Proof of Renormalizability of Sclar Field Theories Using the Epstein-Glaser Scheme and Techniques of Microlocal Analysis

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ABSTRACT

The renormalizability of QFT's is a vastly studied issue, and particularly the results concerning a scalar field theory are well-known through the traditional renormalization approach in the literature. However, in this paper we analyze the problem through a less known approach, which justifies in a more rigorous and mathematically neat manner, the heuristic arguments of standard treatments of divergencies in QFT's. This paper analyzes the renormalizability of an arbitrary scalar field theory with interaction Lagrangean $L(x) =: \varphi^m(x):$ using the method of Epstein-Glaser and techniques of microlocal analysis, in particular, the concept of scaling degree of a distribution. For a renormalizability proof of perturbative models in the Epstein-Glaser scheme one first needs to define an n-fold product of sub-Wick monomials of the interaction Lagrangean. This time ordering is an operator-valued distribution on $\mathbb{R}^{4n}$ and the basic issue is its ill-definedness on a null set. The renormalization of a theory in this scheme amounts to the problem of extension of distributions across null sets.

Keywords

Quantum Field Theory, Renormalization, Epstein-Glaser scheme, Microlocal analysis, scalar field theory, scaling degree, time-ordered product.

1 Introduction

The problem of infinities is already present in classical electromagnetism. It is a known fact that if one tries to calculate the electron self-energy due to its electric field considering the electron as a point-like particle, one is left with an integral which diverges as the radius of the electron goes to zero, i.e. $r \to 0$. Unfortunately[1] , as first noted by Heisenberg and Pauli [4, 6], the problem of infinities emerged even more drastically in Quantum Field Theory. Analagously to the classical case, in Quantum Electrodynamics (QED) there is a problem of divergence that arises when one tries to calculate perturbatively the electron and photon self-energies, for instance. The Renormalization Theory is a process devised initially as a way to eliminate these infinities arising in QED and other QFT's [5, 7, 10, 9, 8, 13, 11, 12, 15, 14, 16, 17]. It states essentially that the parameters appearing in the Lagrangean of the theory have no direct physical interpretation, and one can adjust them in order to absorb the divergencies coming out in the perturbative calculations at each order in the perturbative expansion. A physical interpretation of this redefinition of parameters can be motivated by the following example: Suppose an electron is moving in the interior of a solid substance. Due to its interaction with the internal structure of the substance, the electron's effective mass m^* , which determines its response to the application of an external force, is certainly different from the electron's mass m measured outside of the solid. The electron's mass is changed (renormalized) from m to m^* due to the interaction between the internal structure of the material and the electron. Along the same lines, it may be interpreted that the infinite electron's "self-mass" is produced by interaction of the electron with its own virtual photon cloud. The infinities that have to be renormalized in QFT come from the short distance, or equivalently large momenta, behaviour of the Feynman propagators, and are thus called ultraviolet divergencies.

From almost a century of careful research in QFT, a lot more has been completely understood about renormalizability. However, the main approach to renormalization is still through a redefinition of "infinite parameters" in the Lagrangean of the theory. Yet successful and perfectly understandable, a more rigorous way of dealing with the delicacy of the problem of divergencies in QFT has been highly desirable since its emergence. In the words of Dirac himself (1975) [18]: "Most physicists are very satisfied with the situation. They say: 'Quantum electrodynamics is a good theory and we do not have to worry about it any more.' I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small not neglecting it just because it is infinitely great and you do not want it!"

One important part of renormalization is the process of regularization, which roughly speaking consists in a method for handling divergent integrals by the introduction of a suitable parameter called regulator. The regularization essentially separates divergent integrals into finite and infinite parts, these last ones are ultimately absorbed in the redefinition of parameters with renormalization. Although, many regularization methods have been quite successful, there are ambiguities associated to the extracting of the finite part of a divergent integral [22, 23, 24], which is natural since even the more fundamental axioms of arithmetic are only applied to numbers. However, since every physical process in perturbative quantum field theory (PQFT) is expressed in terms of Feynman propagators, which are not functions but

distributions, we may expect that the most resoanable means to treat Feynman amplitudes is by considering its distributional character and applying the powerful theory of distributions.

The first step towards a more rigorous treatment of PQFT is due to H. Epstein and V. J. Glaser [3]. The perturbative construction of interaction models a la Epstein and Glaser depends fundamentally on time ordered products of sub-Wick polynomials W_k of the interaction Lagrangean L, denoted by $T_n(W_1, W_2, ..., W_n)$. These are operator-valued distributions on $\mathbb{R}^{4n}$ acting on the domain $\mathcal{D}$ of vectors with finite particle number and smooth momentum space wave functions. The T-product is a well-defined distribution outside the subset of $\mathbb{R}^{4n}$ consisting of all coincidental points, called the total diagonal $\Delta_n \doteq \{(x_1, ..., x_n) : x_1 = \cdots = x_n\}$. The problem of renormalization in the Epstein-Glaser scheme corresponds to the extension of the T - product across Δ_n. This extension is, in general not unique, depending on the scaling degree[2] of the distribution involved. However, the distribution is unique up to derivatives of delta distribution (see 4.9). Each derivative of the delta is multiplied by an arbitrary constant. The redefinition of the parameters in standard Renormalization theory corresponds to the fixing of these constants, which are realized by requiring the maintenance of physical principles such as conservation of energy and momentum. Having constructed the T-product, the Bogoliubov's S-matrix is defined as a functional associating with a test function $g \in \mathcal{D}(\mathbb{R}^4)$ and an interaction Lagrangean L the following series

$$S(gL) \doteq \sum_{n=0}^{\infty} \frac{i^n}{n!} T_n(L, L, ..., L)(g, g, ..., g)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n g(x_1) \cdots g(x_n) T_n(L(x_1), \cdots, L(x_n)), \tag{1.1}$$

in the formal sense, i.e. without requiring the convergence of the series. In the so called adiabatic limit, $g(x) \to g = constant$, equation (1.1) becomes the S-matrix of the model.

A very general treatment of Renormalization of QFT's using the method of Epstein-Glaser and techniques of microlocal analysis is exposed in a monumental work by K. Fredenhagen and R. Brunetti [1]. However, in this work, we give a simpler and more detailed analysis restricted to the four dimensional Minkowski spacetime. In particular, we prove the renormalizability of an arbitrary scalar field theory with no derivatives and interaction Lagrangean $L(x) =: \varphi^m(x):$ through the calculation of the scaling degree of the distribution associated to an arbitrary n-fold time-ordered product of sub-Wick monomials of the interaction lagrangean.

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[1] Or fortunately, if we consider the vast amount of development in mathematics and in the fundamentals of physics that sprang from the paradigm of infinities in Quantum Field Theory.

[2] For the definition of scaling degree see 4.3.

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2 Renormalizability of $L(x) =: \varphi^m(x)$:

Let φ be a point-like operator-valued field, then the time-ordering of the product $\varphi(x')\varphi(x)$ is given by[3]

$$T\varphi(x)\varphi(x') = \begin{cases} \varphi(x)\varphi(x') \ if \ x \succcurlyeq x' \\ \varphi(x')\varphi(x) \ if \ x' \succcurlyeq x \end{cases}. \tag{2.1}$$

where $x \succcurlyeq y$ means, by definition, that x is not contained in the past light-cone of y, i.e. $x \succcurlyeq y :\Leftrightarrow x \notin \overline{V_-(y)}$. We may read the symbol "$\succcurlyeq$" as "later than". Furthermore, if the points x and y are space-like separated, then both $x \succcurlyeq y$ and $y \succcurlyeq x$ hold, but even so, by the causality axiom, which asserts the commutativity of fields for space-like separated events, there is no ambiguity to equation (2.1), hence equation (2.1) is well-defined for every pair of points $(x, x') \in \mathbb{R}^8 \setminus \Delta_2$, where $\Delta_2 \doteq \{(x, x') \in \mathbb{R}^8 : x = x'\}$. Thus, the T-product is Lorentz invariant and hence well-defined for non-coincidental points.

Using Wick's expansion theorem, we may rewrite (2.1) as

$$T\varphi(x)\varphi(x') = :\varphi(x)\varphi(x'): +(\Omega, T\varphi(x)\varphi(x')\Omega), \tag{2.2}$$

where the double colon stands for the normal order relation and $(\Omega, T\varphi(x)\varphi(x')\Omega) := \Delta_F(x - x')$ is the Feynman propagator associated with φ. The case in which x coincides with x' is reasonably treated by extending the numerical distribution given by the Feynman propagator in equation (2.2) across the diagonal $\Delta_2 \doteq \{(x, x') \in \mathbb{R}^8 : x = x'\}$. This last step is where the subtleties actually lie.

In order to study the time ordering of an arbitrary product of sub-Wick monomials of the interaction lagrangean one needs to know its fundamental properties. Let us first start with the following informal definition

$$T_n(W_1, \ldots, W_n)(f_1, \ldots, f_n) =: \int_{(\mathbb{R}^4)^{\times n}} dx_1 \cdots dx_n \, T_n(W_1(x_1) \cdots W(x_n)) f_1(x_1) \cdots f_n(x_n),$$

where $f_1, \cdots f_n \in \mathcal{D}(\mathbb{R}^4)$ and T_0 and T_1 are given by $T_0 := 1$ and $T_1(W) := W$. Further requirements are:

i. (Linearity.) The time-ordered product T_n is an n-linear application from the Wick polynomials into operator-valued distributions acting on $\mathcal{D}$[5].

ii. (Symmetry.) $T^n(W_1, \ldots, W_n)$ is symmetric under permutations $W_i \leftrightarrow W_k$.

iii. (Causality.) If $x_i \succcurlyeq x_j$ for all $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, n\}$, then the following factorization holds:

$$T_n(W_1(x_1) \cdots W_n(x_n)) = T_k(W_1(x_1) \cdots W_k(x_k)) \, T_{n-k}(W_{k+1}(x_{k+1}) \cdots W_n(x_n)).$$

iv. (Covariance.) If W_i are scalar Wick polynomials, then for all $(a, \Lambda) \in P_+^\uparrow$

$$U(a, \Lambda) T_n(W_1(x_1) \cdots W_n(x_n)) U(a, \Lambda)^{-1} = T_n(W_1(a + \Lambda x_1) \cdots W_n(a + \Lambda x_n)).$$

v. (Wick expansion – scalar field case.) Let $k_1, \ldots, k_n \in \mathbb{N}_0$ and let $\mathcal{G}(k_1, \ldots, k_n) \doteq \{0, \ldots, k_1\} \times \cdots \times \{0, \ldots, k_n\}$ be the set of multi-indices $(l_1, \ldots, l_n) \in \mathbb{N}_0^{\times n}$ with $0 \le l_i \le k_i$. Then

$$T_n(:\varphi^{k_1}(x_1): \cdots :\varphi^{k_n}(x_n):) = \sum_{G \in \mathcal{G}(k_1, \ldots, k_n)} t_G(x_1, \ldots, x_n) :\varphi^{l_1}(x_1) \cdots \varphi^{l_n}(x_n):, \tag{2.3}$$

where for $G = (l_1, \ldots, l_n)$, t_G is the numerical distribution

$$t_G(x_1, \ldots, x_n) = \binom{k_1}{l_1} \cdots \binom{k_n}{l_n} (\Omega, T_n :\varphi^{k_1-l_1}(x_1): \cdots :\varphi^{k_n-l_n}(x_n): \Omega).$$

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(The product of distributions in Eq. (2.3) exists due to Epstein-Glaser's "Theorem 0"[3].) The time-ordered products can be constructed inductively: If all T_k are known up to some order $k \leq n-1$, then T_n is fixed [3, 1] by causality and symmetry outside the small diagonal Δ_n. Let us prove this claim as a theorem:

Theorem 2.1 Let L be a particular functional lagrangean and denote by $T_n(x_1,...,x_n) \equiv TL(x_1)\cdots L(x_n),\ n \in \mathbb{N}$, the family of symmetric operator-valued distributions on $\mathbb{R}^{4n}$ satisfying the causality property. Then equation

$$T_n(x_1,...,x_n) = T_k(x_1,...,x_k)T_{n-k}(x_{k+1},...,x_n). \tag{2.4}$$

fixes T_n inductively up to the total diagonal $\Delta_n \doteq \{(x_1,...,x_n): x_1 = \cdots = x_n\}$ in the following sense: If T_k is known for every $k < n$, then T_n is fixed on $\mathbb{R}^{4n} \setminus \Delta_n$.

Proof. Suppose $(x_1,...,x_n) \notin \Delta_n$, then there are at least two points in the set $\{x_1, x_2,..., x_n\}$ and hence there exists a partition $P = \{I, I^c\}$ of the set $\{1,...,n\}$ such that $x_i \succeq x_{i'}\ \forall i \in I$ and $\forall i' \in I^c$. Thus, the causality property of the T-product implies

$$T_n(\{1,...,n\}) = T(I)T(I^c) \tag{2.5}$$

where $T(I) := T(x_{i_1},...,x_{i_k})$, $I = \{i_1,...,i_k\}$ and $k = |I|$. If the points x_i are all causally connected, that is $x_i^2 \geq 0$ $\forall i \in \{1,...,n\}$, then I is uniquely defined ($x_{i_1} \succeq x_{i_2} \succeq \cdots \succeq x_{i_n}$ for all reference frames, where $\{i_1,...,i_n\} = \{1,...,n\}$) and we are done. If, on the other hand, $\exists l \in \{1,...,n\}$ such that $x_l^2 < 0$, (that is, x_l is space-like) then I is not uniquely defined, and in this case we must prove the independence of equation (2.5) on the choice of the proper subset I. With that purpose in mind, let $Q = \{J, J^c\}$ be another partition of the set $\{1,...,n\}$ such that $x_j \succeq x_{j'}\ \forall j \in J$ and $\forall j' \in J^c$ Using the identities $I = I \cap J \,\dot\cup\, I \cap J^c$ and $I^c = I^c \cap J \,\dot\cup\, I^c \cap J^c$, we can rewrite the right-hand side of equation (2.5) as

$$T(I)T(I^c) = T\left(I \cap J \,\dot\cup\, I \cap J^c\right)T\left(I^c \cap J \,\dot\cup\, I^c \cap J^c\right). \tag{2.6}$$

It is easy to see that $I \cap J \,\dot\succeq\, I \cap J^c$ and $I^c \cap J \,\dot\succeq\, I^c \cap J^c$ (in the sense of the elements of the corresponding sets) and so (2.6) becomes

$$T(I)T(I^c) = T(I \cap J)T(I \cap J^c)T(I^c \cap J)T(I^c \cap J^c). \tag{2.7}$$

Analagously, for the sets J and J^c we have the following identities $J = J \cap I \,\dot\cup\, J \cap I^c$ and $J^c = J^c \cap I \,\dot\cup\, J^c \cap I^c$, yield

$$T(J)T(J^c) = T\left(J \cap I \,\dot\cup\, J \cap I^c\right)T\left(J^c \cap I \,\dot\cup\, J^c \cap I^c\right), \tag{2.8}$$

and since both $J \cap I \,\dot\succeq\, J \cap I^c$ and $J^c \cap I \,\dot\succeq\, J^c \cap I^c$, we have

$$T(J)T(J^c) = T(J \cap I)T(J \cap I^c)T(J^c \cap I)T(J^c \cap I^c). \tag{2.9}$$

It is clear that the arguments of the second and third terms in equation (2.9) are such that $J \cap I^c \succeq J^c \cap I$ from the perspective of the partition Q. However, from the perspective of the partition P we also have $J^c \cap I \succeq J \cap I^c$. This means that the points in $J^c \cap I$ are causally disjoint from the points in $J \cap I^c$, which enables us to commute the terms $T(J \cap I^c)$ and $T(J^c \cap I)$ in (2.9) obtaining as a result

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$$T(J)T(J^c) = T(J \cap I)T(J^c \cap I)T(J \cap I^c)T(J^c \cap I^c) = T(I)T(I^c) \tag{2.10}$$

From the previous theorem we know that the time-ordering of an n-fold product of any point-like interaction lagrangean $L(x)$ is well-defined everywhere, except in the total (small) diagonal Δ_n.

The "UV problem" of divergencies consists, in this context, in the extension of the T-product across Δ_n. This extension is not unique in general, and the choice of possible extensions is restricted by the requirements of covariance and Wick expansion for the T-product, which therefore may be called (re-) normalization conditions. By covariance, the T_n are fixed up to the origin in $\mathbb{R}^{4n}$. The condition of Wick expansion holds due to Wick's theorem outside the union of all diagonals, i.e., whenever $x_i \neq x_j$, and the requirement that it holds on all $\mathbb{R}^{4n}$ is a normalization condition by virtue of which the extension problem needs to be considered only for the numerical distributions t_G.

Let us now investigate the renormalizability of a scalar theory whose interaction lagrangean is of the form $L(x) =: \varphi^m(x):$. In this case, the Wick expansion (2.3) for n vertices is given by

$$TL(x_1) \cdots L(x_n) = \sum_{G \in \mathcal{G}(m,\ldots,m)} t_G(x_1,\ldots,x_n): \varphi^{b_1}(x_1) \cdots \varphi^{b_n}(x_n):, \tag{2.11}$$

where for $G = (b_1,\ldots,b_n)$, t_G is the numerical distribution

$$t_G(x_1,\ldots,x_n) = \binom{m}{b_1} \cdots \binom{m}{b_n}(\Omega, T_n: \varphi^{m-b_1}(x_1): \cdots : \varphi^{m-b_n}(x_n):\Omega). \tag{2.12}$$

Applying Wick's expansion to both sides of equation (2.5) we get

$$\sum_{G \in \mathcal{G}_n} t_G(x_1, \cdots, x_n): \varphi^{b_1}(x_1) \cdots \varphi^{b_n}(x_n):$$

$$= \sum_{G_1 \in \mathcal{G}_k} \sum_{G_2 \in \mathcal{G}_{n-k}} t_{G_1}(I) t_{G_2}(I^c): \varphi^{b_{i_1}}(x_{i_1}) \cdots \varphi^{b_{i_k}}(x_{i_k})::\varphi^{b_{i_{k+1}}}(x_{i_{k+1}}) \cdots \varphi^{b_{i_n}}(x_{i_n}):$$

$$= \sum_{G_1, G_2} t_{G_1}(I) t_{G_2}(I^c): \varphi^{b_I}(I)::\varphi^{b_{I^c}}(I^c): \tag{2.13}$$

where $\mathcal{G}_n$ is the set of graphs with n vertices $x_1, \cdots, x_n$ and with b_i external lines coming out of the vertex x_i. The set $\{I, I^c\}$ is a partition of the set $\{x_1, \cdots, x_n\}$ with $I = \{x_{i_1}, \cdots, x_{i_k}\}$ and $I^c = \{x_1, \cdots, x_n\} \setminus I = \{x_{i_{k+1}}, \cdots, x_{i_n}\}$. Also, we have defined $\varphi^{b_{i_1}}(x_{i_1}) \cdots \varphi^{b_{i_k}}(x_{i_k}) \overset{def}{=} \varphi^{b_I}(I)$, where $b_I = b_{i_1} + \cdots + b_{i_k}$ and $b_{I^c} = b_{i_{k+1}} + \cdots b_{i_n}$ are the total number of external lines coming out of the I and I^c vertices, respectively, in the graph G.

Using Wick's expansion for the product of the normal ordered terms in equation (2.13) we get

$$\sum_G t_G(x_1, \cdots, x_n): \varphi^{b_1}(x_1) \cdots \varphi^{b_n}(x_n):$$

$$= \sum_{G_0, G_1, G_2} t_{G_1}(I) t_{G_2}(I^c) \prod_{(i,j) \in I \times I^c} [\Delta_F(x_i - x_j)]^{l_{ij}}: \varphi^{b'_I}(I) \varphi^{b'_{I^c}}(I^c): \tag{2.14}$$

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where G_0 runs through the graphs whose internal lines connect only vertices in I with vertices in I^c, and $b := b_I + b_{I^c} = b'_I + b'_{I^c} - 2l_{I,I^c}$, where $l_{I,I^c} = \sum l_{ij}$ is the number of lines connecting vertices in I with vertices in I^c, as ilustrated in figures 1 and 2. The term Δ_F represents the Feynman propagator[4] given by equation (4.7) in appendix. We may now calculate the scaling degree (definition 4.3 in appendix 4) of the distribution t_G by induction:
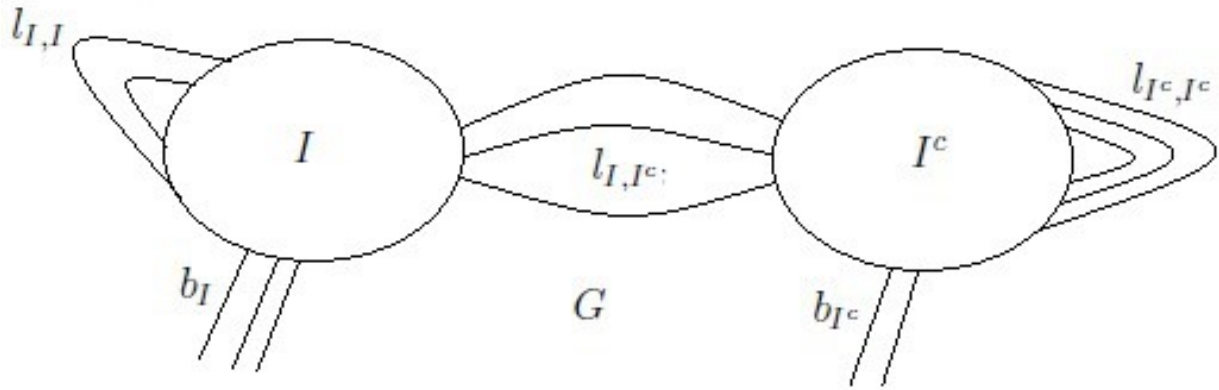

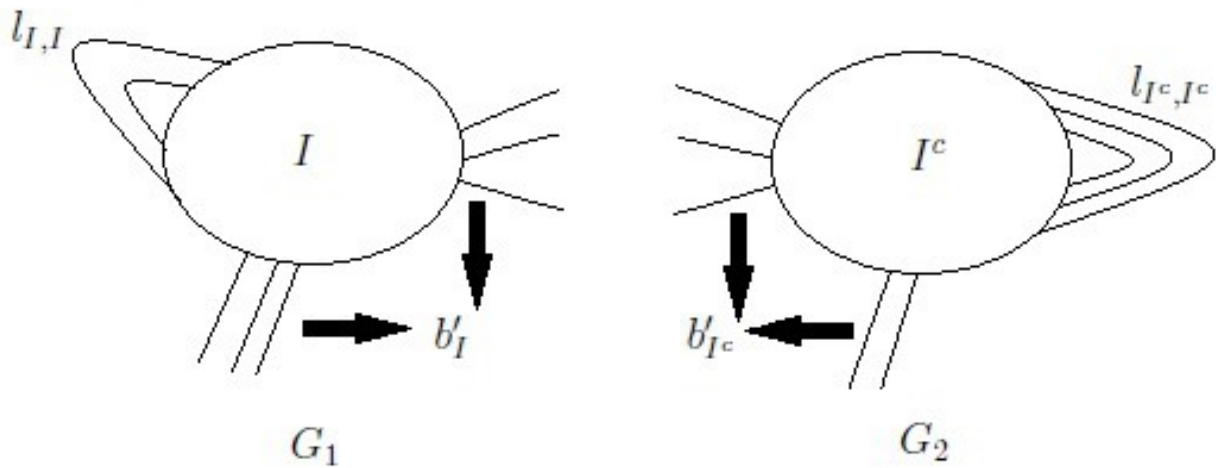Figure 1: Graph G with n vertices $\{x_1, \cdots, x_n\} = I \cup I^c$ and $b_I + b_{I^c}$ external lines.

Figure 2: Dichotomy of the graph G into G_1 and G_2, where graph G_1 has $k = |I|$ vertices and b'_I external lines and graph G_2 has $n - k = |I^c|$ vertices and b'_{I^c} external lines.

Theorem 2.2 For $G \in \mathcal{G}^b_n$, that is, G has n vertices and b external lines, the distribution t_G defined in (2.12) has scaling degree $mn - b$.

Proof. From equation (2.14) we get

$$\mathrm{sd}(t_G) = \mathrm{sd}(t_{G_1}) + \mathrm{sd}(t_{G_2}) + \mathrm{sd}(\Delta_F)\sum l_{ij} \qquad (2.15)$$

where we have used the properties of scaling degree stated in 4.8. The graph G_1 has k vertices and b'_I external lines whilst G_2 has $n - k$ vertices and b'_{I^c} external lines. Hence, using the induction hypothesis for t_{G_1} and t_{G_2} and the result $sd(\Delta_F) = 2$ (obtained in example 4.7 in appendix 4), we get

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$$\mathrm{sd}(t_G) = [mk - b'_I] + [m(n-k) - b'_{I^c}] + 2l_{I,I^c} \tag{2.16}$$

$$= mn - (b'_I + b'_{I^c}) + 2l_{I,I^c} \tag{2.17}$$

$$= mn - b \tag{2.18}$$

To prove the basis step, note that for $n = 1$ we have $t_G(x_1) = 1$ with scaling degree 0 and $b = m$, hence $\mathrm{sd}(t_G) = 0 = m - b$. This completes the proof.

We may now calculate the degree of divergence[6] of the distribution t_G by

$$\mathrm{div}(t_G) := \omega = \mathrm{sd}(t_G) - \mathrm{codim}(\Delta_n) \tag{2.19}$$

where $\mathrm{codim}(\Delta_n) = \dim(\mathbb{R}^{4n}) - \dim(\Delta_n) = 4(n-1)$. Therefore,

$$\omega = (m-4)n + 4 - b \tag{2.20}$$

We may now analyze the renormalizability of a scalar theory with interaction lagrangean of the form $L(x) =: \varphi^m(x):$. By inspecting equation (2.20) we notice that if $m > 4$, the degree of divergence will increase with the number of vertices no matter how large the number of external lines is, which makes the theory non-renormalizable in this case. As an example, for $m = 4$ we have $\omega = 4 - b$ and only graphs with $n = 2$ and $n = 4$ will be superficially divergent, those with $0 \le b \le 4$. Therefore, the theory is renormalizable, since only a finite number of physical constants has to be redefined. The redefinition of physical parameters in standard renormalization schemes corresponds to the non-unique extension of distributions in the Epstein-Glaser scheme. As stated in theorem 4.9 in appendix 4, for $\omega \ge 0$ the extension of the distribution through the origin is unique up to derivatives of the delta distribution and each term contains an undefined constant which can be fixed by the imposition of physical principles, such as conservation of energy and momentum. The larger ω is, the larger will be the number of parameters to be fixed. Furthermore, if $\omega < 0$ the extension is unique.

[3] We consider the case where the field φ is bosonic. In the case it were fermionic, the second line in equation (2.2) would be $-\varphi(x)\varphi(x')$

[4] For the explicit form of the Feynman propagator for scalar fields see 4.7.

[5] We adopt here the "on-shell formalism".

[6] See the definition of degree of divergence in the last paragraph of appendix 4

3 Final comments

In this work, we discussed the renormalizability of a scalar field theory $L(x) =: \varphi^m(x):$, which is a well-known fact in the traditional literature in QFT. However, whilst the traditional approach is realized in a more intuitive and less rigorous way, we have used the more mathematically rigorous approach of Epstein-Glaser with a slight modification extensively treated by K. Fredenhagen et al, which uses techniques of microlocal analysis for treating the extension problem, instead of the more arduous and elaborate technique of splitting of distributions.

We have constructed the time-ordered products of sub-Wick monomials of the interaction Lagrangean. Those T-products were proved by induction in theorem 2.1 to be well-defined distributions outside the meager set of all coincidental points in $(\mathbb{R}^4)^{\times n}$, which is a null subset Δ_n. Later we classified the degree of freedom of the general T-product by calculating its scaling degree, which depended on the power of the field m in the interaction, the number of vertices n, and the total number of external lines in the corresponding graph G associated to the distribution t_G. Finally, the degree of divergence of the theory was shown to be $\omega = (m-4)n + 4 - b$, which emphasizes precisely the non-renormalizability of a scalar field theory with power $m > 4$ in the interaction Lagrangean, since in this case the degree of

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divergence will inevitably increase with the number of vertices of a grapgh G, independently on how large the number of external lines is.

The Epstein-Glaser approach is very fruitful and has been gaining the attention of physicists even from outside the area of mathematical and theoretical physics. More recently, the method has been successfully implemented in the relatively new approach for QFT's using string-localized fields[7] instead of the usual point-localized ones [26].

[7] For the main work on this new approach see [25].

4 Basic Notions on the Extension of Distributions and Scaling Degree

As mentioned previously, the problem of ultraviolet divergence can be solved by the extension of certain distributions which in turn will account for the renormalization of the theory. In order to that one uses the notion of **scaling degree** of a distribution. In this section[8], we shall introduce some properties of the scaling degree and provide some simple examples. Let us start with the following definition:

Definition 4.1 Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$, then we define a **dilataion** Λ of the function φ through λ by

$$\Lambda : \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$$

$$(\lambda, \varphi) \mapsto \varphi^\lambda \doteq \lambda^{-n} \varphi(\lambda^{-1} \cdot) \tag{4.1}$$

By pullback[9], we can define a **dilatation** of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ as

$$(\Lambda^* u)(\varphi) \doteq u_\lambda(\varphi) \doteq u(\varphi^\lambda). \tag{4.2}$$

For the case of $u \in L^1_{loc}(\mathbb{R}^n)$ we can write equation (4.2) as the integral

$$u_\lambda(\varphi) = \int u(\lambda x) \varphi(x) d^n x, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{4.3}$$

The quantity $u(x)$ is referred to as the integral kernel of u and we shall, by the usual abuse of notation, denote a general distribution $u(\varphi)$ as the integral in (4.3).

Let $\mathcal{D}(\mathbb{R}^n \setminus \{0\}) = \{\varphi \in \mathcal{D}(\mathbb{R}^n) : 0 \notin supp(\varphi)\}$ be the subspace of test functions whose support does not contain the origin and $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ its dual[10]. With all that said, the central problem can be stated as follows:

Problem 4.2 Given a distribution $u_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, how can we construct a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $u_0(\varphi) = u(\varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$?

To answer that question one uses the concept of the so called scaling degree of a distribution, which basicly measures how singular a distribution is at the origin.

Definition 4.3 A distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ has **scaling degree** s (in symbols $\mathrm{sd}(u) = s$) with respect to the origin in $\mathbb{R}^n$ if

$$s \doteq \inf\{s' \in \mathbb{R} : \lambda^{s'} u_\lambda \xrightarrow{\lambda \to 0} 0\}. \tag{4.4}$$

Let us see some simple examples.

Example 4.4 Let $f \in C^0(\mathbb{R})$ with $f(0) \neq 0$, then

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$$\lim_{\lambda \downarrow 0} \lambda^{\omega} \int f(\lambda x) \varphi(x) dx = \begin{cases} 0, & \text{if } \omega > 0 \\ f(0) \int \varphi(x) dx, & \text{if } \omega = 0 \\ \infty, & \text{if } \omega > 0 \end{cases} \tag{4.5}$$

and consequently, $\mathrm{sd}(f) = 0$.

Example 4.5 Let $\delta \in \mathcal{D}'(\mathbb{R}^n)$ be the Dirac δ-function. Since $\delta(\lambda x) = \lambda^{-n} \delta(x)$, we have $\mathrm{sd}(\delta) = n$.

Example 4.6 The functions $f(x) = e^{-\frac{1}{x^2}}$ and $g(x) = e^{\frac{1}{x}}$ both define distributions on $\mathbb{R} \setminus \{0\}$. Thus,

$$\begin{cases} \lim_{\lambda \downarrow 0} \lambda^s f(\lambda x) = 0 & \forall s \in \mathbb{R} \Leftrightarrow \mathrm{sd}(f) = -\infty \\ \lim_{\lambda \downarrow 0} \lambda^s g(\lambda x) = \infty & \forall s \in \mathbb{R} \Leftrightarrow \mathrm{sd}(g) = \infty \end{cases} \tag{4.6}$$

Example 4.7 Consider the scalar Feynman propagator Δ_F of a massive scalar theory in four dimensions given by

$$\Delta_F(x, m) = \lim_{\varepsilon \downarrow 0} (2\pi)^{-4} \int d^4 p \frac{e^{-ip \cdot x}}{(p^2 - m^2 + i\varepsilon)}, \tag{4.7}$$

then

$$\Delta_F(\lambda x, m) = \lim_{\varepsilon \downarrow 0} (2\pi)^{-4} \int d^4 p \frac{e^{-ip \cdot \lambda x}}{(p^2 - m^2 + i\varepsilon)}$$

$$= \lim_{\varepsilon \downarrow 0} (2\pi)^{-4} \lambda^{-2} \int d^4 p \frac{e^{-ip \cdot x}}{(p^2 - (\lambda m)^2 + i\varepsilon)}$$

$$= \lambda^{-2} \Delta_F(x, \lambda m). \tag{4.8}$$

Since for $m \to 0$, $\Delta_F(x, m)$ converges to the massless scalar propagator of the theory, we have $\mathrm{sd}(\Delta_F) = 2$.

The following theorem highlights some important properties of the scaling degree. It will be presented without proof (see [1]).

Theorem 4.8 Let $t \in \mathcal{D}'(\mathbb{R}^n)$ have $\mathrm{sd}(t) = s$ at 0, then the scaling degree obeys the following properties:

- $\mathrm{sd}(\partial^{\alpha} t) \le s + |\alpha|$, where $\alpha \in \mathbb{N}^n$ is any multiindex,

- $\mathrm{sd}(x^{\alpha} t) \le s - |\alpha|$, where $\alpha \in \mathbb{N}^n$ is any multiindex,

- $\mathrm{sd}(ft) \le \mathrm{sd}(t)$, where $f \in \mathcal{E}^n(\mathbb{R}^n)$,

- $\mathrm{sd}(t_1 \otimes t_2) = \mathrm{sd}(t_1) + \mathrm{sd}(t_2)$, for $t_i \in \mathcal{D}'(\mathbb{R}^{n_i})$, $i = 1, 2$.

A precise answer to problem of uniqueness in the extension of distributions is given by the following theorem, which will be presented without proof[11].

Theorem 4.9 Let $u_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ [12].

1. If $\mathrm{sd}(u_0) < n$, then there exists a unique extension $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $\mathrm{sd}(u_0) = \mathrm{sd}(u)$.

2. If $n \leq \mathrm{sd}(u_0) < \infty$, then there exist several extensions $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $\mathrm{sd}(u_0) = \mathrm{sd}(u)$. Given a particular solution u_p, the most general solution reads

$$u = u_p + \sum_{|\alpha| \leq \mathrm{sd}(u_0)-n} c_\alpha \partial^\alpha \delta^{(n)} \tag{4.9}$$

with arbitray constants $c_\alpha \in \mathbb{C}$.

3. $\mathrm{sd}(u_0) = \infty$ If , then there exists no extension $u \in \mathcal{D}'(\mathbb{R}^n)$.

It is convinient to define the notion of **degree of divergence** of a distribution u, which is given by $\mathrm{div}(u) \doteq \mathrm{sd}(u) - n$. It is worth mentioning that the non-unique extension case is given by (4.9) due to the fact that the most general distribution supported at the origin is given by an arbitrary differential polynomial applied to the δ distribution, see example 3.1.2 in [21].

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[8] For more detailed description of the problem of extension of distributions consult [19, 20, 1].

[9] The pullback essencialy transfers the effect of an operation over the test functions to an operation over the distributions by a composition. That is, let X and Y be open sets, Λ be the mapping $\Lambda : \mathbb{R}_+ \times X \subseteq \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^n) \to Y \subseteq \mathcal{D}(\mathbb{R}^n)$ and u be the distribution $u : Y \subseteq \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$, then $u \circ \Lambda := \Lambda^* u : \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$.

[10] The extension exists by Hahn-Banach theorem, see [21, Chap. 3.2].

[11] For a detailed proof see [1].

[12] In the case the extension is not through the origin, but through an arbitrary submanifold D of $\mathbb{R}^n$ we must replace n in this theorem by the codimension of the submanifold, which is defined as $\mathrm{codim}(D) = \dim(\mathbb{R}^n) - \dim(D) = n - \dim(D)$.

Acknowledgments

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