In case of the transition from  $n$ -th exergy state to the equilibrium state, the oscillator will emit  $n\hbar\omega_0$  energy, thus (87) can be written as

$$n\hbar\omega_0 = \hbar \omega_0 \beta n \int_0^{\infty} \frac{d\omega'}{(\omega - \omega')^2 + \beta^2} = \frac{1}{\beta} \hbar \omega_0 \beta n \tag{99}$$





Consequently the spectral density of the dissipated energy is

$$w_{diss}(\omega') = \frac{n\hbar\omega_0\beta}{(\omega - \omega')^2 + \beta^2} \quad (100)$$

This is a Lorentz distribution about the shifted circular frequency  $\omega$ . This result agrees well with corresponding result derived from two state atom models of Wigner and Weisskopf [32], [33] and system-plus-reservoir model [34], [10], [31]. It is well known that the half value width  $\Delta\omega$  of this distribution is

$$\Delta\omega = 2\beta. \quad (101)$$

### Natural width of the spectral line

The natural line width of the spectral line is the result of the dissipative quantum process which accompanies the spontaneous emission of an atom. We shall treat this emission process in a dissipative two state model. To do it we consider the two states of the atom as the first and second exergy state of a damped oscillator. In this case the spontaneous emission of a photon is the consequence of the transition from the first exergy state to the equilibrium state of the damped oscillator. In that model the spectrum density of the emitted photon follows from (89)

$$w(\omega') = \frac{\hbar\omega_0\beta}{(\omega - \omega')^2 + \beta^2}. \quad (102)$$

Also the width of the frequency spectrum of a spontaneous emission of the atom is a direct consequence of the dissipative self-force on the atom. This back-reaction of the emitted photon can be written with two physical quantities, namely: the frequency shift  $\omega_0 \rightarrow \omega$  and the half value width  $\Delta\omega$ . If we consider  $\hbar \frac{\Delta\omega}{2}$  as energy uncertainty  $\Delta E$  and the time constant of the emission process  $\Delta t = \beta^{-1}$  as time uncertainty, we obtain the

$$\Delta E \Delta t \geq \frac{\hbar}{2\beta} \beta = \frac{\hbar}{2} \quad (103)$$

uncertainty relation. The quantum mechanical interpretation of natural line should be based on this relation, in which the physical quantities  $\Delta E$  and  $\Delta t = \beta^{-1}$  have precise meaning. In our model the natural line width occurs in wave length  $\lambda$  and can be calculated as

$$\Delta\lambda = \left| \Delta \frac{2\pi c}{\omega} \right| = \frac{2\pi c}{\omega^2} \Delta\omega = \frac{2\pi c}{\omega^2} \beta, \quad (104)$$

where  $c$  is the vacuum velocity of light. It is well-known that in the classical dipole model of light emission the natural line width can be calculated as

$$\Delta\lambda = \frac{4\pi\epsilon_0}{3mc^2}, \quad (105)$$

where  $\epsilon_0$  is the vacuum permittivity and  $r_e := \frac{\epsilon_0}{3mc^2} = 2,818 \cdot 10^{-15} m$  is the so called classical electron radius. From the above two equations it follows that

$$\beta = \frac{2}{3} \frac{\omega^2}{c} r_e = \frac{4\pi}{3} \frac{r_e}{\lambda} \omega, \quad (106)$$

in dipole radiation model.

## ADDENDUM

### The strange quantum Poisson bracket

The classical Poisson bracket in case of damped oscillator is not invariant throughout the motion. It depends exponentially on time. To show this property, let us suppose that the commutation relationship is true at  $t = 0$  and let us generate the commutation relationship at  $t = 0 + dt$ . Then we get



$$(\pi, u)_K = \left( \pi(0) + \frac{d\pi}{dt} \Big|_{t=0} dt, u(0) + \frac{du}{dt} \Big|_{t=0} dt \right)_K = (\pi, u)_{Kt=0} - 2\beta(\pi, u)_{Kt=0} dt. \tag{107}$$

Here we used that

$$\begin{aligned} \frac{du}{dt} \Big|_{t=0} &= (u, H)_{t=0} = \frac{\pi - cu}{m} \Big|_{t=0} \\ \frac{d\pi}{dt} \Big|_{t=0} &= (\pi, H)_{t=0} = -ku \Big|_{t=0} \end{aligned} \tag{108}$$

Then our statement follows from (107), namely:

$$(\pi, u)_K = (\pi, u)_{Kt=0} e^{-2\beta t}. \tag{109}$$

Now, we can see that in case of the simple oscillator the constitutive Poisson bracket will be constant in time throughout the motion of the undamped oscillator. Due to the behaviour of the constitutive Poisson bracket we should choose  $(\pi, u)_K \Big|_{t=0} = (\pi, u)$ .

From the Poisson bracket  $(u, u) = 0$  we can conclude that

$$(p, u)_K = (p, u)_{Kt=0} e^{-2\beta t} = (p, u) e^{-2\beta t}. \tag{110}$$

where  $p = m \frac{du}{dt}$  is the Newtonian momentum.

From (109) it follows that we can formulate the canonical equations (14) by using the constitutive Poisson brackets as

$$\begin{aligned} \frac{du}{dt} &= e^{2\beta t} (u, H)_K, \\ \frac{d\pi}{dt} &= e^{2\beta t} (\pi, H)_K \end{aligned} \tag{111}$$

Evidently, these equations must be evaluated with the fundamental brackets (17).

To proceed towards the correct form of the QPB we must find a time invariant classical PB. The following PB can be defined as

$$(p, q) = e^{2\beta t} (p, q)_K \tag{112}$$

It remains a constant throughout the motion. As we shown above, the constitutive Poisson bracket corresponds to the

$$(p, q)_K \xrightarrow{\text{canonical quantisation}} \frac{1}{i\hbar} (\mathbf{p}\mathbf{q} - \mathbf{q}\mathbf{p}) := \frac{1}{i\hbar} [\mathbf{p}, \mathbf{q}] \tag{113}$$

commutation relation, and also to the Dirac's one. Consequently, the classical Poisson brackets in case of the damped oscillator corresponds to the following explicit time dependent commutator

$$(p, q) \longrightarrow e^{2\beta t} \frac{1}{i\hbar} [\mathbf{p}, \mathbf{q}] \tag{114}$$

Since the classical equations of motion of the damped oscillator can be given by classical restricted Poisson bracket as

$$\begin{aligned} \frac{du}{dt} &= (u, H), \\ \frac{d\pi}{dt} &= (\pi, H) \end{aligned} \tag{115}$$

The quantum damped oscillator has the corresponding



$$\frac{d\mathbf{u}}{dt} = e^{2\beta t} \frac{1}{i\hbar} [\mathbf{u}, \mathbf{H}] \tag{116}$$

$$\frac{d\boldsymbol{\pi}}{dt} = e^{2\beta t} \frac{1}{i\hbar} [\boldsymbol{\pi}, \mathbf{H}]$$

operator differential equations. The above presented theory of canonical quantisation should be simplified. To do it, let us write the canonical equations of the oscillator (115) in the next form

$$\frac{du}{dt} = (u, H) \tag{117}$$

$$\frac{dp}{dt} = (p, H) - c(u, H)$$

which are the classical equations of damped oscillator, formulated with Newtonian momentum and classical Poisson brackets. By the method of canonical quantisation using (113) it follows that the equations of quantum damped oscillator are

$$\frac{d\mathbf{u}}{dt} = e^{2\beta t} \frac{1}{i\hbar} [\mathbf{u}, \mathbf{H}] \tag{118}$$

$$\frac{d\mathbf{p}}{dt} = e^{2\beta t} \frac{1}{i\hbar} [\mathbf{p}, \mathbf{H}] - ce^{2\beta t} \frac{1}{i\hbar} [\mathbf{u}, \mathbf{H}]$$

It is evident that the equations (115) and (118) are equivalent.

In the above deduction of the quantum mechanical equations of damped oscillator must be assumed that the time and also the pure time functions, must necessarily be considered as ordinary numbers in the sense of operator algebra.

### The modified uncertainty relation

The Uncertainty Principle of quantum mechanics states that both the position and the momentum cannot simultaneously be measured with complete precision. However, the position (alone) of a free moving particle is precisely measurable, creating an eigenstate of the position with a wave-function of a Dirac delta at a particular position. This is called an eigenstate of position, where its momentum is completely unknown. On the other hand, if the particle is in an eigenstate of momentum, then its position is completely unknown. More accurately, the uncertainty assumes two, constant initial states in time multitude of quantum states; and the momentum and position are measured independently on these. As a result of this, one decomposes on momentum and the other on coordinate position eigenstates, and the deviation is determined

independently in the two sets of states. The product of these uncertainties (deviations) is larger than  $\frac{\hbar}{2}$  in conservative systems. In our case neither the initial state nor the result of the measurement are constant in time. Our new uncertainty principle formulates different conditions. We perform - in various time-points- momentum and position measurement on the eigenvalue sets of momentum and position which was created by a measurement made at  $t = 0$  initial time. These measurements on the initially constructed sets do not change the momentum and position states of the multitude, shortly speaking, they do not generate newer set of eigenvalues. With this method we determine the momentum and position deviations in the various measured time-points and we have the modified relation of the uncertainty principle:

$$\Delta p \Delta u \geq \frac{\hbar}{2} e^{-2\beta t}$$

This is correct because the eigenvalues of momentums and positions decrease exponentially in the

set while the ratio of  $\langle p_i | \Psi(0) \rangle^2, \langle u_i | \Psi(0) \rangle^2, (i = 1,2,3,..)$  eigenvalues in the sets are unchanged.

This construction of the “measurement” does not satisfy the request of quantum-mechanical measurement principle, when the measurements have to be performed on sets of identical initial states independently for momentum and position.

Supporting our present results let us perform another kind of measurement. Consider four sets of quantum-states having identical initial states. Perform the momentum and position measurements at  $t = 0$  and at a later  $t$  time on an independent part of the quantum-sets. These measurements show the well-known Heisenberg uncertainty by these pairs. However, the two sets of momentum and position will be statistically different in the  $t$  time-point, because probability of the eigenstates will be  $\langle p_i | \Psi(0) \rangle^2, \langle u_i | \Psi(0) \rangle^2, (i = 1,2,3,..)$ , in one; and  $\langle p_i | \Psi(t) \rangle^2, \langle u_i | \Psi(t) \rangle^2, (i = 1,2,3,..)$  in the other one. ( $\langle p_i |, \langle u_i |$  are the corresponding eigen-kets, while  $|\Psi(0)\rangle, |\Psi(t)\rangle$  the corresponding state-functions in  $t = 0$  and  $t$  time-points.



## The coordinate representation of the momentum operator

According to Fong [35], [36] let us proceed from the (22) with  $g(t) = 1$ , also from the

$$[\boldsymbol{\pi}, \mathbf{u}] = \frac{i\hbar}{g(t)} \boldsymbol{\delta} \quad (119)$$

commutation relation. It is well-known that the

$$\{1, u, u^2, \dots, u^n\} \quad (120)$$

set of polynomials is dense in the space of the  $L^2$  functions. Also, any  $L^2$  function can be constructed as linear combination of these polynomials. We obtain that for the basis of any  $L^2$  function holds the

$$\boldsymbol{\pi}(u) = u\boldsymbol{\pi}(1) - \frac{i\hbar}{g(t)} \quad (121)$$

operator identity followed from (119) and from the chosen correspondence:  $\mathbf{u} = u$ . Let us assume that

$$\boldsymbol{\pi}(u^{n-1}) = u^{n-1}\boldsymbol{\pi}(1) - \frac{i\hbar}{g(t)}(n-1)u^{n-2} \quad (122)$$

holds for  $u^n$ . Then by the law of mathematical induction from (121) and (A.16) we conclude that

$$\boldsymbol{\pi}(u^n) = \boldsymbol{\pi}(uu^{n-1}) = u\boldsymbol{\pi}(u^{n-1}) - \frac{i\hbar}{g(t)}u^{n-1} = u^n\boldsymbol{\pi}(1) - n\frac{i\hbar}{g(t)}u^{n-1} \quad (123)$$

From this identity the result is the following

$$\boldsymbol{\pi} = -\frac{i\hbar}{g(t)} \frac{d}{du} + A(u) \quad (124)$$

where  $A(u) = \boldsymbol{\pi}(1)$ .

## About the emission problem

The rate of energy dissipation should be connected with the transition probability per unit time of the light emission process, which was introduced by Dirac [37], [33]. Dirac showed that the rate of transition probability in an emission process is proportional to the square of the transition velocity [37]. Also, in our problem, the rate of transition probability is

$$\frac{dp_{10}}{dt} \propto \left| \left( \frac{du}{dt} \right)_{10} \right|^2 \propto D_{10} \quad (125)$$

From this the probability that the emission of photon occurs during the time interval  $[0, t]$  is

$$p_{10}(t) \propto \int_0^t |e^{\gamma^* t'}|^2 dt' \propto \int_0^t D_{10}(t') dt' \quad (126)$$

So the probability that the emission of photon occurs during the time interval  $[0, t]$  is proportional to the energy dissipation during the same time interval. A consequence of this expression is that the frequency spectrum of the transition probably is the same as the spectrum of the energy dissipation, i.e.

$$w_{dissz}(\omega') \propto p_{10}(\omega') = \frac{\beta}{(\omega - \omega')^2 + \beta^2} \quad (127)$$

## Speculation about the strange commutation relation and the irreversibility

The inequality of





$$\overline{(\Delta p)^2} \overline{(\Delta u)^2} > \left[ \overline{(\mathbf{p} - \bar{\mathbf{p}})(\mathbf{u} - \bar{\mathbf{u}})} \right]^2 \tag{128}$$

uncertainty relation is true in general. Split the operator  $\mathbf{pu}$  to a symmetric and an asymmetric part, i.e.

$$\mathbf{pu} = \frac{\mathbf{pu} + \mathbf{up}}{2} + \frac{\mathbf{pu} - \mathbf{up}}{2} \tag{129}$$

And pursuant to the foregoing:

$$\frac{\mathbf{pu} - \mathbf{up}}{2} = \frac{\hbar}{2} e^{-2\beta t} \tag{130}$$

Therefore, we get that in case of non-commuting operators the uncertainty relation looks like this:

$$\overline{(\Delta p)^2} \overline{(\Delta u)^2} > \left[ \frac{\mathbf{pu} + \mathbf{up}}{2} - \overline{\mathbf{pu}} \right]^2 + \frac{\hbar^2}{4} e^{-4\beta t} \tag{131}$$

From this we may see that the second member of the right side keeps getting smaller with the progress of time and becomes negligible. Also, the uncertainty relation is transformed into the

$$\overline{(\Delta p)^2} \overline{(\Delta u)^2} > \left[ \frac{\mathbf{pu} + \mathbf{up}}{2} - \overline{\mathbf{pu}} \right]^2 \tag{132}$$

inequality. But, in this limited case, as a consequence of the commutation relation (A.8),  $\mathbf{pu} \cong \mathbf{up}$  holds. Thus the above relation is transformed into the inequality

$$\overline{(\Delta p)^2} \overline{(\Delta u)^2} > \left[ \overline{\mathbf{pu}} - \overline{\mathbf{pu}} \right]^2 \tag{133}$$

which is the uncertainty relation of the classical statistical theory and it follows from the simple fact that the regression coefficient is smaller than one. Namely, the quantum mechanics of the dissipative processes gradually turns into a classical statistical theory.

This way, we get a new speculative interpretation of irreversibility: *in an irreversible quantum process the non-commuting quantum operators proceed to commuting classical observables, which are submitted the laws of classical statistic theory.*

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