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On the Harmonic Oscillations for the Motion of a Dynamical System

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Abstract

This work touches two important cases for the motion of a pendulum called Sub and Ultra-harmonic cases. The small parameter method is used to obtain the approximate analytic periodic solutions of the equation of motion when the pivot point of the pendulum moves in an elliptic path. Moreover, the fourth order Runge-Kutta method is used to investigate the numerical solutions of the considered model. The comparison between both the analytical solution and the numerical ones shows high consistency between them.

Keywords: Sub-harmonic oscillations, Ultra-harmonic oscillations, Perturbation methods.

MSC2010: 70B10, 70E20, 70K28

1 Introduction

The motion of a pendulum whether it has a rigid arm or elastic one has shed the interest of many researchers during the last century. The study of this motion has been widely spread in the last three decades due to its great applications in different fields like, clinical studies [1-2], physics [3-4], military [5] and engineering applications [6].

Numerous perturbation methods [7] were used to obtain the solutions of such models. The approximate analytical solution of a pendulum with rigid arm was investigated in [8] using the small parameter methods, while the small oscillations besides rotational motions of a parametric pendulum under a vertical harmonic force are studied in [9]. On the other hand, the harmonic balance method was used in [10] to utilize the solution of an excited spring pendulum. This motion was studied in [11] when the pivot point of a spring pendulum moves in a circular path using the multiple scales (MS) technique. The chaotic responses of agitated spring pendulum are studied. The attained solutions are obtained up to the third order of approximations. The Sub-harmonic and homoclinic bifurcations in a parametrically forced pendulum system using Melnikov [12] and averaging methods [7] are achieved in [13].

Many researchers have investigated the chaotic motion of multi-DOF for the nonlinear dynamical systems from the point of view of the resonance conditions. For example, the resonances for the Sub-harmonic case of 2DOF dynamical system in the presence of 3:1 internal resonance have been studied in [14]. Another system was investigated in [14] in the presence of 1:2 internal resonances. Moreover, in the presence of 1:3 internal resonances a dynamical system with cubic nonlinearity was treated in [16]. On the other hand, the parametrically excited spring [17], quadratic nonlinear oscillators [18], and also nonlinear behavior of spring pendulum [19] have been investigated. Some of the perturbation methods like MS technique in [20], averaging method in [21] and harmonic balance method [22] were used to study the dynamical motion of a spring pendulum.

The main aim of this paper is to obtain the approximate periodic solutions for two different cases of nonlinear oscillations. It is worthwhile to mention that every case of oscillations depends on the value of the angular velocity Ω, whether it is an integer or not. For each case, the method of small parameter is used to construct the required approximate periodic solutions. On the other hand, the fourth order Runge-Kutta method is used to achieve the numerical solutions of the equation of motion to assert the accuracy of the analytical method. The comparison between the analytical and the numerical solution reveals a good agreement between them.

2 Description of the problem

Let us consider the motion of a particle of unit mass attached to one end of a massless rod of length ℓ, while the other end of the rod is attached to a point moves on an elliptic path with a constant angular velocity ω. Applying Lagrange's equation of motion, the governing equation of motion takes the form [8]

$$\begin{aligned}
&\varphi'' + \Omega^2\varphi = \sin\tau - \varepsilon\nu\varphi\cos\tau - \frac{1}{2}\varepsilon^2\varphi^2\sin\tau + \frac{1}{6}\varepsilon^3\nu\varphi^3\cos\tau,\\
&\varepsilon = \frac{b}{\ell} << 1,\ \nu = \frac{a}{b} = \sqrt{1-e^2},\ \Omega = \frac{\omega_n}{\omega},\\
&\omega_n^2 = \frac{g}{\ell},\ \Phi = \varepsilon\varphi,\ \tau = \varepsilon t,
\end{aligned} \tag{1}$$

where ε is a small parameter, ν is the ratio between a and b while a and b are the major and minor axes of the ellipse, e is the eccentricity of the ellipse, ω_n is the natural angular velocity, g is the magnitude of gravitational acceleration, τ is the time dimensionless quantity, φ is the angle between the arm of the pendulum and the vertical axis and represents the generalized coordinate.

In order to simulate the dynamic behavior of the considered model for the resonance case, we assume that Ω is either differs from an integer n "sub-harmonic case" or equal n "ultra-harmonic case".

3 The Sub-Harmonic Solution $n=\frac{1}{2}$

In this section we consider one of the important cases for the pendulum motion called sub-harmonic case. To deal with this case, we shall assume the mistuning $(n^2-\Omega^2)$ is of the order of smallness of ε [23], that is;

$$\Omega^2 = n^2 - \varepsilon\lambda, \tag{2}$$

where λ is a finite magnitude.

Our concern will be confined to obtain the solution if $(n=\frac{1}{2})$ which is the commonest practical. Substituting from (2) into (1) with $(n=\frac{1}{2})$, yields

$$\varphi'' + \frac{1}{4}\varphi = \sin\tau + \varepsilon(\lambda\varphi - \nu\varphi\cos\tau) - \frac{1}{2}\varepsilon^2\varphi^2\sin\tau + \frac{1}{6}\varepsilon^3\nu\varphi^3\cos\tau. \tag{3}$$

In order to obtain a uniformly valid expansion for all times of the solution of (3), we assume that there exist a uniformly valid asymptotic representation of $\varphi(\tau;\varepsilon)$ in the form

$$\varphi(\tau;\varepsilon) = \varphi_0(\tau) + \varepsilon\varphi_1(\tau) + \varepsilon^2\varphi_2(\tau) + \varepsilon^3\varphi_3(\tau) + \ldots \tag{4}$$

Substituting from (4) into (3) and equating coefficients of like powers of ε in both sides, we obtain

$$\varphi_0'' + \frac{1}{4}\varphi_0 = \sin\tau, \tag{5}$$

$$\varphi_1'' + \frac{1}{4}\varphi_1 = \lambda\varphi_0 - \nu\varphi_0\cos\tau, \tag{6}$$

$$\varphi_2'' + \frac{1}{4}\varphi_2 = \lambda\varphi_1 - \nu\varphi_1\cos\tau - \frac{1}{2}\varphi_0^2\sin\tau, \tag{7}$$

$$\varphi_3'' + \frac{1}{4}\varphi_3 = \lambda\varphi_2 - \nu\varphi_2\cos\tau - \varphi_0\varphi_1\sin\tau + \frac{\nu}{6}\varphi_0^2\cos\tau, \tag{8}$$

$$\varphi_4'' + \frac{1}{4}\varphi_4 = \lambda\varphi_3 - \nu\varphi_3\cos\tau - \frac{1}{2}\varphi_1^2\sin\tau - \varphi_0\varphi_2\sin\tau + \frac{\nu}{2}\varphi_0^2\varphi_1\cos\tau. \tag{9}$$

It should be noticed that equations (5)-(9) can be solved successively in addition to the secular terms are eliminated when $\lambda \neq \pm\frac{\nu}{2}$. Consequently, we have

$$\varphi_0(\tau) = -\frac{4}{3}\sin\tau, \tag{10}$$

$$\varphi_1(\tau) = -\frac{16}{9}\lambda \sin\tau - \frac{8}{45}\nu \sin 2\tau, \tag{11}$$

$$\varphi_2(\tau) = -2c_1 \sin\tau + \frac{64}{15\times 15}\lambda\nu \sin 2\tau - 2c_2 \sin 3\tau, \tag{12}$$

$$\varphi_3(\tau) = -\frac{4}{3}c_3 \sin\tau - \frac{4}{15}c_4 \sin 2\tau - \frac{4}{35}c_5 \sin 3\tau - \frac{4}{63}c_6 \sin 4\tau, \tag{13}$$

in which

$$\begin{aligned}
c_1 &= -\frac{4}{3\times 45}(2\nu^2 - 15 + 40\lambda^2),\\
c_2 &= \frac{4}{35\times 45}(2\nu^2 + 5),\\
c_3 &= -2\lambda(c_1 + \frac{16}{15\times 45}\nu^2 - \frac{8}{9}),\\
c_4 &= \nu(\frac{64}{15\times 15}\lambda^2 + c_1 + c_2 - \frac{88}{405}),\\
c_5 &= -2\lambda(c_2 + \frac{16}{15\times 15}\nu^2 + \frac{8}{27}),\\
c_6 &= \nu(c_2 + \frac{44}{405}).
\end{aligned} \tag{14}$$

Substituting from (10)-(13) into (4), to yield

$$\begin{aligned}
\varphi(\tau;\varepsilon) &= -\frac{4}{3}\sin\tau + \frac{9}{8}\varepsilon(2\lambda\sin\tau - \frac{\nu}{5}\sin 2\tau) + 2\varepsilon^2(-c_1\sin\tau + \frac{32}{15\times 15}\lambda\nu\sin 2\tau - c_2\sin 3\tau)\\
&\quad - 4\varepsilon^3(\frac{1}{3}c_3\sin\tau + \frac{1}{15}c_4\sin 2\tau + \frac{1}{35}c_5\sin 3\tau + \frac{1}{63}c_6\sin 4\tau) + \ldots
\end{aligned} \tag{15}$$

It is remarked that, the above approximate periodic solution (15) describes the behavior of the pendulum motion according to the series (4) and is given as a function of $\sin m\tau; m$ is a positive integer. It clear that, the series of $\sin mt$ is so rapidly convergent that is; one can determine only some few terms as in (15) which tell the convergence appear obviously.

4 Simulation of the results

This section is devoted to discuss the analytical and the numerical results for the max value of the attained solution φ for the considered problem for both the analytical results and the numerical ones. It is worthwhile to mentioned that the numerical results are obtained using the fourth order Runge-Kutta method [6]. Now, let us consider the maximum value of both the analytical and the numerical solutions included in tables 1,2,3 and 4 through the following cases.

Case 1: $(\varepsilon = 0.05, \lambda = 0.008)$

Table (1): Data for case when $(\varepsilon = 0.05, \lambda = 0.008)$

ν	φ_{max} analytical	φ_{max} numerical
0.0	1.323055	1.32306
0.2	1.323435	1.32344
0.4	1.323835	1.323841
0.6	1.324258	1.324263
0.8	1.324702	1.324706
1.0	1.325168	1.325171

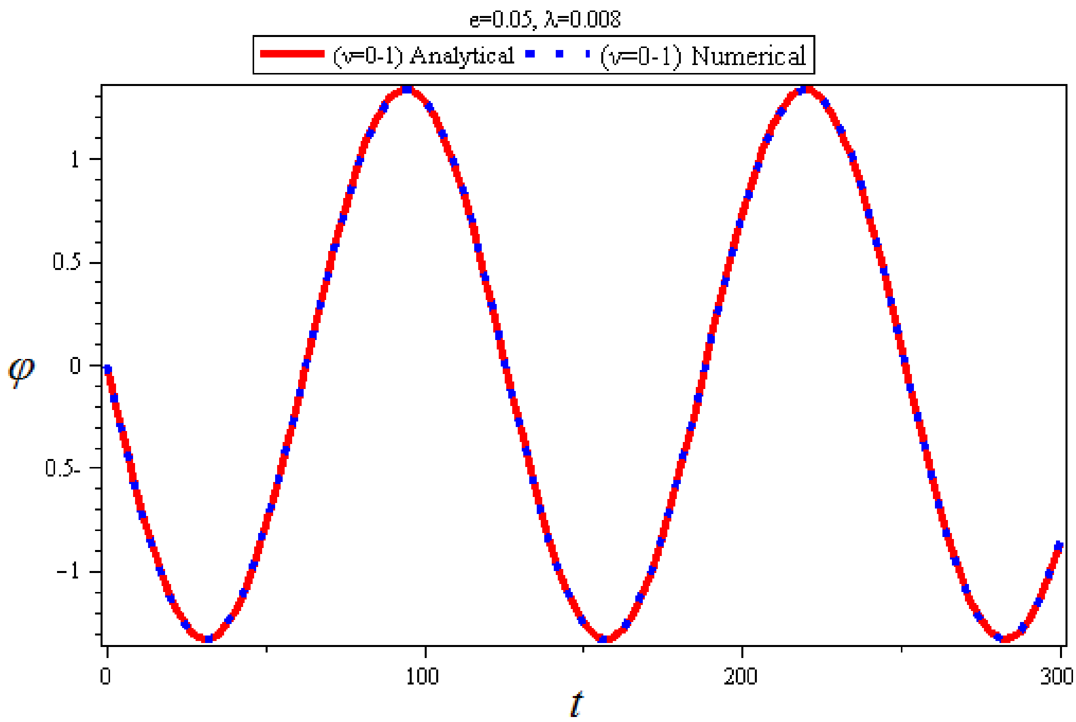

Figure 1: Illustrates the variation of $\varphi(rad)$ **via** $t(s)$ **when** $\varepsilon = 0.05, \lambda = 0.008$.

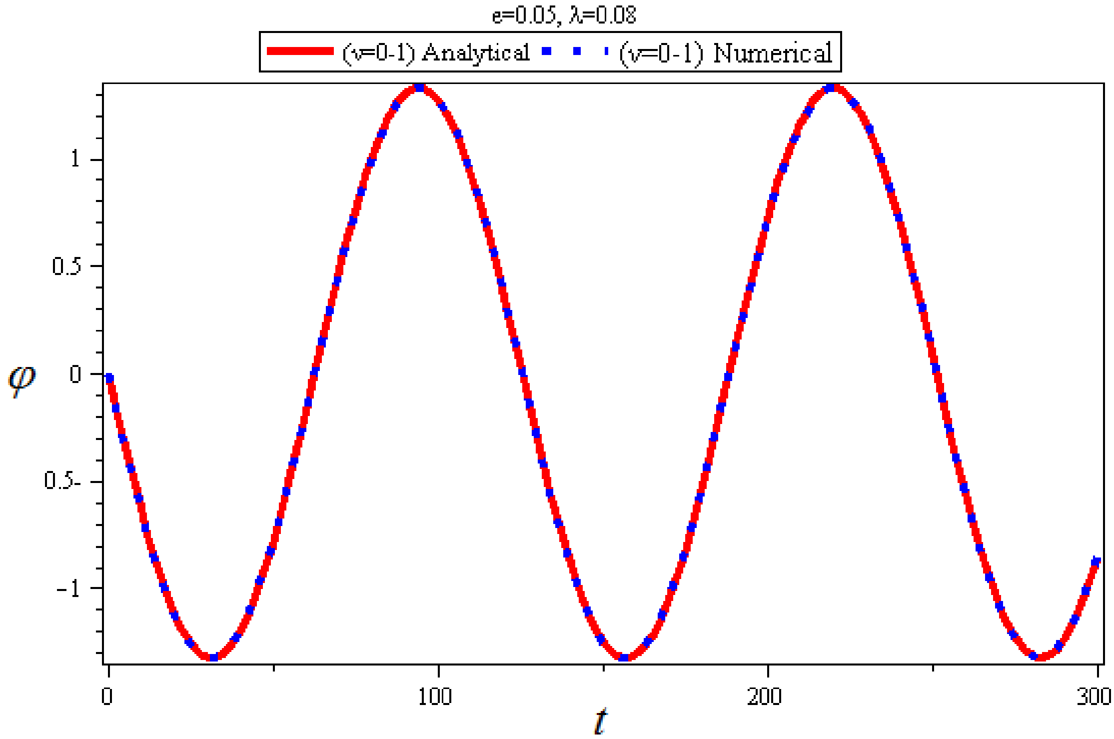

Figure 2: Illustrates the variation of $\varphi(rad)$ **via** $t(s)$ **when** $\varepsilon = 0.05, \lambda = 0.08$.

Case 2: $(\varepsilon = 0.05, \lambda = 0.08)$

Table (2): Data for case when $(\varepsilon = 0.05, \lambda = 0.08)$

ν	φ_{max} analytical	φ_{max} numerical
0.0	1.31676	1.316762
0.2	1.317136	1.317141
0.4	1.317535	1.317538
0.6	1.317955	1.317957
0.8	1.318396	1.318398
1.0	1.318858	1.318859

Case 3: $(\varepsilon = 0.25, \lambda = 0.008)$

Table (3): Data for case when $(\varepsilon = 0.25, \lambda = 0.008)$

ν	φ_{max} analytical	φ_{max} numerical
0.0	1.1266184	1.277605
0.2	1.268173	1.280105
0.4	1.270702	1.28331
0.6	1.273774	1.2867
0.8	1.277392	1.2902
1.0	1.281557	1.293748

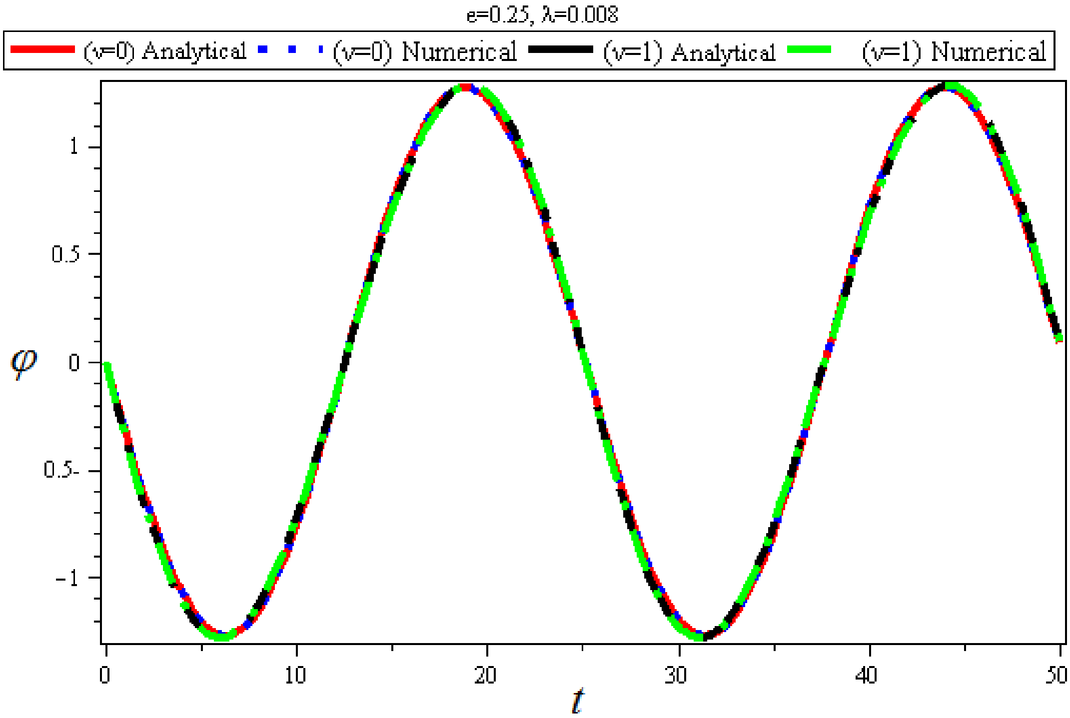

Figure 3: Illustrates the variation of $\varphi(rad)$ **via** $t(s)$ **when** $\varepsilon = 0.25, \lambda = 0.008, \nu = 0$ **and** $\nu = 1$**.**

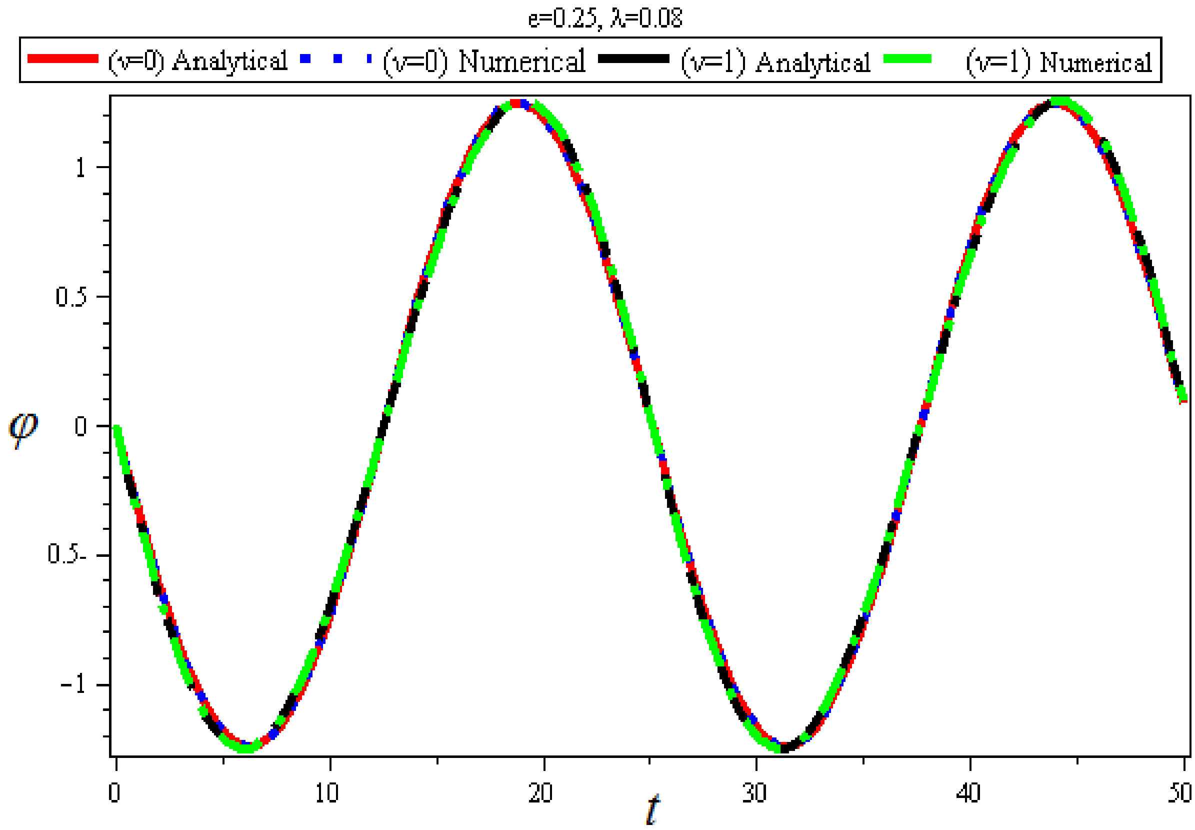

Figure 4: Illustrates the variation of $\varphi(rad)$ **via** $t(s)$ **when** $\varepsilon = 0.25, \lambda = 0.08, \nu = 0$ **and** $\nu = 1$

Case 4: $(\varepsilon = 0.25, \lambda = 0.08)$

Table (4): Data for case when $(\varepsilon = 0.25, \lambda = 0.08)$

ν	φ_{max} analytical	φ_{max} numerical
0.0	1.239246	1.248933
0.2	1.241167	1.251498
0.4	1.243599	1.254612
0.6	1.246541	1.257907
0.8	1.249998	1.261305
1.0	1.253972	1.264734

The previous tables 1-4 describe the variation of ν from zero to one with step 0.2. It is clear that the amplitude of the wave increases slowly when ν increases. Moreover when ε and λ increase, the amplitude of the waves decreases due to that ε is a small parameter.

Figures 1 and 2 represent the analytical and numerical solutions when $\varepsilon = 0.05$ and ν takes its value from 0 to 1 for different values of $\lambda = 0.008$ and $\lambda = 0.08$ respectively, while Figures 3 and 4 are calculated when

$\varepsilon = 0.25, \lambda = 0.008$ and $\varepsilon = 0.25, \lambda = 0.08$ respectively for different values of $\nu = 0$ and $\nu = 1$.

An inspection of these figure we can conclude that the number of oscillations remains unchanged when λ increases as seen from Figures 1 and 2. On the other side the amplitude of the waves increases to some extent when ν increases see Figures 3 and 4.

The comparison between the analytical and the numerical solutions in each figure shows that, the deviation between them is very slightly and can be neglected, that is; the numerical solutions are in quite agreement with the analytical ones.

5 The Ultra-Harmonic Solution $n = 2$

The aim of this section is to study the ultra-harmonic case of oscillations of the considered model. For this purpose we consider $n = 2$. The substitution from (2) into (1) yields

$$\varphi'' + 4\varphi = \sin\tau + \varepsilon(\lambda\varphi - \nu\varphi\cos\tau) - \frac{\varepsilon^2}{2}\varphi^2 \sin\tau + \frac{\varepsilon^3}{6}\nu\varphi^3\cos\tau. \tag{16}$$

As in the previous section, we seek the solution of (16) as in the form of series (4). Substituting from (4) into (16), then equating coefficients of like powers of ε in both sides, we obtain

$$\varphi_0'' + 4\varphi_0 = \sin\tau, \tag{17}$$

$$\varphi_1'' + 4\varphi_1 = \lambda\varphi_0 - \nu\varphi_0\cos\tau, \tag{18}$$

$$\varphi_2'' + 4\varphi_2 = \lambda\varphi_1 - \nu\varphi_1\cos\tau - \frac{1}{2}\varphi_0^2\sin\tau, \tag{19}$$

$$\varphi_3'' + 4\varphi_3 = \lambda\varphi_2 - \nu\varphi_2\cos\tau - \varphi_0\varphi_1\sin\tau + \frac{\nu}{6}\varphi_0^2\cos\tau, \tag{20}$$

$$\varphi_4'' + 4\varphi_4 = \lambda\varphi_3 - \nu\varphi_3\cos\tau - \frac{1}{2}\varphi_1^2\sin\tau - \varphi_0\varphi_2\sin\tau + \frac{\nu}{2}\varphi_0^2\varphi_1\cos\tau. \tag{21}$$

These equations can be solved successively, to obtain

$$\varphi_0(\tau) = \frac{1}{3}(\sin\tau + \frac{\nu}{2\lambda}\sin 2\tau), \tag{22}$$

$$\varphi_1(\tau) = \frac{1}{36\lambda}(4\lambda^2 - \nu^2)\sin\tau + B_1\sin 2\tau + \frac{\nu^2}{60\lambda}\sin 3\tau, \tag{23}$$

$$\varphi_2(\tau) = B_2\sin 2\tau + \frac{C_1}{3}\sin\tau - \frac{C_2}{5}\sin 3\tau - \frac{C_3}{12}\sin 4\tau - \frac{C_4}{21}\sin 5\tau, \tag{24}$$

$$\varphi_3(\tau) = B_3\sin 2\tau + \frac{C_5}{3}\sin\tau - \frac{C_6}{5}\sin 3\tau - \frac{C_7}{12}\sin 4\tau - \frac{C_8}{21}\sin 5\tau - \frac{C_9}{32}\sin 6\tau - \frac{C_{10}}{45}\sin 7\tau. \tag{25}$$

Here,

$$
\begin{aligned}
B_1 &= \frac{\nu}{360\lambda^2}[5(4\lambda^2-\nu^2)+3\nu^2+10],\\
B_2 &= \frac{1}{\lambda}[\frac{\nu C_1}{6}-\frac{\nu C_2}{10}+\frac{\nu}{12\times 36\lambda^2}(4\lambda^2-\nu^2)+\frac{\nu^3}{360\times 4\lambda^2}+\frac{B_1}{6}-\frac{\nu}{24\times 27}-\frac{\nu^3}{64\times 9\lambda^2}],\\
B_3 &= \frac{1}{6\lambda}(\nu C_5+B_2)-\frac{\nu}{10\lambda}C_6+\frac{B_1}{72\lambda^2}(4\lambda^2-\nu^2)+\frac{B_1\nu^2}{240\lambda^2}+\frac{\nu C_1}{36\lambda^2}-\frac{\nu C_2}{120\lambda^2}+\frac{\nu C_4}{21\times 24\lambda^2}\\
&\quad+\frac{C_3}{144\lambda}-\frac{\nu^3}{16\times 12\times 36\lambda^2}(4\lambda^2-\nu^2)-\frac{\nu^5}{48\times 240\lambda^3}-\frac{\nu}{72\times 36\lambda}(4\lambda^2-\nu^2)-\frac{\nu^3}{18\times 480\lambda}\\
&\quad-\frac{\nu^2 B_1}{48\lambda},\\
C_1 &= \frac{1}{36}(4\lambda^2-\nu^2)-\frac{\nu}{2}B_1-\frac{\nu^2}{18\times 8\lambda^2}-\frac{1}{24},\\
C_2 &= \frac{\nu^2}{60}-\frac{\nu}{2}B_1-\frac{\nu^2}{18\times 16\lambda^2}+\frac{1}{72},\\
C_3 &= \frac{\nu}{24\times 15\lambda}(5-3\nu^2),\\
C_4 &= \frac{\nu^2}{18\times 16\lambda^2},\\
C_5 &= \frac{C_1\lambda}{3}-\frac{\nu B_2}{2}-\frac{\nu B_2}{12\lambda}+\frac{\nu^2}{720\lambda}+\frac{\nu^4}{48\times 72\lambda^3}+\frac{\nu^2}{16\times 27\lambda}-\frac{1}{144\lambda}(4\lambda^2-\nu^2),\\
C_6 &= -\frac{C_2\lambda}{5}-\frac{\nu B_2}{2}+\frac{C_3\nu}{24}-\frac{\nu B_1}{24\lambda}+\frac{1}{12\times 36\lambda}(4\lambda^2-\nu^2)-\frac{\nu^2}{360\lambda}+\frac{\nu^4}{48\times 72\lambda^3}+\frac{\nu^2}{32\times 27\lambda},\\
C_7 &= -\frac{C_3\lambda}{12}+\frac{C_2\nu}{10}+\frac{C_4\nu}{42}+\frac{\nu}{24\times 36\lambda^2}(4\lambda^2-\nu^2)-\frac{\nu^3}{1440\lambda^2}+\frac{B_1}{12}-\frac{\nu}{48\times 27},\\
C_8 &= -\frac{C_4\lambda}{21}+\frac{C_3\nu}{24}+\frac{\nu B_1}{24\lambda}+\frac{\nu^2}{720\lambda}-\frac{\nu^4}{48\times 216\lambda^3}-\frac{\nu^2}{32\times 27\lambda},\\
C_9 &= \frac{\nu C_4}{42}+\frac{\nu^3}{1440\lambda^2}-\frac{\nu^3}{64\times 26\lambda^2},\\
C_{10} &= -\frac{\nu^4}{48\times 216\lambda^3}.
\end{aligned} \tag{26}
$$

Substituting from (22)-(25) into (4), we get

$$
\begin{aligned}
\varphi(\tau;\varepsilon) &= \frac{1}{3}(\sin\tau+\frac{\nu}{2\lambda}\sin 2\tau)+\varepsilon[\frac{1}{36\lambda}(4\lambda^2-\nu^2)\sin\tau+B_1\sin 2\tau+\frac{\nu^2}{60\lambda}\sin 3\tau]\\
&\quad+\varepsilon^2[\frac{1}{3}C_1\sin\tau+B_2\sin 2\tau-\frac{1}{5}C_2\sin 3\tau-\frac{1}{12}C_3\sin 4\tau-\frac{1}{21}C_4\sin 5\tau]\\
&\quad+\varepsilon^3[\frac{1}{3}C_5\sin\tau+B_2\sin 2\tau-\frac{1}{5}C_6\sin 3\tau]-\frac{1}{12}C_7\sin 4\tau-\frac{1}{21}C_8\sin 5\tau\\
&\quad-\frac{1}{32}C_9\sin 6\tau-\frac{1}{45}C_{10}\sin 7\tau]+....
\end{aligned} \tag{27}
$$

This equation represents the desired solution for the ultra-harmonic case and is given as a function of ε and $\sin m\tau; m=1,2,3,...$. It is remarked that it is similar to that one of the sub-harmonic case.

Discussion of the results

As before, we investigate here the comparison between the analytical and numerical solutions for some cases of different parameters of the considered dynamical system. So, we consider the maximum value of the attained solution φ .

Case 5: $(\nu = 0, \varepsilon = 0.2)$

Table (5): Data for case when $(\nu = 0, \varepsilon = 0.2)$

λ	φ_{max} analytical	φ_{max} numerical
0.08	0.3328299	0.3328683
0.12	0.3337218	0.3337601
0.16	0.3346186	0.3346566
0.20	0.3355202	0.335558

Case 6: $(\nu = 0.6, \varepsilon = 0.2)$

Table (6): Data for case when $(\nu = 0.6, \varepsilon = 0.2)$

λ	φ_{max} analytical	φ_{max} numerical
0.08	2.084974	2.084319
0.12	1.324771	1.324566
0.16	1.007065	1.0069532
0.20	0.8344201	0.8343426

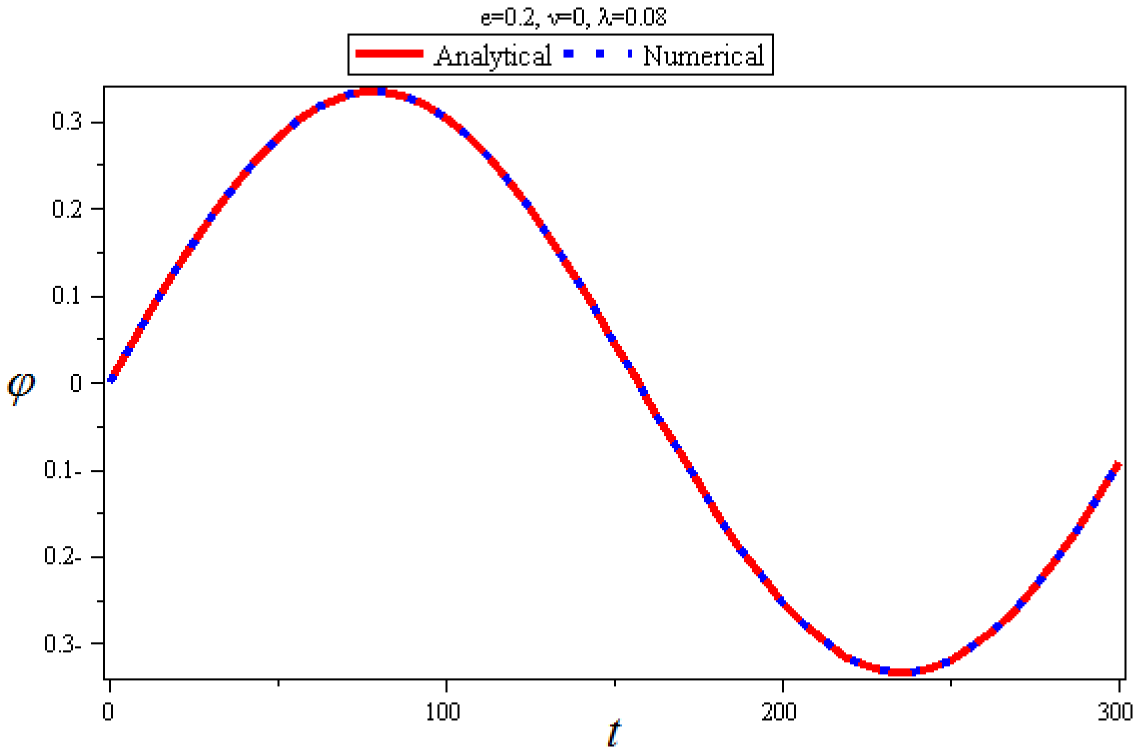

Figure 5: Describes the variation of $\varphi(rad)$ **via** $t(s)$ **when** $\varepsilon = 0.2, \lambda = 0.08$ **and** $\nu = 0$**.**

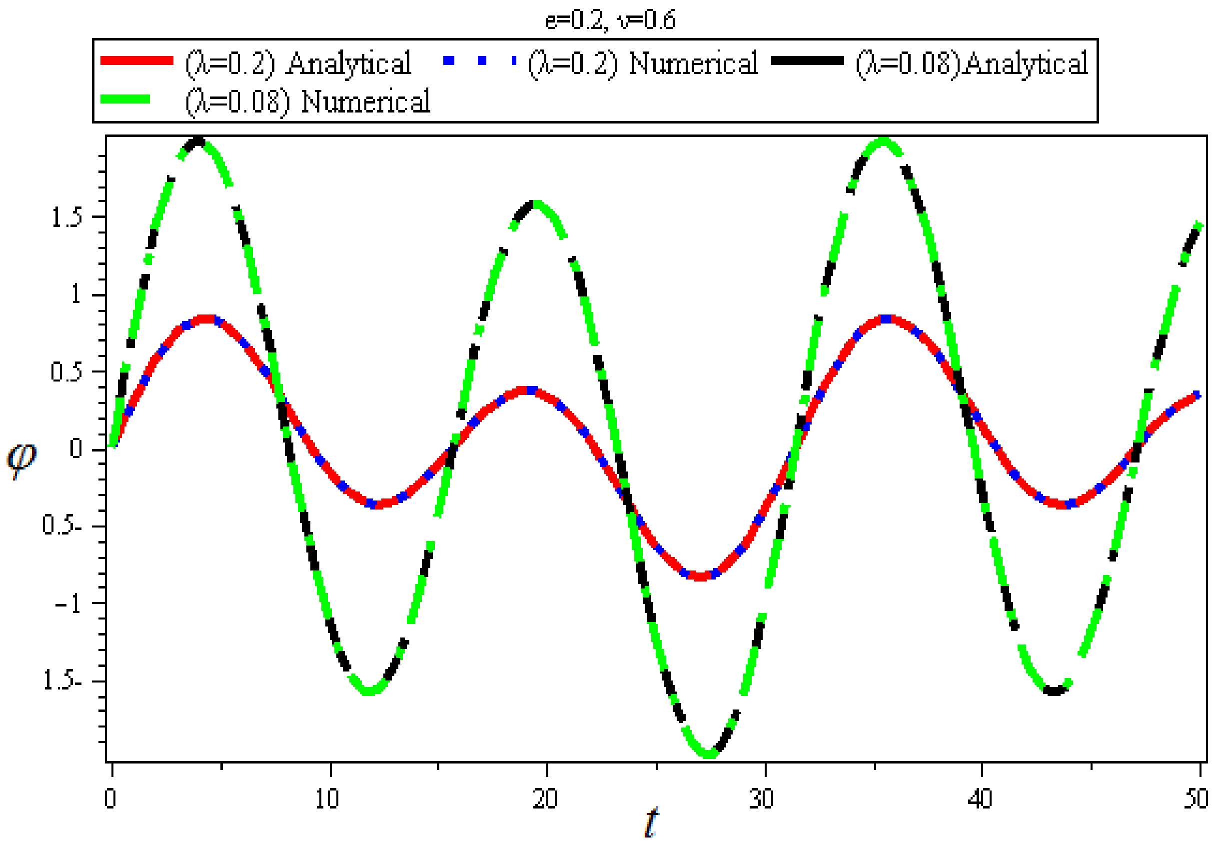

Figure 6: Describes the variation of $\varphi(rad)$ **via** $t(s)$ **when** $\varepsilon = 0.2, \nu = 0.6, \lambda = 0.2$ **and** $\lambda = 0.08$

Case 7: $(\nu = 1, \varepsilon = 0.2)$

Table (7): Data for case when $(\nu = 1, \varepsilon = 0.2)$

λ	φ_{max} analytical	φ_{max} numerical
0.08	2.382134	2.720321
0.12	1.859742	1.860713
0.16	1.428068	1.428307
0.20	1.175988	1.176041

Tables (5), (6) and (7) represent the values of φ_{max} for both the analytical and the numerical results. It is quite clear from table (5) that, the amplitude of the wave increases slowly when the constant λ increases and $\nu = 0$, On the other hand it decreases when λ and ν increases, as seen from the considered date in tables (6) and (7).

Figures 5,6 and 7 illustrate the analytical solutions and the corresponding numerical one for the different values of the physical parameters of the considered model. It is not difficult to conclude that, when λ changes from 0.08 to 0.2, the solutions fluctuate between increasing and decreasing to produce periodic waves as seen from Figures 6 and 7. It is clear that the amplitude of the waves decreases and the number of oscillations remains unchanged.

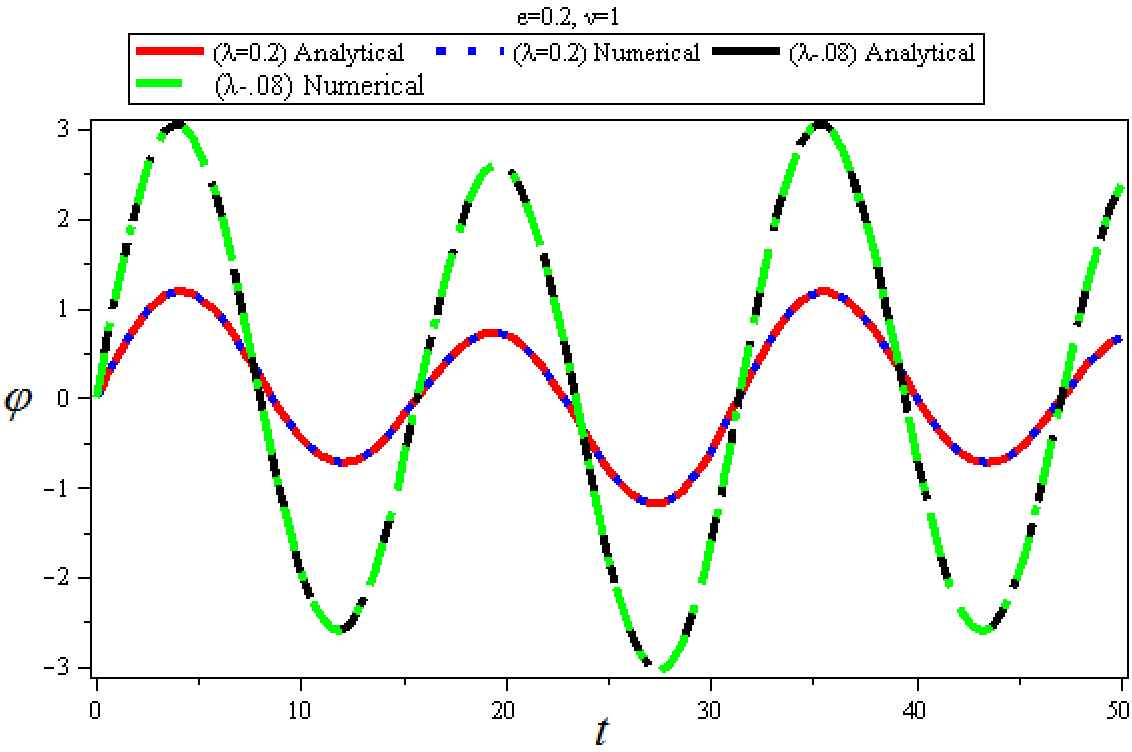

Figure 7: Describes the behavior of the solution $\varphi(rad)$ versus $t(s)$ when $\varepsilon = 0.2, \nu = 1, \lambda = 0.2$ and $\lambda = 0.08$.

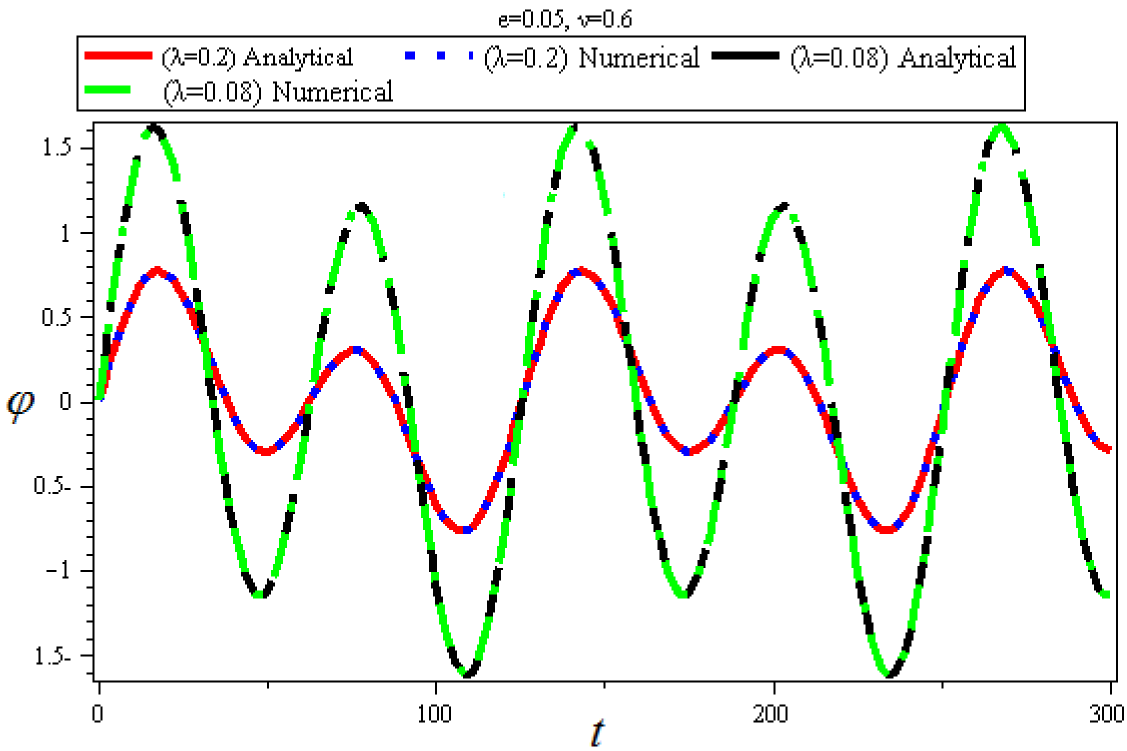

Figure 8: Describes the behavior of the solution $\varphi(rad)$ versus $t(s)$ when $\varepsilon = 0.05, \nu = 0.6, \lambda = 0.2$ and $\lambda = 0.08$.

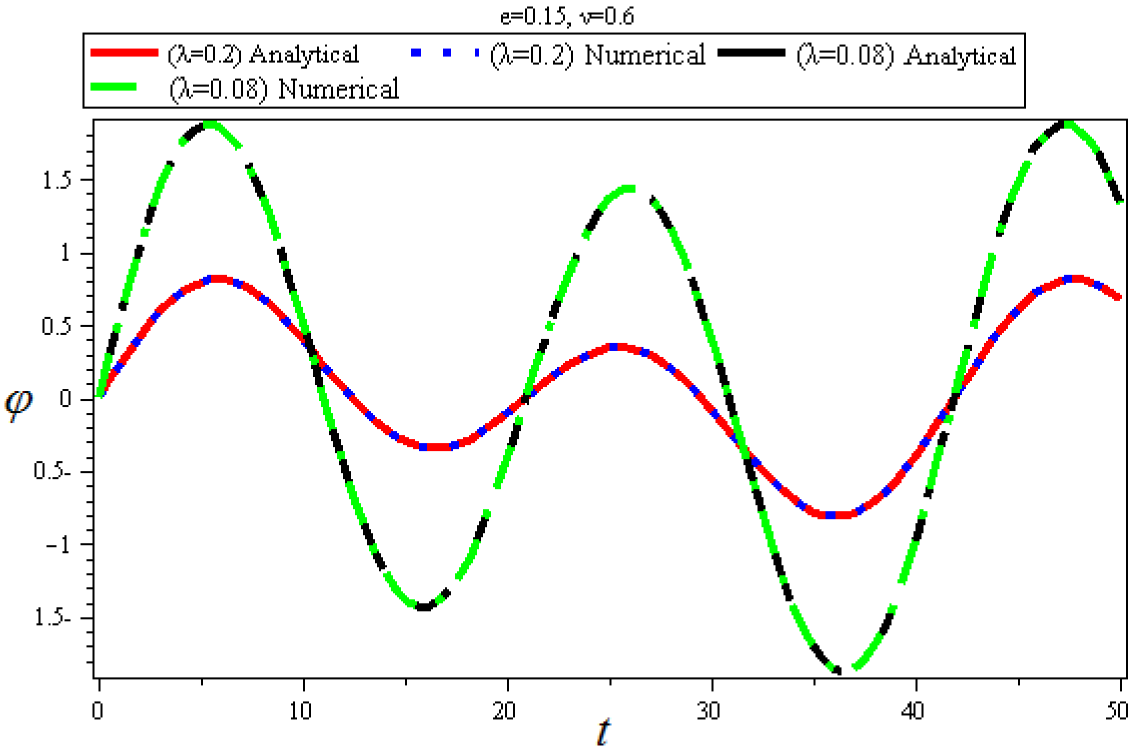

Figure 9: Describes the behavior of the solution $\varphi(rad)$ **against** $t(s)$ **when** $\varepsilon = 0.15, \nu = 0.6, \lambda = 0.2$ **and** $\lambda = 0.08$.

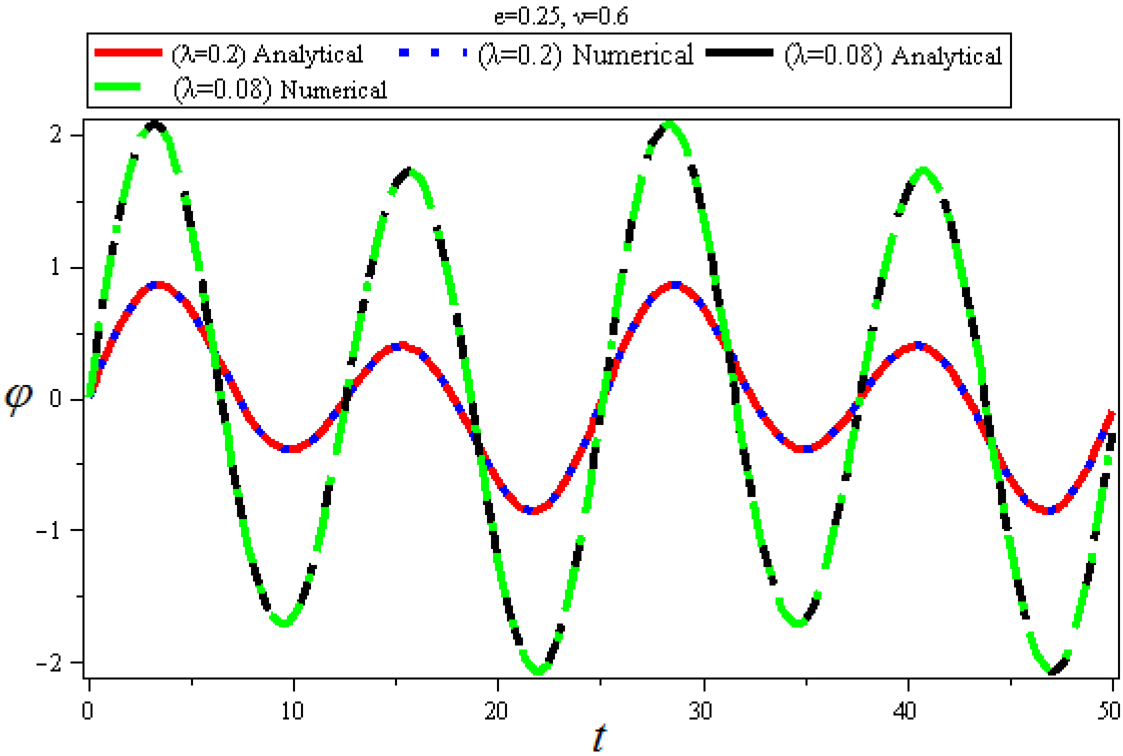

Figure 10: Describes the behavior of the solution $\varphi(rad)$ **against** $t(s)$ **when** $\varepsilon = 0.25, \nu = 0.6, \lambda = 0.2$ **and** $\lambda = 0.08$.

Case 8: $(\varepsilon = 0.05, \nu = 0.6)$

Table (8): Data for case when $(\varepsilon = 0.05, \nu = 0.6)$

λ	φ_{max} analytical	φ_{max} numerical
0.08	1.617575	1.617424
0.12	1.132432	1.132333
0.16	0.9012577	0.9011834
0.20	0.7661203	0.7660601

Case 9: $(\varepsilon = 0.15, \nu = 0.6)$

Table (9): Data for case when $(\varepsilon = 0.15, \nu = 0.6)$

λ	φ_{max} analytical	φ_{max} numerical
0.08	1.914201	1.913874
0.12	1.1256532	1.256394
0.16	0.9701444	0.9700551
0.20	0.8108487	0.8107807

Case 10: $(\varepsilon = 0.25, \nu = 0.6)$

Table (10): Data for case when $(\varepsilon = 0.25, \nu = 0.6)$

λ	φ_{max} analytical	φ_{max} numerical
0.08	2.27129	2.269989
0.12	1.397191	1.396857
0.16	1.045635	1.045484
0.20	0.8587878	0.8586939

Tables (8), (9) and (10) give the values of φ_{max} for both the analytical and the numerical results when ε takes the values $0.05, 0.25$ and 0.25 respectively. For these cases, we notice that, when λ increases the amplitude of the wave decreases to some extent value. For the same values of λ, the amplitude of the wave increases when ε increases.

Figure 8 is plotted when $\varepsilon = 0.05, \nu = 0.6$ at the different values of λ from 0.08 to 0.2. The amplitude of the waves decreases when λ increases. On the other hand the number of oscillations remains unchanged which indicate the stability of the attained solutions. For different values of $\varepsilon = 0.05, 0.15, 0.25$ and with the stationary value of ν, Figures 8, 9 and 10 for analytical and numerical solutions are plotted. It is evident that when ε increases the amplitude of the waves increases.

An inspection of the previous Figures (1-10) confirm that the analytical results coincide with the solutions obtained by solving the equation of motion (1) numerically, which assert accuracy of the achieved analytical solution.

6 Conclusion

In this paper, we have considered the motion of supported point of a pendulum model with rigid arm on elliptic path. The small parameter method is used to obtain the analytical solutions for two important cases namely; the sub-harmonic case and the ultra-harmonic ones. The attained approximate solutions are obtained in terms of periodic functions $\sin(m\tau); m = 1,2,3,...$ On other hand the numerical solutions for both two cases are achieved using the fourth order Runge-Kutta method. The comparison between the analytical and the numerical results shows high consistency between them.

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