ISSN 2347-3487
Volume 13 Number 6
Journal of Advances in Physics

Analytical Investigation of Projectile Motion in Midair

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ABSTRACT

Here is studied a classic problem of the motion of a projectile thrown at an angle to the horizon. The air drag force is taken into account as the quadratic resistance law. An analytic approach is used for the investigation. Equations of the projectile motion are solved analytically. All the basic functional dependencies of the problem are described by elementary functions. There is no need for to study the problem numerically. The found analytical solutions are highly accurate over a wide range of parameters. The motion of a baseball and a badminton shuttlecock are presented as examples.

Keywords

Classic problem, projectile motion, quadratic drag force, analytic formulas

Academic Disciplines

Classical Physics

1. INTRODUCTION

The problem of the motion of a projectile thrown at an angle to the horizon in midair has a long history. It is one of the great classical problems. The number of works devoted to this task is immense. It is a constituent of many introductory courses of physics. With zero air drag force, the analytic solution is well known. The trajectory of the projectile is a parabola. In real tasks, such as throwing a ball, the impact of the medium is taken into account. Usually quadratic drag low is used. In this case the mathematical complexity of the task strongly grows. The problem probably does not have an exact analytic solution. Therefore the attempts are being continued to construct approximate analytical solutions for this problem. In the given paper an analytic approach is used for the investigation of the projectile motion in a medium with quadratic resistance. The proposed analytical solution differs from other solutions by simplicity of formulae, ease of use and high accuracy. The proposed formulas make it possible to study the motion of a projectile in a medium with the resistance in the way it is done for the case without drag. These formulae are available even for first-year undergraduates.

The problem of the motion of a projectile in midair arouses interest of authors as before [1–8]. For the construction of the analytical solutions various methods are used – both the traditional approaches [1], and the modern methods [2, 5]. All proposed approximate analytical solutions are rather complicated and inconvenient for educational purposes. In addition, many approximate solutions use special functions, for example, the Lambert W function. This is why the description of the projectile motion by means of a simple approximate analytical formulae under the quadratic air resistance is of great methodological and educational importance. The purpose of the present work is to give a simple formulas for the construction of the trajectory of the projectile motion with quadratic air resistance. The conditions of applicability of the quadratic resistance law are deemed to be fulfilled, i.e. Reynolds number Re lies within 1×10^3 < Re < 2×10^5.

2. EQUATIONS OF PROJECTILE MOTION

We now state the formulation of the problem and the equations of the motion according to [8]. Suppose that the force of gravity affects the projectile together with the force of air resistance ***R*** (Fig.1). Air resistance force is proportional to the square of the velocity of the projectile and is directed opposite the velocity vector. For the convenience of further calculations, the drag force will be written as $R = mgkV^2$. Here m is the mass of the projectile, g is the acceleration due to gravity, k is the proportionality factor. Vector equation of the motion of the projectile has the form

$$m\boldsymbol{w} = m\boldsymbol{g} + \boldsymbol{R},$$

where ***w*** – acceleration vector of the projectile. Differential equations of the motion, commonly used in ballistics, are as follows [9]

$$\frac{dV}{dt} = -g\sin\theta - gkV^2, \quad \frac{d\theta}{dt} = -\frac{g\cos\theta}{V}, \quad \frac{dx}{dt} = V\cos\theta, \quad \frac{dy}{dt} = V\sin\theta. \tag{1}$$

Here V is the velocity of the projectile, θ is the angle between the tangent to the trajectory of the projectile and the horizontal, x, y are the Cartesian coordinates of the projectile,

$$k = \frac{\rho_a c_d S}{2mg} = \frac{1}{V_{term}^2} = const,$$

ISSN 2347-3487
Volume 13 Number 6
Journal of Advances in Physics

ρ_a is the air density, c_d is the drag factor for a sphere, S is the cross-section area of the object, and V_{term} is the terminal velocity. The first two equations of the system (1) represent the projections of the vector equation of motion on the tangent and principal normal to the trajectory, the other two are kinematic relations connecting the projections of the velocity vector projectile on the axis x, y with derivatives of the coordinates.

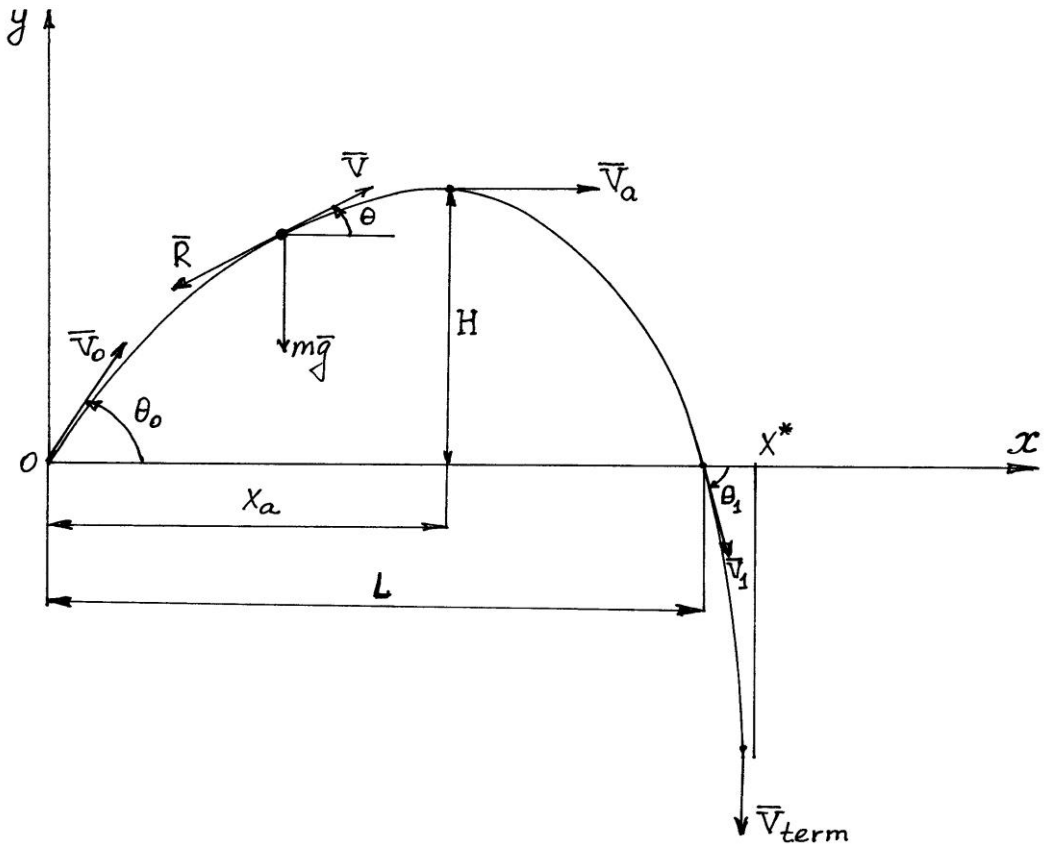

Figure 1: Basic motion parameters.

The well-known solution of system (1) consists of an explicit analytical dependence of the velocity on the slope angle of the trajectory and three quadratures

$$V(\theta)=\frac{V_0\cos\theta_0}{\cos\theta\sqrt{1+kV_0^2\cos^2\theta_0\left(f(\theta_0)-f(\theta)\right)}},\quad f(\theta)=\frac{\sin\theta}{\cos^2\theta}+\ln tg\left(\frac{\theta}{2}+\frac{\pi}{4}\right), \tag{2}$$

$$x=x_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2d\theta,\quad y=y_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2tg\theta d\theta.\quad t=t_0-\frac{1}{g}\int_{\theta_0}^{\theta}\frac{V}{\cos\theta}d\theta\ . \tag{3}$$

Here V_0 and θ_0 are the initial values of the velocity and of the slope of the trajectory respectively, t_0 is the initial value of the time, x_0, y_0 are the initial values of the coordinates of the projectile (usually accepted $t_0=x_0=y_0=0$). The derivation of the formulae (2) is shown in the well-known monograph [10]. The integrals on the right-hand sides of formulas (3) cannot be expressed in terms of elementary functions. Hence, to determine the variables t, x and y we must either integrate system (1) numerically or evaluate the definite integrals (3).

3. OBTAINING AN ANALYTICAL SOLUTION OF THE PROBLEM

The task analysis shows, that equations (3) are not exactly integrable owing to the complicated nature of function (2)

$$f(\theta)=\frac{\sin\theta}{\cos^2\theta}+\ln tg\left(\frac{\theta}{2}+\frac{\pi}{4}\right).$$

The odd function $f(\theta)$ is defined in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Therefore, it can be assumed that a successful approximation of this function will make it possible to calculate analytically the definite integrals (3) with the required accuracy. In Ref. [1], the function $f(\theta)$ on the interval $[0,\theta_0]$ is approximated by the function

$$f_a(\theta)=a_1\tan\theta+b_1\tan^2\theta.$$

The function $f_a(\theta)$ well approximates the function $f(\theta)$ only on the specified interval $[0,\theta_0]$. The function $f(\theta)$ is symmetric to the point $\theta=0$. Therefore, in the present paper we approximate the function $f(\theta)$ on the whole interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ with a function $f_1(\theta)$ of the following form

$$f_1(\theta)=\begin{cases} a_1\tan\theta+b_1\tan^2\theta, & \text{on condition} \quad \theta\geq 0, \\ a_1\tan\theta-b_1\tan^2\theta, & \text{on condition} \quad \theta\leq 0. \end{cases}$$

The coefficients a_1 and b_1 can be chosen in such a way as to smoothly connect the functions $f(\theta)$ and $f_1(\theta)$ to each other with the help of conditions

$$f_1(\theta_0)=f(\theta_0),\quad f_1'(\theta_0)=f'(\theta_0). \tag{4}$$

From the conditions (4) we find

$$a_1=\frac{2\ln\tan(\frac{\theta_0}{2}+\frac{\pi}{4})}{\tan\theta_0},\quad b_1=\frac{1}{\sin\theta_0}-\frac{\ln\tan(\frac{\theta_0}{2}+\frac{\pi}{4})}{\tan^2\theta_0}.$$

Such a function $f_1(\theta)$ well approximates the function $f(\theta)$ throughout the whole interval of its definition for any values θ_0. As an example, we give graphs of functions $f(\theta)$, $f_1(\theta)$ in the interval $-80^\circ\leq\theta\leq 80^\circ$. Coefficients a_1, b_1 are calculated at a value $\theta_0=60^\circ$.

ISSN 2347-3487
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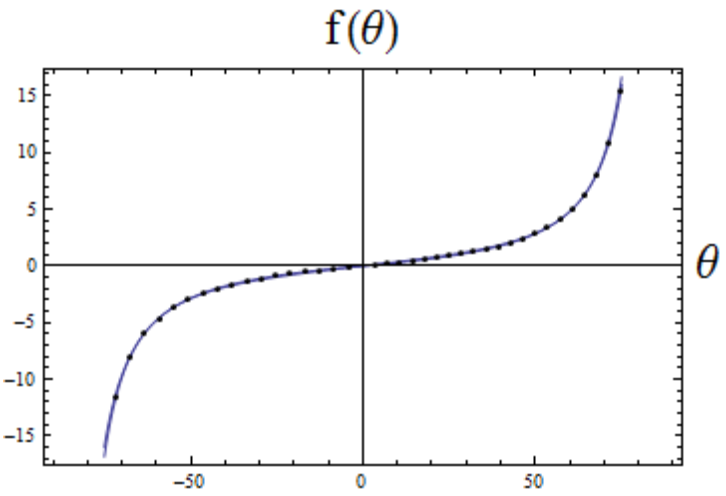

Figure 2: Approximation of the function $f(\theta)$.

The solid curve in Figure 2 is a graph of the function $f(\theta)$, the dot curve is a graph of the function $f_1(\theta)$. The graphs practically coincide. Hence, the function $f_1(\theta)$ can be used instead of the function $f(\theta)$ in calculating the integrals (3).

Now the quadratures (3) are integrated in elementary functions. Since the function $f_1(\theta)$ has a different form on the gaps $\theta \geq 0$, $\theta \leq 0$, the integrals (3) also have a different form at these intervals. For the ascending branch of the trajectory $\theta \geq 0$, for the descending branch of the trajectory $\theta \leq 0$. In calculating the integrals we take $t_0 = x_0 = y_0 = 0$. We integrate the first of the integrals (3). For the coordinate x we obtain:

$$x_1(\theta) = -\frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta = A_1 \arctan\left(\frac{1+2b_2\tan\theta}{b_3}\right) \qquad \text{in case of} \quad \theta \geq 0,$$

$$x_2(\theta) = -A_2 \arctan\left(\frac{-1+2b_2\tan\theta}{b_4}\right) \qquad \text{in case of} \quad \theta \leq 0.$$

Here we introduce the following notation:

$$A_1 = \frac{2}{gka_1b_3},\ A_2 = \frac{2}{gka_1b_4},\ a_2 = \frac{a}{a_1},\ a = \frac{1}{kV_0^2\cos^2\theta_0} + f(\theta_0),$$

$$b_2 = \frac{b_1}{a_1},\ b_3 = \sqrt{-1-4a_2b_2},\ b_4 = \sqrt{-1+4a_2b_2}.$$

Thus, the dependence $x(\theta)$ has the following form:

$$x(\theta) = x_1(\theta) - x_1(\theta_0) \qquad \text{in case of} \quad \theta \geq 0,$$

$$x(\theta) = x_1(0) - x_1(\theta_0) + x_2(\theta) - x_2(0) \qquad \text{in case of} \quad \theta \leq 0. \qquad (5)$$

We integrate the second of the integrals (3). For the coordinate y we obtain:

$$y_1(\theta) = -\frac{1}{g}\int_{\theta_0}^{\theta} V^2 \tan\theta d\theta = -B_1 \arctan\left(\frac{1+2b_2\tan\theta}{b_3}\right) + B_2 \ln\left|-a_2 + \tan\theta + b_2\tan^2\theta\right| \quad \text{in case of} \quad \theta \geq 0,$$

$$y_2(\theta) = -B_3 \arctan\left(\frac{-1+2b_2\tan\theta}{b_4}\right) - B_2 \ln\left|a_2 - \tan\theta + b_2\tan^2\theta\right| \qquad \text{in case of} \quad \theta \le 0.$$

Here we introduce the following notation:

$$B_1 = \frac{1}{kga_1b_2b_3},\ B_2 = \frac{1}{2kga_1b_2},\ B_3 = \frac{1}{kga_1b_2b_4}.$$

Thus, the dependence $y(\theta)$ has the following form:

$$y(\theta) = y_1(\theta) - y_1(\theta_0) \qquad \text{in case of} \quad \theta \ge 0,$$

$$y(\theta) = y_1(0) - y_1(\theta_0) + y_2(\theta) - y_2(0) \qquad \text{in case of} \quad \theta \le 0. \tag{6}$$

For the variable t we get:

$$t_1(\theta) = B_4 \arctan\left[\frac{(1+2b_2\tan\theta)\sqrt{a_2 - \tan\theta - b_2\tan^2\theta}}{2\sqrt{b_2}\left(-a_2 + \tan\theta + b_2\tan^2\theta\right)}\right] \qquad \text{in case of} \quad \theta \ge 0,$$

$$t_2(\theta) = -B_4 \ln\left|-1 + 2b_2\tan\theta + 2\sqrt{b_2}\sqrt{a_2 - \tan\theta + b_2\tan^2\theta}\right| \qquad \text{in case of} \quad \theta \le 0.$$

Here we introduce the notation:

$$B_4 = \frac{1}{g\sqrt{ka_1b_2}}.$$

Thus, the dependence $t(\theta)$ has the following form:

$$t(\theta) = t_1(\theta) - t_1(\theta_0) \qquad \text{in case of } \theta \ge 0,$$

$$t(\theta) = t_1(0) - t_1(\theta_0) + t_2(\theta) - t_2(0) \qquad \text{in case of } \theta \le 0. \tag{7}$$

Consequently, the basic functional dependencies of the problem $x(\theta), y(\theta), t(\theta)$ are written in terms of elementary functions.

The main characteristics of the projectile's motion are the following (Fig. 1):

- H – the maximum height of ascent of the projectile ,
- T – motion time,
- L – flight range ,
- x_a – the abscissa of the trajectory apex,
- t_a – the time of ascent,
- θ_1 – impact angle with respect to the horizontal .

Using formulas (5) - (7), we find:

$$x_a = x_1(0) - x_1(\theta_0), \quad H = y_1(0) - y_1(\theta_0), \quad t_a = t_1(0) - t_1(\theta_0). \tag{8}$$

Then formulas (5) - (7) can be rewritten as:

$$x(\theta) = x_1(\theta) - x_1(\theta_0) \qquad \text{in case of} \quad \theta \ge 0,$$

$$x(\theta) = x_a + x_2(\theta) - x_2(0) \qquad \text{in case of} \quad \theta \le 0.$$

$$y(\theta) = y_1(\theta) - y_1(\theta_0) \qquad \text{in case of} \quad \theta \ge 0,$$

$$y(\theta) = H + y_2(\theta) - y_2(0) \qquad \text{in case of} \quad \theta \le 0.$$

$$t(\theta) = t_1(\theta) - t_1(\theta_0) \qquad \text{in case of} \quad \theta \ge 0,$$

$$t(\theta) = t_a + t_2(\theta) - t_2(0) \qquad \text{in case of} \quad \theta \le 0.$$

The angle of incidence of the projectile θ_1 is determined from the condition $y(\theta_1) = 0$. Then we have

$$L = x_a + x_2(\theta_1) - x_2(0), \; T = t_a + t_2(\theta_1) - t_2(0). \tag{9}$$

We note that formulas (5) - (7) also define the dependences $y = y(x)$, $y = y(t)$, $x = x(t)$ in a parametric way.

4. THE RESULTS OF THE CALCULATIONS. FIELD OF APPLICATION OF THE OBTAINED SOLUTIONS

Proposed formulae have a wide region of application. We introduce the notation $p = kV_0^2$. The dimensionless parameter p has the following physical meaning – it is the ratio of air resistance to the weight of the projectile at the beginning of the movement. As calculations show, trajectory of the projectile $y = y(x)$ and the main characteristics of the motion L, H, T, x_a, t_a have accuracy to within 1% for values of the launch angle and for the parameter p within ranges

$$0° < \theta_0 < 90°, \quad 0 < p \le 60.$$

Figure 3 presents the results of plotting the projectile trajectories with the aid of formulas (5) – (6) over a wide range of the change of the initial angle θ_0 with the following values of the parameters

$$V_0 = 80 \text{ m/s}, \quad k = 0.000625 \text{ s}^2/\text{m}^2, \quad g = 9.81 \text{ m/s}^2, \quad p = 4.$$

The used value of the parameter k is the typical value of the baseball drag coefficient.

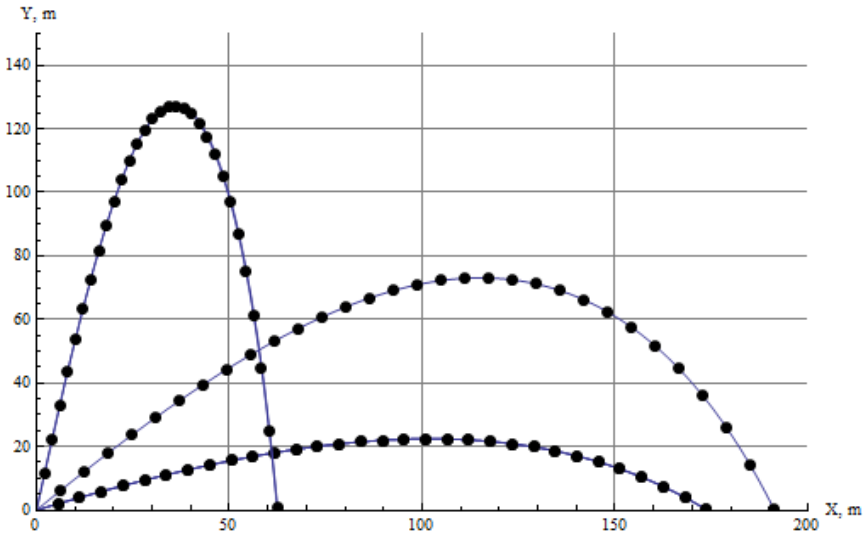

Figure 3: The graphs of the trajectory $y = y(x)$ at launching angles $\theta_0 = 20°, 45°, 80°$.

Analytical solutions are shown in Fig. 3 by dotted lines. The thick solid lines in Fig. 3 are obtained by numerical integration of system (1) with the aid of the 4-th order Runge-Kutta method. As it can be seen from Fig. 3, the analytical solutions (dotted lines) and a numerical solutions are the same.

Figure 4 represents the results of plotting the projectile trajectories with the aid of formulas (5) – (6) over a wide range

of the change of the initial velocity V_0 . In this case the values of the parameter p vary from 1 to 9.

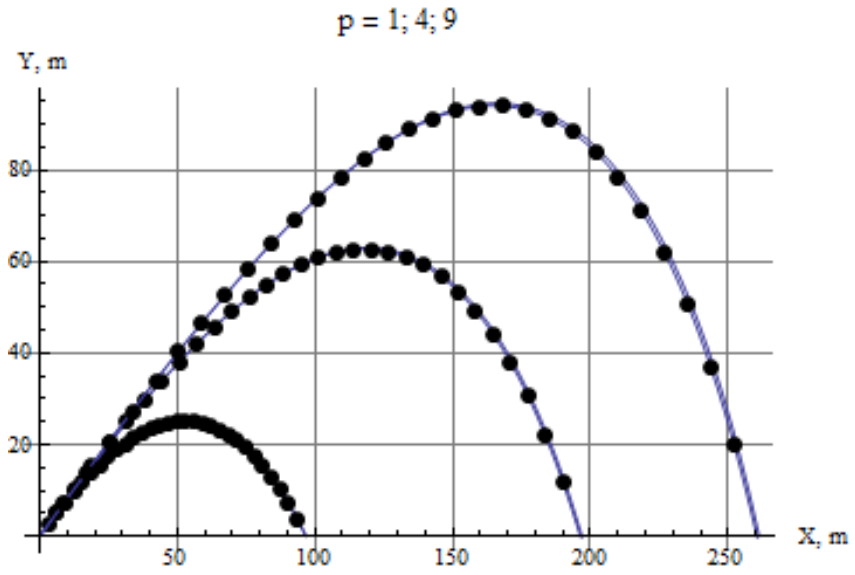

Figure 4: The graphs of the trajectory $y= y(x)$ under the initial conditions $\theta_0 = 40°$, $V_0 = 40$ m/s, 80 m/s, 120 m/s.

As an example of a specific calculation using formulas (5) – (7), we give the trajectory and the values of the basic parameters of the motion L, H, T, x_a , t_a , θ_1 for shuttlecock in badminton. Of all the trajectories of sport projectiles, the trajectory of the shuttlecock has the greatest asymmetry. This is explained by the relatively large value of the drag coefficient k and, accordingly, by the large values of the parameter p. Initial conditions of calculation are

$$k = 0.022 \text{ s}^2/\text{m}^2 ; \quad V_0 = 50 \text{ m/s} ; \quad \theta_0 = 40° ; \quad p = 55.$$

Table 1. Basic parameters of the shuttlecock movement

Parameter	Analytical value	Numerical value	Error (%)
L , (m)	11.33	11.34	-0.1
H , (m)	5.09	5.06	0.4
T , (s)	1.97	1.93	2.0
x_a , (m)	7.89	7.84	0.6
t_a , (s)	0.72	0.71	0.4
θ_1	-79.3°	-78°	1.7

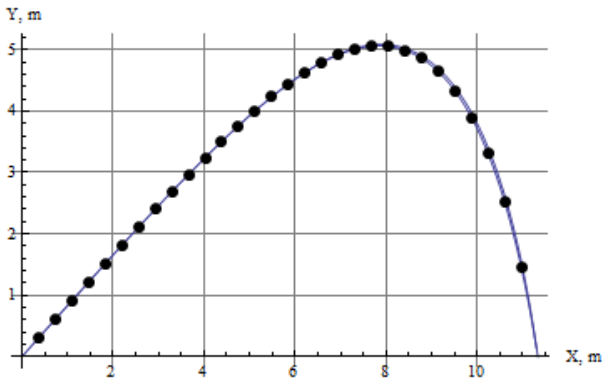

Figure 5: The trajectory of the shuttlecock.

The second column of Table 1 contains range values calculated analytically with formulae (8) – (9). The third column of Table 1 contains range values from the integration of the equations of system (1). The fourth column presents the error of the calculation of the parameter in the percentage. The error does not exceed 2 %.

Thus, a successful approximation of the function $f(\theta)$ made it possible to calculate the integrals (3) in elementary functions and to obtain a highly accurate analytical solution of the problem of the motion of the projectile in the air.

5. CONCLUSIONS

The proposed approach based on the use of analytic formulae makes it possible to simplify significantly a qualitative analysis of the motion of a projectile with the air drag taken into account. All basic variables of the motion are described by analytical formulae containing elementary functions. Moreover, numerical values of the sought variables are determined with high accuracy. It can be implemented even on a standard calculator. Thus, proposed formulae make it possible to study projectile motion with quadratic drag force even for first-year undergraduates.

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