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Finite difference code for velocity and surface traction of a Fluid between Two Eccentric Spheres

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Abstract

The Numerical study of the flow of a fluid in the annular region between two eccentric sphere susing PHP Code is investigated. This flow is created by considering the inner sphere to rotate with angular velocity Ω_1 and the outer sphere rotate with angular velocity Ω_2 about the axis passing through their centers, the z-axis, using the three dimensional Bispherical coordinates (α, β, φ).The velocity field of fluid is determined by solving equation of motion using PHP Code at different cases of angular velocities of inner and outer sphere. Also Finite difference code is used to calculate surface tractions at outer sphere.

Indexing terms/Keywords

PHP Code; Finite difference code; eccentric spheres; Bispherical coordinates; surface traction

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1 Introduction

The determination of the rheological properties of fluids depends, in general, on solution of suitable boundary value problems based on a specific rheological model that represents the fluid under consideration. The theoretical and experimental studies concerning the flow of viscous or viscoelastic fluids in the annular narrow gaps between two rotating bodies are very interesting boundary value problems in rheology. These problems represent the keystone in the high developing today industries and technology such as the flow in rotation turbo machinery, in journal-bearing lubrication, socket joints, petroleum and so on.

One of these problems, for two concentric spheres the numerical and experimental studies are carried out, say by Wimmer [1] and Yamaguchi et. al.[2,3,4] . A large number of theoretical and experimental works are done on the viscous flow between two eccentric spheres; Jeffery [5], Stimson and Jeffery [6], Majumdar [7], Munson [8], Menguturk and Munson [9]. The analytical study of the flow of viscoelastic fluid between two eccentric spheres is investigated by Abu-El Hassan et .al. [10,11] . Force and torque at outer stationary sphere are studied using neural network system and genetic programming by Mostafa Y.Elbakry et .al. [12,13].

Finite difference method one of the important numerical methods in solving many problems in newtonian and non-newtonian flow in fluid mechanics [14-18].

The present work is concerned with the numerical solution of this boundary value problem using PHP programming. The velocity field of a fluid between two eccentric spheres is investigated. Also surface tractions at outer stationary sphere is calculated using Finite difference code.

2 Formulation of the problem

A viscous fluid is assumed to perform steady and isochoric motion in the annular region between two eccentric spheres . This flow is created by considering the inner sphere to rotate with angular velocity Ω about the axis connecting their centers, the z-axis, while the outer sphere is kept at rest.

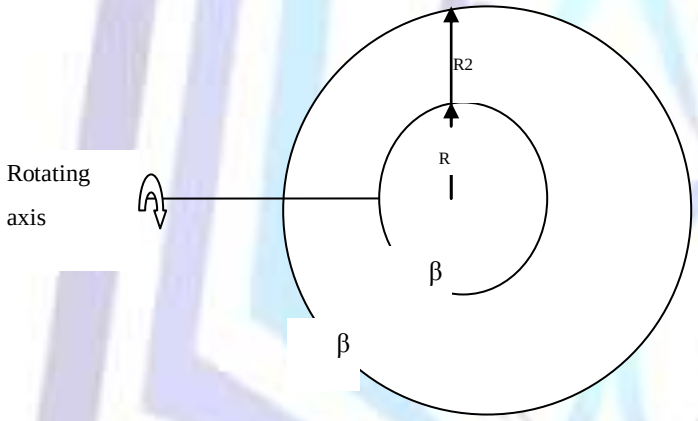

Fig.(1) Configuration of the two eccentric spheres

Using the Bispherical coordinates (α,β,ϕ) the inner sphere is the surface defined by $\beta=\beta_1$, while the outer one is defined by $\beta=\beta_2$. The radii of the inner and the outer spheres are then given by; [19,20],

$$R_1 = \frac{c}{sh\beta_1} \qquad and \qquad R_2 = \frac{c}{sh\beta_2}$$

Where *c* is a parameter related to the scale factors of the coordinates by the following relations:

$$h_\alpha = h_\beta = h = \frac{c}{\cosh\beta - \cos\alpha}$$

$$h_\varphi = h\sin\alpha$$

Symmetry about the z-axis implies that the velocity field $\dot{\underline{X}}$ is independent of the coordinate φ. Hence, the velocity field can be stated in the form

$$\dot{\underline{x}}(\alpha,\beta) = u(\alpha,\beta)\hat{\alpha} + v(\alpha,\beta)\hat{\beta} + W(\alpha,\beta)\hat{\varphi} \ . \qquad (1)$$

where the components $u(\alpha,\beta),\ v(\alpha,\beta) and\ W(\alpha,\beta)$ are only functions of the coordinates α and β.

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The non-slip at the boundaries β_1 and β_2 imposes the boundary conditions

$$\left.\begin{aligned}\underline{\dot{x}}\big|_{\beta=\beta_1} &= \hat{\varphi}\Omega_1 h_\varphi(\alpha,\beta_1)\\ \underline{\dot{x}}\big|_{\beta=\beta_2} &= \hat{\varphi}\Omega_2 h_\varphi(\alpha,\beta_2)\,.\end{aligned}\right\} \tag{2}$$

The equation of continuity ,

$$\nabla \bullet \underline{\dot{x}} = \frac{1}{h^3 \sin\alpha}\left[\frac{\partial}{\partial\alpha}(h^2 \sin\alpha\ u) + \frac{\partial}{\partial\beta}(h^2 \sin\alpha\ v)\right] = 0\,.$$

is satisfied identically if u and v are derivable from a stream function ψ by the expression

$$u = \frac{-1}{h^2 \sin\alpha}\frac{\partial\psi}{\partial\beta} \text{ and } v = \frac{1}{h^2 \sin\alpha}\frac{\partial\psi}{\partial\alpha} \tag{3a}$$

Or in the compact form we can write

$$\underline{U} = u(\alpha,\beta)\hat{\alpha} + v(\alpha,\beta)\hat{\beta} = \nabla x(\hat{\varphi}\frac{c\psi}{h_\varphi})\ . \tag{3b}$$

The Cauchy dynamical equation of motion for a stationary flow is being

$$\nabla.\underline{\underline{T}}(\alpha,\beta) = \nabla.\underline{\underline{T}}_E(\alpha,\beta) - \nabla\chi(\alpha,\beta) = \rho\underline{\dot{x}}(\alpha,\beta).\nabla\underline{\dot{x}}(\alpha,\beta)\,, \tag{4}$$

where $\underline{\underline{T}}$ is the stress tensor, $\underline{\underline{T}}_E$ is the extra stress tensor, χ is the hydrostatic pressure function and ρ is the density of the fluid .

The constitutive equation for a viscous fluid is stated as follow

$$\underline{\underline{T}}_E = \mu\left[\nabla\underline{\dot{x}} + (\nabla\underline{\dot{x}})^T\right], \tag{5}$$

where μ is the coefficient of viscosity.

Substitution from equations (1)and (3b)into equation (5)the extra stress tensor becomes

$$\underline{\underline{T}}_E = \mu\left[\nabla\underline{U} + (\nabla\underline{U})^T + \nabla(W\hat{\varphi}) + (\nabla(W\hat{\varphi}))^T\right], \tag{6}$$

Substitution from equation (6) into equation (4) we get two equations of velocity components,

The axial velocity satisfies the harmonic equation,

$$\nabla^2(W\hat{\varphi}) = 0\,. \tag{7}$$

with the boundary conditions ,

$$W(\alpha,\beta_1) = h_\varphi(\alpha,\beta_1)\ , W(\alpha,\beta_2) = h_\varphi(\alpha,\beta_2). \tag{8}$$

This boundary value problem has the solution which determined by finite difference iteration method.

On the other hand, the stream function ψ satisfies the boundary value problem,

$$\nabla^4\left(\frac{c\psi(\alpha,\beta)}{h_\varphi}\hat{\varphi}\right) = 0\,. \tag{9}$$

with the boundary conditions

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$$\psi\ (\alpha,\beta)=\partial_\beta\psi(\alpha,\beta)=\begin{Bmatrix}0\\0\end{Bmatrix}\quad at\quad \beta=\begin{Bmatrix}\beta_1\\\beta_2\end{Bmatrix}. \tag{10}$$

It can be easily shown that the only solution for the boundary value problem defined by equations (9) and (10) is the trivial solution ψ =0.

The velocity field reduces to,

$$\underline{\dot{x}}=\Omega W\ (\alpha,\beta)\hat{\varphi}\,, \tag{11}$$

3Surface Tractions at outer stationary sphere

The surface traction at the boundary $\beta=\beta_2$ .when the inner sphere rotate with angular velocity Ω and the outer sphere at rest, is defined by

$$\underline{S}(\alpha,\beta_2)=-p(\alpha,\beta_2)\hat{\beta}+\underline{\underline{T}}_E(\alpha,\beta_2).\hat{\beta} \tag{12}$$

This expression describes the stress vector per unit area on the surface of a spherical shell

$$\beta=\beta_2$$

The constitutive equation for a fluid of grade two is defined by the relation,[21],

$$\underline{\underline{T}}_E=\mu\underline{\underline{A}}_1+\alpha_1\underline{\underline{A}}_2+\alpha_2\underline{\underline{A}}_1^2\,, \tag{13}$$

Where, $\underline{\underline{A}}_1=\nabla\underline{\dot{x}}+(\nabla\underline{\dot{x}})^{\mathrm{T}}$, (14)

$$\underline{\underline{A}}_2=\underline{\dot{x}}\cdot\nabla\underline{\underline{A}}_1+\nabla\underline{\dot{x}}\cdot\underline{\underline{A}}_1+(\nabla\underline{\dot{x}}\cdot\underline{\underline{A}}_1)^{\mathrm{T}} \tag{15}$$

The velocity field, as expressed by equation (11) leads to the following expression

$$\left[\underline{\underline{A}}_1\right]=\begin{bmatrix}0&0&Q_\alpha\\0&0&Q_\beta\\Q_\alpha&Q_\beta&0\end{bmatrix}, \tag{16}$$

$$\left[\underline{\underline{A}}_2\right]=2\begin{bmatrix}Q_\alpha^2&Q_\alpha Q_\beta&0\\Q_\alpha Q_\beta&Q_\beta^2&0\\0&0&0\end{bmatrix}, \tag{17}$$

$$\left[\underline{\underline{A}}_1^2\right]=\begin{bmatrix}Q_\alpha^2&Q_\alpha Q_\beta&0\\Q_\alpha Q_\beta&Q_\beta^2&0\\0&0&Q_\alpha^2+Q_\beta^2\end{bmatrix}. \tag{18}$$

where

$$Q_\alpha=h^{-1}\left[W_{,\alpha}-\frac{h(1-\cosh\beta\cos\alpha)}{c\sin\alpha}W_1\right]$$

$$Q_\beta=h^{-1}\left[W_{,\beta}-\frac{h\,sh\beta}{c}W_1\right]$$

Substituting from equations(16), (17) and (18) into the constitutive equation (13) and multiplying by the unit vector $\hat{\beta}$; we obtain the surface tractions ,

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$$\underline{s}(\alpha,\beta)= s_\alpha\hat{\alpha} + s_\beta\hat{\beta} + s_\varphi\hat{\varphi} \ , \tag{19}$$

where

$$\underline{s}_\alpha = \Omega^2(\alpha_2 + 2\alpha_1)Q_\alpha Q_\beta$$

$$\underline{s}_\beta = \Omega^2(\alpha_2 + 2\alpha_1)Q_\beta^2 \tag{20}$$

$$\underline{s}_\varphi = \Omega\ \mu Q_\beta$$

At the surface of the stationary outer sphere, $\beta = \beta_2$, the velocity field vanishes; i.e. $W\big|_{\beta_2} = W_{,\alpha}\big|_{\beta_2} = 0$ therefore, equations (19) reduces to

$$\underline{s}(\alpha,\beta_2)= \left[s_\beta\hat{\beta} + s_\varphi\hat{\varphi}\right]_{\beta=\beta_2} . \tag{21}$$

$$s_\beta\big|_{\beta_2} = \Omega^2(\alpha_2 + 2\alpha_1)(h^{-2}W_{,\beta}^2)\big|_{\beta_2} \tag{22}$$

$$s_\varphi\big|_{\beta_2} = \Omega\mu(h^{-1}W_{,\beta})\big|_{\beta_2} \tag{23}$$

The surface traction at the stationary outer sphere, $\beta = \beta_2$ in equations (22) and (23) can be normalized as follows

$$S_\beta = \frac{C^2 s_\beta\big|_{\beta_2}}{\Omega^2(\alpha_2 + 2\alpha_1)} = C^2(h^{-2}W_{,\beta}^2)\big|_{\beta_2} \tag{24}$$

$$S_\varphi = \frac{Cs_\varphi\big|_{\beta_2}}{\Omega\mu} = C(h^{-1}W_{,\beta})\big|_{\beta_2} \tag{25}$$

4 Finite Difference code For the problem

4.1Numerical calculation:

In our numerical calculation we used the finite differences schema to handle this boundary value problem, this schema is

$$\frac{\partial^2 W}{\partial\beta^2} = \frac{W_{i+1,j} - 2W_{i,j} + W_{i-1,j}}{h^2} + O(h^3)$$

$$\frac{\partial^2 W}{\partial \propto^2} = \frac{W_{i,j+1} - 2W_{i,j} + W_{i,j-1}}{k^2} + O(h^3)$$

$$\frac{\partial W}{\partial\beta} = \frac{W_{i+1,j} - W_{i-1,j}}{2h} + O(h^3)$$

$$\frac{\partial W}{\partial \propto} = \frac{W_{i,j+1} - W_{i,j-1}}{2k} + O(h^3)$$

Where h and k are the increments in α and β directions and i,j =0,1,2…….n

We used the PHP code to carry the numerical calculation, we found this code is useful and very fast to perform the calculations.

The following algorithm can be written in a different programing languages

1-defining the increment in α and β directions

n=20;

m=n-1;

C=4 , (C is the half distance betweenthe two poles of the bispherical coordinates)

the boundaries of α:

α_0 =0;

α_n=π;

the two radii of the spheres:

R_1=2;

R_2=4;

the boundaries of β:

$\beta_1 = \sinh^{-1}(C/ R_1)$;

$\beta_2 = \sinh^{-1}(C/ R_2)$;

the increment in β direction is h=(β_2- β_1)/m;

the increment in α direction is k=pi/m;

$K=k^2$;

$H=h^2$;

2- Calculating the boundary conditions

```
i=0;
While(i<=n)
{
α(i)=i*k;
β(i)= β1 +i*h;
h1=cosh(β(0))-cos(α(i));
u(0,i)=sin(α(i))/h1;
W(n,i)=0;
W(i,0)=0;
W(i,n)=0;
i++;
}
3- beginning the iteration
GG=10;
HH=H/K;
while (abs(GG)>0)
{
i=1;
while( i<=m)
{
j=1;
while (j<=m)
{
h2=cosh(β(j))-cos(α(i));
h3=1-cosh(β(j))*cos(α(i));
f1=1;
f2=1;
f3=0;
f4=-sinh(β(j))/h1;
f5=-h3/(h1*sin(α(i)));
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f6=-1/power(sin(α(i)),2);
D1=2*(f1+f2*HH)+H*f6;
D2=f1+h*f4/2;
D3=f1-h*f4/2;
D4=f2+k*f5/2;
D5=f2-k*f5/2;
W(i,j)=(D2* W(i+1,j)+D3* W(i-1,j)+HH*D4* W(i,j+1)+HH*D5* W(i,j-1))/D1;
W_{β}(i,j)=(W(i+1,j)- W(i-1,j))/(2*h);
W_{α}(i,j)=(W(i,j+1)- W(i,j-1))/(2*k);
j++;
}
i++;
}
For(i=5:1:5)
{
sum=0;
for j=1:1:m-1
sum=sum+ W(i,j);
}
area(i,l)=sum*h/2;
GG=area(i,l)-area(i,l-1);
l++;
}
}
5- plot the results
Plot(β, W);
Plot(α, W);

5Results and discussion

The present work represents a numerical investigation of the isochoric and isothermal flow of a viscous fluidin the annular region between two eccentric spheres using finite difference method. The results show that the axial component,W(α,β)is determined for different cases of angular velocities of two spheres, while the planar secondary velocity field $\underline{U}$ vanishes. The surface tractions at stationary sphere is obtained.

We have different cases of flow as follows:

Case(1):If the inner sphere β_1 is being at rest;i.e, $\Omega_1 = 0$ and the outer sphere β_2 rotates with angular velocity Ω, Fig.(2)show that the distribution velocity field W (α,β) versus α at different values of β, while Fig.(3) shows the velocity field W (α,β) versus β at different values of α.

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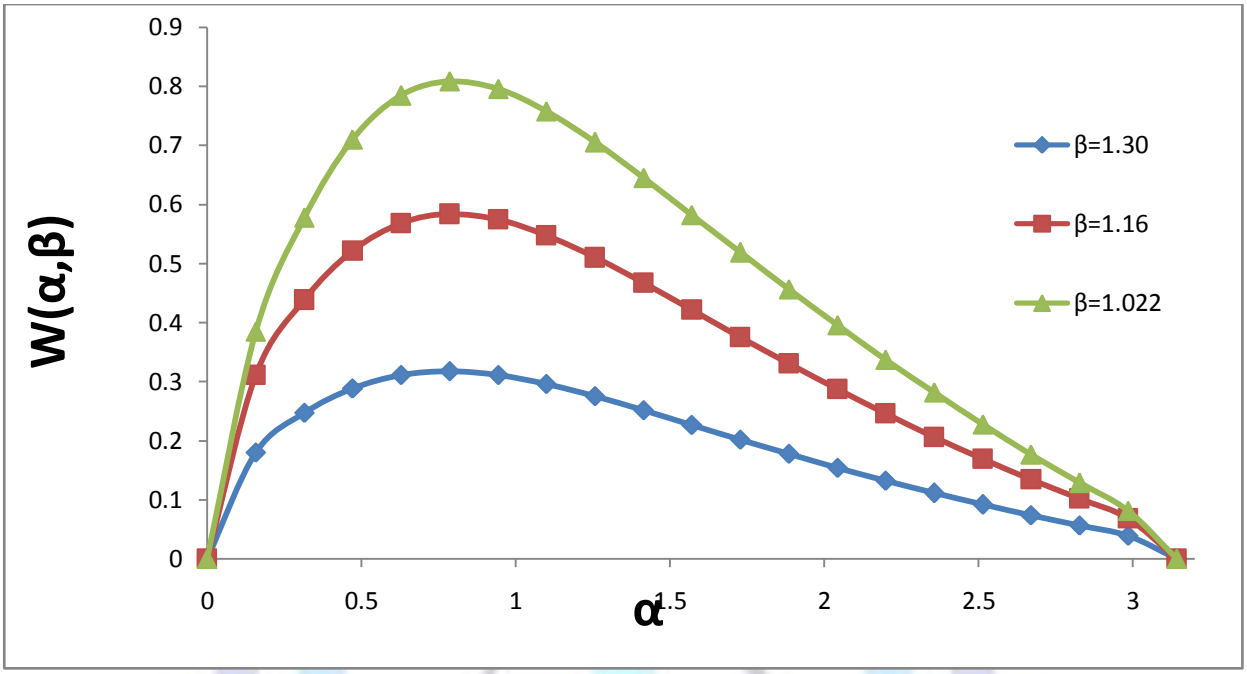

Fig.(2) velocity field W (α, β) versus α at different values of β when the outer sphere rotate with angular velocity Ω and the inner sphere at rest for radii are $R_1 = 2cm, R_2 = 4cm$ and the distance between two poles c=4cm.

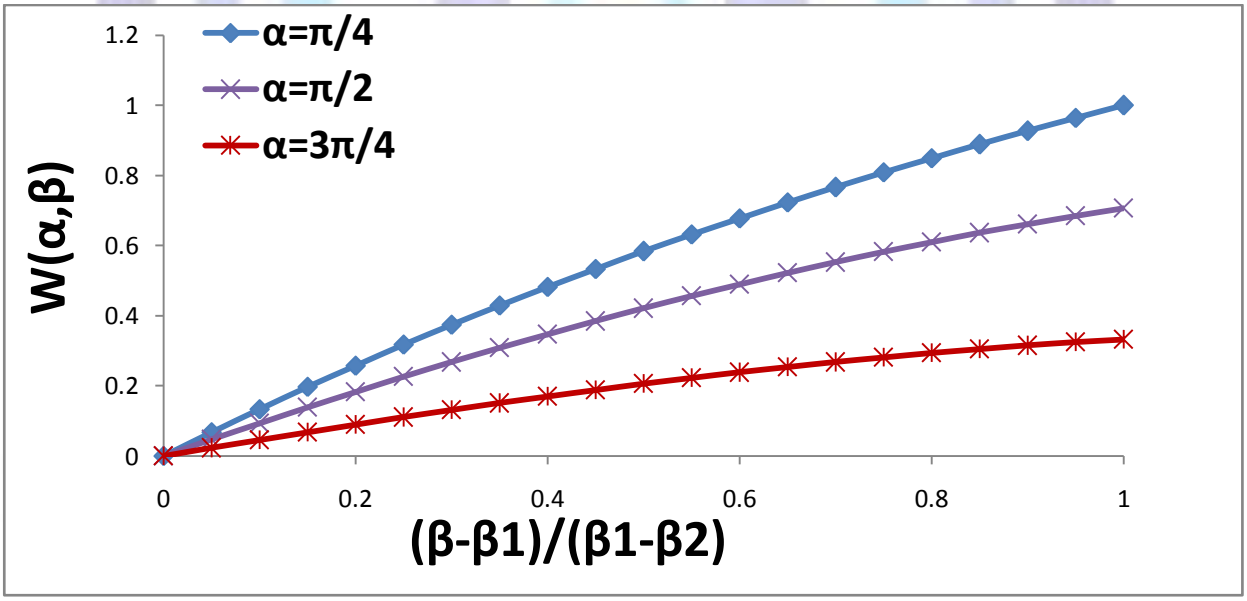

Fig.(3) velocity field W (α, β) versus $(\beta - \beta_1)/\beta_1 - \beta_2$) at different values of α when the outer sphere rotate with angular velocity Ω and the inner sphere at rest for radii are $R_1 = 2cm, R_2 = 4cm$ and distance between two poles c=4cm.

Case(2):In contrast to case(1) the outer sphere β_2 is being at rest ;i.e, $\Omega_2 = 0$ and the inner sphere β_1 rotates with angular velocity Ω , Fig.(4)show that the distribution velocity field W (α, β) behaves as the same manner as in case (1) with α while it is inversely proportional to β =const. ;Fig.(4),while Fig.(5) shows the velocity field W (α, β) versus β at different values of α .

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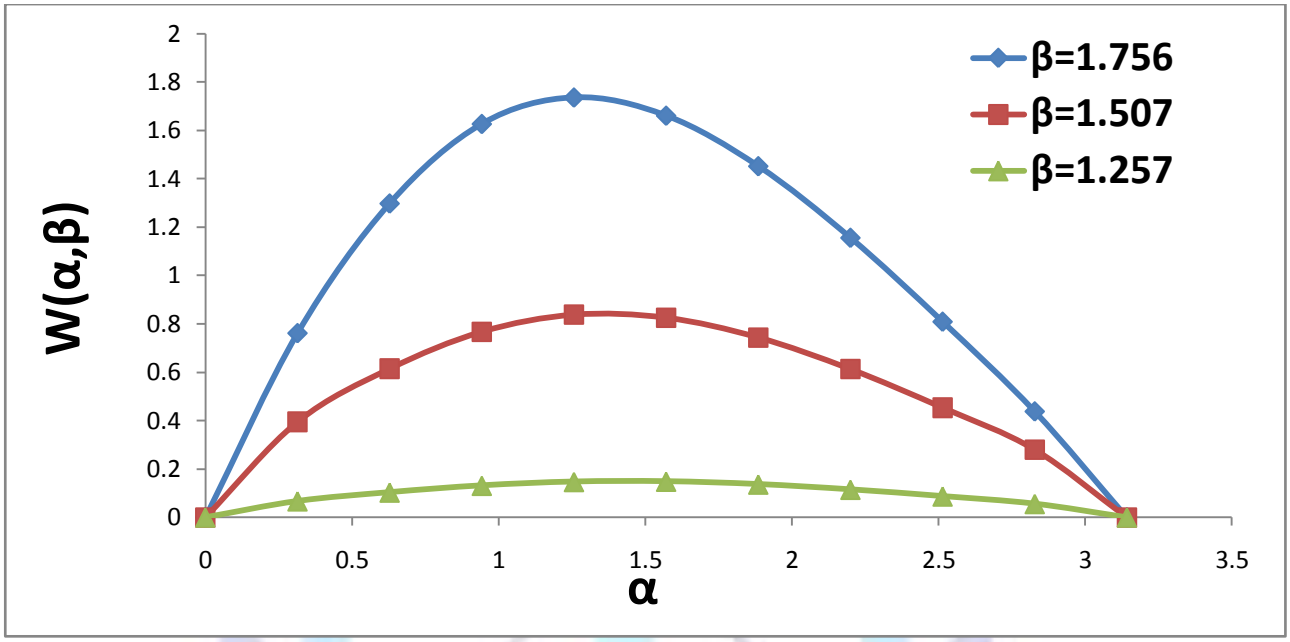

Fig.(4) velocity field W (α, β) versus α at different values of β when inner sphere rotate with angular velocity Ω and outer sphere at rest for radii are $R_1 = 2cm, R_2 = 4cm$ and distance between two poles c=4cm.

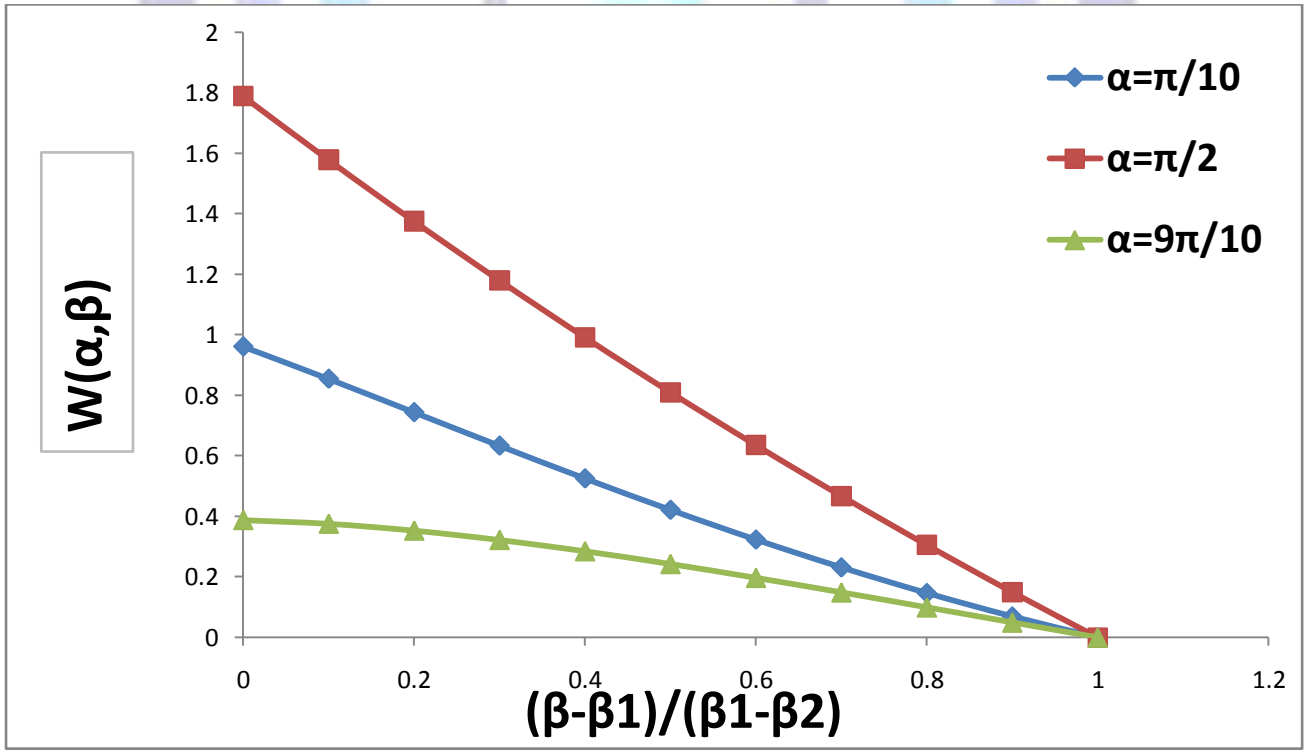

Fig.(5) velocity field W (α, β) versus $(\beta - \beta_1)/\beta_1 - \beta_2$) at different values of α when inner sphere rotate with angular velocity Ω and outer sphere at rest for radii are $R_1 = 2cm, R_2 = 4cm$ and distance between two poles c=4cm.

Case(3):If the two spheres β_1 and β_2 are rotating with the same angular velocity but with opposite directions;i.e, $\Omega_2 = -\Omega_1$, Fig.(6) ,while Fig.(7) shows the velocity field W (α, β) versus $(\beta - \beta_1)/\beta_1 - \beta_2$) at different values of α .

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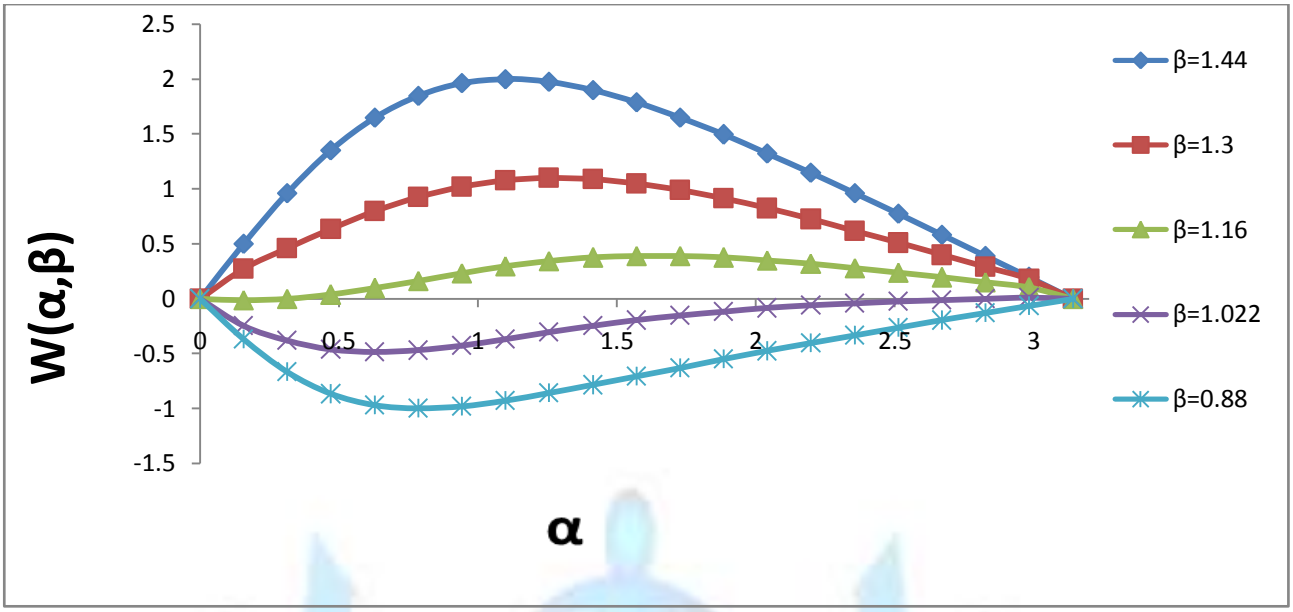

Fig.(6) velocity field W (α, β) versus α at different values of β when the inner sphere rotate with angular velocity Ω_1 and the outer sphere $\Omega_2 = -\Omega_1$ for radii are $R_1 = 2cm, R_2 = 4cm$ and distance between two poles c=4cm.

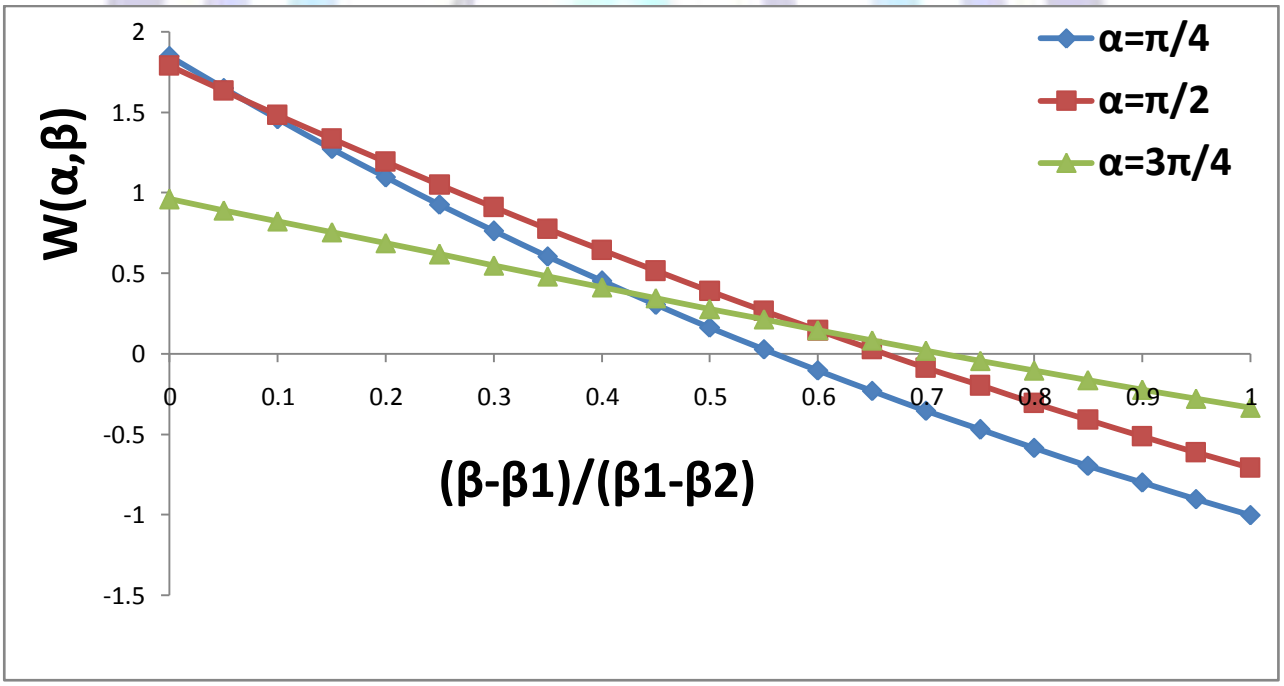

Fig.(7) velocity field W (α, β) versus $(\beta - \beta_1)/\beta_1 - \beta_2$) at different values of α when inner sphere rotate with angular velocity Ω_1 and outer sphere $\Omega_2 = -\Omega_1$ for radii are $R_1 = 2cm, R_2 = 4cm$ and distance between two poles c=4cm.

Surface tractions

We have two components of normalized surface tractions S_β and S_φ Fig.(8)show the distribution of normalized surface traction S_β versus α at different values of radii of inner and outer spheres, while Fig(9)show the normalized surface traction S_φ versus α at different values of radii of inner and outer spheres.

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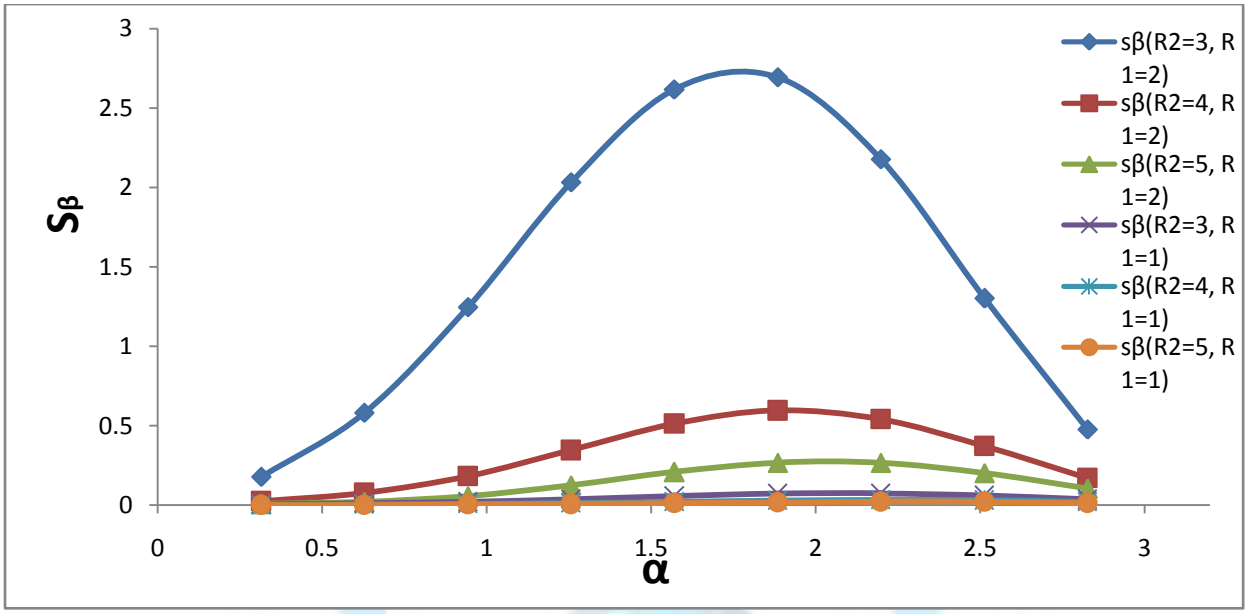
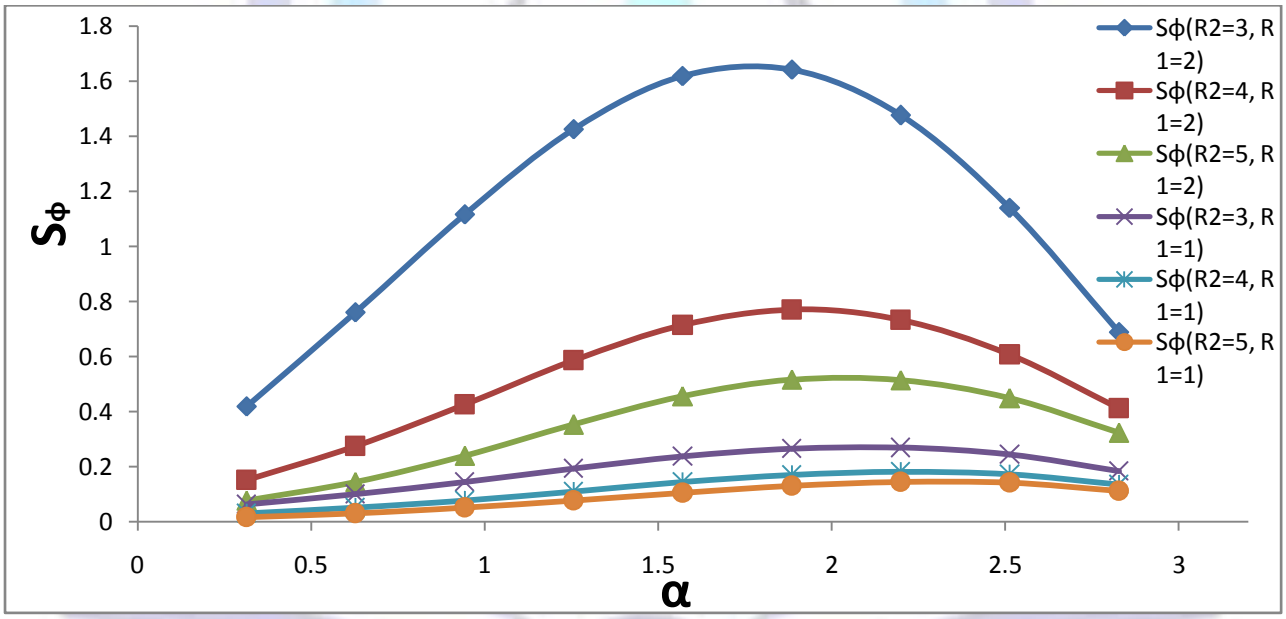


Fig(8)The normalized surface traction S_β versus α at different values of radii of inner and outer spheres



Fig(9)the normalized surface traction S_φ versus α at different values of radii of inner and outer spheres.

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