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Nambu-Goto Strings via Weierstrass Representation.

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Abstract

A description of string model of gauge theory are related to minimal surfaces. notations of minimal surface and related mean and Gauss curvature discussed. The Weierstrass representation for a surface conformally which immersed in R used to represent Nambu-Goto action, action of Nambu-Goto is calculated using Weierstrass representation which can be used to calculate the Partition Function and potential, then a non-perturbative solution for action is aimed and fulfilled and a consequences of that are investigated and its mathematical and physical properties are discussed.

1. Introduction

The study of the geometric aspects and differential geometry treatment for properties of the surface of the world sheet in the pseudo-Riemannian and Mankowski spaces is very important to solve many mathematical problems. The Nambu-Goto string action is the most simple model, however, major diffculties arise in its treatment mainly due to its nonlinearity ,i.e its square root formula. First of all, we focus our study on the three-dimensional model, the geometry of immersed surfaces in R^3 is reviewed considering Enneper-Weierstrass representation of minimal surfaces. A different approaches has been introduced [11] [7],we show three dimensional Nambu-Goto string and show its equivalent in the minimal surface theory. At this point, a formula of linearized Numbu Goto action depending on Weierstrass parametrization can be established.

We restrict ourself to three dimensional case, it is most simple in hope to generalized many dimensional in future, in the beginning we revise some notations and concepts in the geometry of immersed surfaces in R^3 which shall be used later, and the will introduce Enneper-Weierstrass representation of minimal surfaces then we recall three dimensional Nambu-Goto string and see how it is equivalent to minimal surface theory. from these point it is ready to go further treatment to get a formula of linear of Numbu-Goto action depending on parametrization of Weierstrass. In this paper we introduce a new formula for Nambu Goto Action based on Weierstrass representation which introduced in surface of R^3 where can given by a linear equation solved according to boundary conditions, Extending this work give us abilities to express the Partition function of system of bosonic string to get potential function. In first section of this paper a brief notation about surface is introduced and in second section a Weierstrass representation and plateau problem is presented and definitions revisited, then in section four a Nambu Goto action and the Dirichlet's boundary conditions corresponding to fixed ends in the spatial directions (the static quark and anti-quark) finally the formula based on Weierstrass representation is calculated.

2 A brief notations about surface theory.

Let be $\sum$ an oriented two-dimensional connected Riemannian manifold X : $\sum \Rightarrow R^3$ isometric of $\sum$ into R^3, at any point of $\sum$ a basis for tangent plane is provided by $\partial_\alpha X^i$. The induced metric is giving by:

$$g_{\alpha\beta} = \partial_\alpha X \partial_\beta X \qquad (1)$$

The first fundamental form I of X is Riemannian metric on S defined as.

$$\mathrm{I}_p(\omega_1\omega_2) = \quad \langle d_pX(\omega_1), d_pX\rangle(\omega_2)\rangle \qquad for\ p \in S$$
$$\omega_1, \omega_2 \in T_pS \qquad (2)$$

Let N denotes a unit normal field to S that is to say $N: S \longrightarrow S^2 \quad \subset R^3$ and *N(p)* is orthogonal to $\langle d_pX(T_pS\,)$ for each $p \in S, S^2$ denotes the unit sphere centered at the origin in R^2. The second fundamental denoted by II is the field of bilinear symmetric forms defined as follows :

$$\mathrm{II}_p(\omega_1\omega_2)=\langle d_p N(\omega_1), d_p X\rangle(\omega_2)\rangle \qquad for\ p \in S$$
$$\omega_1, \omega_2 \in T_p S \qquad (3)$$

The shape operator ***B*** of the immersion ***X*** in the field symmetric endomorphism defined by

$$\mathrm{II}_p(\omega_1\omega_2) = \mathrm{I}_p(\omega_1\omega_2) \qquad (4)$$

The mean curvature ***H*** of the immersion ***X*** and its Gaussian curvature ***K*** are defined as follows [12] :

$$\boldsymbol{H} = \frac{1}{2} tr(\boldsymbol{B}) \quad \boldsymbol{K} = det(\boldsymbol{B}) \qquad (5)$$

Proposition .1 *Let $\boldsymbol{X}: \boldsymbol{S} \longrightarrow \boldsymbol{R^3}$ be an immersed connected orientable surface in the Euclidean space, then its Gauss map :*

*$\boldsymbol{N}: \boldsymbol{S} \longrightarrow \boldsymbol{R^3}$ is almost conformal if and only if **X** is minimal or **X(S)** is a subset of a round sphere.*

Definition 1 *An immersed orientable surface in $\boldsymbol{R^3}$ is called minimal if its mean curvature is identically zero i.e **H = 0**.*

3 Generalized Weierstrass representation.

If the surface is expressed basically as a graph x; y; $h(x;\ y)$, (x; y) $\in \Omega$ of some function $h(x;\ y): \longrightarrow \boldsymbol{R^3}$, then the minimality means the function *h* satisfies in Ω the minimal surface equation;

$$(1 + \mathrm{h}_\mathrm{y}^2)\mathrm{h}_{\mathrm{xx}} - 2\,\mathrm{h}_\mathrm{x}\,\mathrm{h}_\mathrm{y}\mathrm{h}_{\mathrm{xy}} + (1 + \mathrm{h}_\mathrm{x}^2)\mathrm{h}_{\mathrm{yy}} = 0 \qquad (6)$$

then may be rewritten in follows:

$$div\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) \qquad (7)$$

This is a nonlinear elliptic partial differential equation, a natural problem is to solve the Dirichelet problem for the minimal surface equation [11], the problem has a solution for any boundary data if and only if domain is convex.

3.1 Weierstrass representation.

Let $\boldsymbol{X}: \boldsymbol{S} \longrightarrow \boldsymbol{R^3}$ be an immersed surface in Euclidean space **S** is thus endowed with the induced metric $\boldsymbol{dS^2}$. Recall the classical result that around each point of S we can find conformal coordinates which metric $S(u,v)$

$$\boldsymbol{dS^2} = \lambda^2(du^2 + dv^2) \qquad (8)$$

Proposition 2

Let $\boldsymbol{X}: \boldsymbol{S} \longrightarrow \boldsymbol{R^3}$ be an immersed connected surface oriented by a unit normal ***N*** and let (u,v) be local conformal coordinates on ***S***, then there coordinates:

$$\triangle \boldsymbol{X} = 2\boldsymbol{H}\lambda^2\ \mathbf{N} \qquad (9)$$

In particular, the surface is minimal if and only if its coordinates functions x_1; x_2; x_3 are harmonic in any conformal coordinates.

Proposition 3

Let $\boldsymbol{X}: (\underline{u}, v) \longrightarrow \boldsymbol{R^3}$ define a minimal immersion with (u, v) conformal coordinates, then the function

$$\Phi_i(z) = \frac{\partial x_i}{\partial u} - i\frac{\partial x_i}{\partial v}$$

$$\sum_i^3 {\Phi_i}^2 = 0 \qquad (10)$$

given $\Phi_1(z), \Phi_2(z), \Phi_3(z)$ analytical satisfy the precious condition in a simply connected domain $\boldsymbol{D} \subset \boldsymbol{C}$ then

$$\boldsymbol{X}(z) = Re \int_{z_o}^{z} (\Phi_1(z), \Phi_2(z), \Phi_3(z)\,)\, \mathrm{dz} \qquad (11)$$

where $z_o \in \boldsymbol{D}$ is any fixed point defines a conformal minimal immersion satisfying eq.[11]

Definition 2

$$g = \frac{\Phi_3}{\Phi_1 - i\Phi_2}, \quad \eta = \Phi_1 - i\Phi_2$$

where ***g*** is a meromorphic function and holomorphic function, which form

$$\Phi_1 = \frac{1}{2}(1 - \bar{g}^2)\eta$$

$$\Phi_2 = \frac{1}{2}(1 + \bar{g}^2)\eta \qquad (14)$$

$$\Phi_3 = g\eta$$

Theorem 1 (The Weierstrass representation)

Let $\boldsymbol{X}: \boldsymbol{S} \longrightarrow \boldsymbol{R^3}$ be a minimal immersion of an orientable surface, let $g = \sigma \otimes \boldsymbol{N}$ be the composition of the stereographic projection from the point (0; 0; 1) of thesphere to extended complex plane $C \cup \infty$, with Gauss map ***N*** of ***X***.

then *g* is meromorphic and there exist a holomorphic form on ***S*** such that.

$$\boldsymbol{X}(p) - \boldsymbol{X}(p_o) = \int_{p_o}^{p} (\frac{1}{2}(1 - g^2), \frac{1}{2}(1 + g^2), g)\, \eta \qquad (15)$$

for $p, p_o \in \boldsymbol{S}$ the integration being taken on any path from $p \; to \; p_o \in \boldsymbol{S}$ moreover the zeros of η coincide with the pole of *g* and have twice order. All the geometric quantities associated to a minimal surface can be expressed by Weierstrass data $(g; \eta\,)$. If *z* is a local conformal coordinate, the Metric and Gaussian curvature are expressed as follows

$$\boldsymbol{dS^2} = \frac{1}{4}|f(z)|^2(1+|g(z)|^2)|dz|^2 \tag{16}$$

Another generalization of the Weierstrass formulae to generic surfaces in R^3 have been proposed independently by Konopelchenko in 1993. It starts with the linear system (two-dimensional Dirac equation)

$$\psi_z = p\varphi$$
$$\psi_z = p\varphi \tag{17}$$

this is called Kenmotsu formulae where ψ *and* φ are complex-valued functions of $\mathrm{z}; \bar{\mathrm{z}} \in \mathrm{C}$ and $\mathrm{p}(\mathrm{z}; \bar{\mathrm{z}})$ is a real-valued function.

Then one defines the three real-valued functions $X_1(\mathrm{z}; \bar{\mathrm{z}}), X_2(\mathrm{z}; \bar{\mathrm{z}})$, and $X_3(\mathrm{z}; \bar{\mathrm{z}})$, .

$$X_1 + \mathrm{i}\, X_2 = \mathrm{i}\int_\Gamma(\bar{\psi}^2 dz' - \bar{\varphi}^2 d\bar{z}') \tag{18}$$
$$X_1 - \mathrm{i}\, X_2 = \mathrm{i}\int_\Gamma(\psi^2 dz' - \varphi^2 d\bar{z}')$$
$$X_3 = \mathrm{i}\int_\Gamma(\bar{\psi}\varphi dz' - \overline{\psi\varphi} d\bar{z}')$$

where is an arbitrary curve in ***C***. In virtue of eq.17 the r.h.s. in eq.18 do not depend on the choice of Γ. If one now treats $X_1(\mathrm{z}; \bar{\mathrm{z}})$ as the coordinates in R^3 then the formulae of eq.17 and eq.18 define a conformal immersion of surface into R^3 with the induced metric of the form

$$ds^2 = u^2 dz\bar{z}$$
$$u = |\psi|^2 + |\varphi|^2 \tag{19}$$

with the Gauss curvature

$$K = -\frac{4}{u^2}[log(u)]_{z\bar{z}} \tag{20}$$
$$H = 2\frac{p}{u} \tag{21}$$

At $\mathrm{p} = 0$ one gets minimal surfaces and the formulae eq.18 are reduced to the old Weierstrass formulae, generalized Weierstrass formulae the functions φ and ψ obey linear equations, the Kenmotsu formulae has a nonlinear constraint is hard to treat. In particular, that the Willmore functional has or the Helfrich-Polyakov action

$$W = \int H^2 ds \tag{22}$$

a very simple form

$$W = \int p^2 dx\, dy \qquad ;\ \mathrm{z} = \mathrm{x} + \mathrm{iy} \tag{23}$$

the generalized Weierstrass representation (18) has been proved to be an effective tool to study surfaces in R^3 and their integrable deformations.

3.2 The Plateau Problem.

Let Γ be a jordan curve in R^3, i.e a continuous curve which is homomorphic to the circle **S**. The plateau find the surface of last area spanning, for solving this problem, we introduce some notations.

$$\boldsymbol{B} = (u, v) \quad \in R^2, du^2 + dv^2 \leq 1 \tag{25}$$

will denote the closed unit disk in the Euclidean plane.

Let $\boldsymbol{C}(\boldsymbol{\omega}) = \boldsymbol{X}{:}\,\boldsymbol{B} \longrightarrow \boldsymbol{R}^3, \boldsymbol{X}$ is piecewise $\boldsymbol{S}$ is a monotone parameterization of $\boldsymbol{\omega}$, we define the area functional $A{:}\,\boldsymbol{C}(\boldsymbol{\omega}) \longrightarrow \boldsymbol{R}$ by the following integral.

$$A(x) = \int_B |X_u \wedge X_v| du\, dv \tag{26}$$

Where $|X_u \wedge X_v|^2 = |X_u|^2 |X_v|^2 - \langle X_u, X_v \rangle^2$ Let

$$A_\Gamma = inf\ A(\boldsymbol{X})$$
$$\boldsymbol{X} \in (\boldsymbol{\Gamma})$$

therefore our problem is to find a $\boldsymbol{X} \in (\boldsymbol{\Gamma})$such that $A_\Gamma = A(\boldsymbol{X})$, note that Jordan curve bounding a finite surface $A_\Gamma < \infty$ It is requisite to control the parametrization to get minimizing area,i.e curves in riemannian manifold, this done by minimizing the energy integral, the case of surfaces corresponding energy is so called Dirichlet integral.

$$A(x) = \int_B (|X_u|^2 + |X_v|^2) du\, dv \tag{28}$$

this holds if and only if

$$|X_u| = |X_v| \tag{29}$$
$$\langle X_u\,,\, X_v \rangle = 0$$

everywhere in **B** and $|X_u| > 0$ such a map is conformal and induces a metric on B of the form

$$\boldsymbol{dS^2} = \lambda^2 (du^2 + dv^2) \tag{30}$$

where $\lambda = |X_u| = |X_v|$. the parameters (u, v) are thus conformal coordinates for the surface.

3.2.1 The first variation formula.

An immersion $X{:}\,S \longrightarrow R^3$ in a minimal if and only if for every variation $X_t\ of\ X$ with compact support the derivative of th area $\mathrm{A(x)}$ vanishes at $\mathrm{t} = 0$.

$$\frac{d^2 A}{dt^2}(0) > 0 \tag{31}$$

3.2.2 The stability of the minimal surface.

the stability of the minimal surface is related to the second variation formula of the area. A compact domain $\boldsymbol{D} \subset \boldsymbol{S}$ on minimal immersion $X{:}\,S \longrightarrow R^3$ is said stable for every nontrivial normal boundary preserving variation X_t on D.

$$\int_D (|\nabla \Phi|^2 + 2k\ \Phi^2) dA > 0 \tag{32}$$

4. Nambu-Goto Action.

the action is the length swept out by the point, the action of the string now is defined to be the surface area of this world sheet, the action of the Nambu-Goto model is proportional to the area of the string world-sheet which we restrict our attention to the d + 1 dimension case, which is particularly simple, [3],[4] as there is only one transverse degree of freedom (***X***):

$$S[X] = \sigma \int_0^L dv \int_0^R du \sqrt{1 + (\partial_\tau X)^2 + (\partial_\zeta X)^2} \tag{34}$$

Where σ is the string tension which appears as a parameter of the effective model [3]. the effective string world-sheet associated with a two-point Polyakov loop correlation function obeys periodic b.c. in the compactified direction and Dirichlet b.c. along the interquark axis direction:

$$X^i(\tau + L, \zeta) = X^i(\tau, \zeta) \tag{35}$$

$$X^i(\tau, 0) = X^i(\tau, R) = 0 \tag{36}$$

Substituting By Weierstrass representation. The Nambu-Goto action

$$S_{NG} = \int \ (|\psi|^2 + |\varphi|^2)^2 \ dz^2 \tag{37}$$

becomes $dz^2 = i/2\, dz\, d\bar{z}$ The generalized Weierstrass representation gives allowance to linearize NambuGoto action and extract linear formulae from square root as we seen, And give an opportunity for further research into bosonic strings model, can also more search on Willmore surface differentiable other middle of the surfaces which provide extremum to the Willmore functional Polyakov action [5], the solution of Nambu-Goto Action is associated Variational principle for the minimum mean curvature, by using conformal gauge (equivalently isothermal coordinate), the solution of harmonic oscillator of action is given by

$$X(\mathrm{z};\ \bar{\mathrm{z}}) = \mathbf{x} - \mathbf{i/4}\, p\, \mathrm{z}\bar{\mathrm{z}} + \mathrm{i}/2 \sum \frac{1}{n}\ \alpha_n z^{-n} + \mathrm{i}/2 \sum \frac{1}{n}\ \alpha_n \bar{z}^{-n}$$

there another constraints come from the reparameterization invariance of the action. They can be written in terms

$$\partial X . \partial X = \bar{\partial} X . \bar{\partial} X$$

In terms of these fields one may write the Enneper-Weierstrass map, the solution space associated with the Nambu-Goto equation of motions can be described as the solution space of the equation

$$\partial \bar{\partial} X = 0$$

with Dirichlet's boundary conditions, subjected to constrain 39, then the linearizion of Nambu-Goto action by Weierstrass representation give us a new tool to investigate the action, partition function and squared width of flux tube by new tool Along with taking into consideration fluctuations resulting from quantum effect and self - interaction of string Apart from the difficulty of using perturbative treatments

$$X_3 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{c\ Sin(\frac{\pi\, v}{b}(2n-1))}{(2n-1) Sinh(\frac{\pi\, a}{b}(2n-1))} [Sinh(\frac{\pi\,(a-u)}{b}(2n-1)) + Sinh(\frac{\pi\, a}{b}(2n-1))] \tag{41}$$

$$X_2 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{c\ Sin(\frac{\pi\, v}{b}(2n-1))}{(2n-1) Sinh(\frac{\pi\, a}{b}(2n-1))} [(c+R) Sinh(\frac{\pi\,(a-u)}{b}(2n-1)) + c Sinh(\frac{\pi\, a}{b}(2n-1))] \tag{42}$$

$$i\,\overline{\psi}^2 = \frac{1}{2}(\frac{\partial(i\,X_2 + X_3)}{\partial u} - i\frac{\partial(i\,X_2 + X_3)}{\partial v}),$$

$$-i\,\overline{\varphi}^2 = \frac{1}{2}(\frac{\partial(i\,X_2 + X_3)}{\partial u} + i\frac{\partial(i\,X_2 + X_3)}{\partial v}),$$

$$i\,\varphi^2 = \frac{1}{2}\left(\frac{\partial(\,X_3 - iX_2)}{\partial u} - i\frac{\partial(\,X_3 - iX_2)}{\partial v}\right),$$

$$-i\,\psi^2 = \frac{1}{2}\left(\frac{\partial(\,X_3 - iX_2)}{\partial u} + i\frac{\partial(\,X_3 - iX_2)}{\partial v}\right) \tag{43}$$

where a,b are constants determined from boundary conditions eq.35,eq.36.

$$S_{NG} = \int \,(|\psi|^2 + |\varphi|^2)^2\ dz^2$$

$$= -\frac{i\,32\,a}{b}(\mathrm{c}+\mathrm{R})\int_0^b\int_0^a e^{\frac{2n-1}{b}\pi\,u}du\,dv \tag{44}$$

Then Nambu-Goto" action for which is simply the area of the two dimensional worldsheet they trace out in space is acquired as a solution of clynderical coordinates which subjected to Dirichlet's boundary conditions, this action can be employed in Partition function to get potential function.

5. Conclusion

Non-linearly realized at the field level, provoking ordering problems that seem only to be fixed for the critical dimension. These approach that we present here is depending on minimal surface representation via Weierstrass, in the sense that the procedure go through a complete equivalent of the Nambu-Goto string prior to quantization. Nevertheless, invariance all along the way. The key ingredient to the construction will be the Enneper-Weierstrass representation of minimal surfaces. As we show, following otherwise completely standard geometrical constructions, of the three-dimensional closed Nambu-Goto string can be locally identified with the space of complex analytic functions . In the reduction process the conformal structure is completely fixed by choosing a geometrical parameterization of the surface in terms of its Gauss map. a standard linear fractional transformations acting on the Riemann sphere, and realized linearly on the physical fields.

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References

1. Eddington A. S. The mathematical theory of relativity. Cambridge University Press, Cambridge, 1924.
2. Bondi H. Negative mass in General Relativity. Review of Modern Physics, 1957, v. 29 (3), 423–428.
3. Pezzaglia W. Physical applications of generalized Clifford Calculus: Papatetrou equations and metamorphic curvature. arXiv: gr-qc/9710027.
4. Lambiase G., Papini G., Scarpetta G. Maximal acceleration corrections to the Lamb shift of one electron atoms. Nuovo Cimento, v. B112, 1997, 1003. arXiv: hep-th/9702130.
5. A. M. Polyakov, "Fine Structure of Strings," Nucl. Phys. B 268 (1986) 406. doi:10.1016/0550-3213(86)90162-8
6. V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, "Fractal Structure of 2D Quantum Gravity," Mod. Phys. Lett. A 3, 819 (1988). doi:10.1142/S0217732388000982
7. R. Parthasarathy and K. S. Viswanathan, "A Realization of W gravities in the conformal gauge through generalized Gauss maps," Int. J. Mod. Phys. A 7, 317 (1992). doi:10.1142/S0217751X92000193
8. M. Caselle, R. Fiore, F. Gliozzi, P. Guaita and S. Vinti, "On the behavior of spatial Wilson loops in the high temperature phase of LGT," Nucl. Phys. B 422, 397 (1994) doi:10.1016/0550-3213(94)00147-2 [hep-lat/9312056].
9. M. Caselle, F. Gliozzi, U. Magnea and S. Vinti, "Width of long color flux tubes in lattice gauge systems," Nucl. Phys. B 460, 397 (1996) doi:10.1016/0550-3213(95)00639-7 [hep-lat/9510019].
10. M. Luscher and P. Weisz, "Quark confinement and the bosonic string," JHEP 0207, 049 (2002) doi:10.1088/1126-6708/2002/07/049 [hep-lat/0207003].
11. B. G. Konopelchenko and G. Landolfi, "Quantum effects for extrinsic geometry of strings via the generalized Weierstrass representation," Phys. Lett. B 444, 299 (1998) doi:10.1016/S0370-2693(98)01412-9 [hep-th/9810209].
12. Elsa Abbena, Simon Salamon, Alfred Gray, "Modern Differential Geometry of Curves and Surfaces with Mathematica,"