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On solving the nonlinear Biswas-Milovic equation with dual-power law nonlinearity using the extended tanh-function method

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Abstract

In this article, we apply the extended tanh-function method to find the exact traveling wave solutions of the nonlinear Biswas-Milovic equation (BME), which describes the propagation of solitons through optical fibers for trans-continental and trans-oceanic distances. This equation is a generalized version of the nonlinear Schrödinger equation with dual-power law nonlinearity. With the aid of computer algebraic system Maple, both constant and time-dependent coefficients of BME are discussed. Comparison between our new results and the well-known results is given. The given method in this article is straightforward, concise and can be applied to other nonlinear partial differential equations (PDEs) in mathematical physics.

Keywords

Nonlinear PDEs; Exact traveling wave solutions; Biswas-Milovic equation (BME); Extended tanh-function method.

Mathematics Subject Classification

35K99, 35P05, 35P99, 35C05.

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1. Introduction

The investigation of exact traveling wave solutions to nonlinear PDEs plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent decades, many effective methods have been established to obtain exact solutions of nonlinear PDEs, such as the inverse scattering transform [1], the Hirota method [2], the truncated Painlevé expansion method [3], the Bäcklund transform method [1,4,5], the exp-function method [6-8], the simplest equation method [9,10], the Weierstrass elliptic function method [11], the Jacobi elliptic function method [12-16], the tanh-function method [17-21], sine-cosine method [22-24], the $\left(\frac{G'}{G}\right)$-expansion method [25-30], the modified simple equation method [31-36], the Kudryashov method [37-39],the multiple exp-function algorithm method [40,41], the transformed rational function method [42], the Frobenius decomposition technique [43], the local fractional variation iteration method [44], the local fractional series expansion method [45], the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method [46-51] and so on.

The objective of this article is to use the extended tanh-function method to construct the exact traveling wave solutions of the Biswas-Milovic equation (BME) with dual-power law nonlinearity [52] in the following two forms:

(i) The (1+1)-dimensional Biswas-Milovic equation (BME) with constant coefficients

$$i\left(q^m\right)_t + a\left(q^m\right)_{xx} + b\left(|q|^{2n} + k\,|q|^{4n}\right)q^m = 0, \quad m,n \geq 1, \tag{1.1}$$

where $q = q(x,t)$ is a complex function, the variable x is interpreted as the normalized propagation distance, t - retarded time, a is the coefficient of group-velocity dispersion (GVD) and $b,\ k$ are the coefficients of the nonlinear terms, such that $a, b,\ k$ are all constants.

(ii) The (1+1)-dimensional Biswas-Milovic equation (BME) with time-dependent coefficient

$$i\left(q^m\right)_t + a(t)\left(q^m\right)_{xx} + b(t)\left(|q|^{2n} + k(t)|q|^{4n}\right)q^m = 0, \quad m,n \geq 1, \tag{1.2}$$

Here $a(t)$ represents the coefficient of GVD while $b(t)$ and $k(t)$ are the coefficients of nonlinear terms, such that $a(t),\ b(t),\ k(t)$ are all functions of the time t .

If $m=1$, then Eqs. (1.1) and (1.2) can be reduced to the nonlinear Schrödinger equation, with dual-power law nonlinearity [53]. Mirzazadeh et al [52] have discussed Eqs. (1.1) and (1.2) using the $\left(\frac{G'}{G}\right)$-expansion method and found few of the exact solutions.

This paper is organized as follows: In Sec. 2, the description of the extended tanh-function method is given. In Sec. 3, we use the extended tanh-function method described in Sec. 2, to find exact traveling wave solutions of Eqs. (1.1) and (1.2). In Sec. 4, some conclusions are obtained.

2. Description of the extended tanh-function method

Suppose that we have the following nonlinear PDE:

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method [17-21]:

Step 1. Using the wave transformation

$$u(x,t) = u(\xi), \quad \xi = x - \lambda t, \tag{2.2}$$

where λ is a constant, to reduce Eq. (2.1) to the following nonlinear ordinary differential equation ODE:

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$$P(u,u',u'',...)=0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives while $'=d/d\xi$.

Step 2. Assume that Eq.(2.3) has the formal solution

$$u(\xi)=a_0+\sum_{i=1}^{N}\left[a_iY^{i}(\xi)+a_{-i}Y^{-i}(\xi)\right], \tag{2.4}$$

where a_0, a_i, a_{-i} are constants to be determined, such that $a_N\neq 0$ or $a_{-N}\neq 0$, while $Y(\xi)$ is given by

$$Y(\xi)=\tanh(\mu\xi), \tag{2.5}$$

where μ is a constants to be determined later. The independent variable (2.5) leads to the following derivatives:

$$\begin{aligned}\frac{d}{d\xi}&=\mu(1-Y^2)\frac{d}{dY},\\ \frac{d^2}{d\xi^2}&=\mu^2(1-Y^2)\left[-2Y\frac{d}{dY}+(1-Y^2)\frac{d^2}{dY^2}\right],\\ \frac{d^3}{d\xi^3}&=\mu^3(1-Y^2)\left[(6Y^2-2)\frac{d}{dY}-6Y(1-Y^2)\frac{d^2}{dY^2}+(1-Y^2)^2\frac{d^3}{dY^3}\right],\end{aligned} \tag{2.6}$$

And so on.

Step 3. We determine the positive integer N in (2.4) by using the homogeneous balance between the highest-order derivative and the highest nonlinear term in Eq. (2.3). More precisely we define the degree of $u(\xi)$ as $D\left[u(\xi)\right]=N$ which gives rise to the degree of other expressions as follows:

$$\begin{aligned}D\left[\frac{d^q u}{d\xi^q}\right]&=N+q,\\ D\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right]&=Np+s(q+N).\end{aligned} \tag{2.7}$$

Therefore, we can get the value of N in (2.4). In some nonlinear equations the balance number N is not a positive integer. In this case, we make the following transformations [54]:

(a) When $N=\frac{q}{p}$ where $\frac{q}{p}$ is a fraction in the lowest terms, we let

$$u(\xi)=v^{\frac{q}{p}}(\xi), \tag{2.8}$$

then substituting (2.8) into (2.3) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

(b) When N is a negative number, we let

$$u(\xi)=v^{N}(\xi), \tag{2.9}$$

and substituting (2.9) into (2.3) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

Step 4. We substitute (2.4) along with Eq. (2.6) into Eq. (2.3), collect all the terms with the same powers of $Y(\xi)$ and set them to zero, we obtain a system of algebraic equations, which can be solved by Maple to get the values of a_0, a_i, a_{-i} and λ. Consequently, we obtain the exact traveling wave solutions of Eq. (2.1).

3. Applications

In this section, we will apply the method described in Sec. 2 to find the exact traveling wave solutions of Biswas-Milovic equation with dual-power law nonlinearity Eqs. (1.1) and (1.2).

3.1. Exact traveling wave solutions of Eq. (1.1)

In this subsection, we consider the exact traveling wave solutions of Biswas-Milovic equation (1.1) with constant coefficients. To this end, we assume that the solution of Eq. (1.1) can be written as:

$$q(x,t)=u(\xi)e^{i(-k_1x+\omega t+\theta)},\quad \xi=x-\lambda t, \tag{3.1}$$

where $u(\xi)$ is a real function of ξ while k_1,ω,θ and λ represent the frequency, wave number, phase constant and the speed of the wave respectively. Substituting (3.1) into Eq. (1.1) and separating the real and imaginary parts, we obtain

$$\lambda=-2mak_1, \tag{3.2}$$

and the following nonlinear ODE:

$$a\left(u^m\right)''-\left(m\omega+am^2k_1^2\right)u^m+bu^{2n+m}+bku^{4n+m}=0. \tag{3.3}$$

By balancing between $\left(u^m\right)''$ with u^{4n+m} in (3.3) we get $mN+2=N\left(4n+m\right)\Rightarrow N=\dfrac{1}{2n}$. According to step 3, we use the transformation

$$u(\xi)=v^{\frac{1}{2n}}(\xi), \tag{3.4}$$

where $v(\xi)$ is a new function of ξ. Substituting (3.4) into (3.3), we get the new ODE

$$2namvv''+am(m-2n)(v')^2-4n^2m\left(\omega+amk_1^2\right)v^2+4n^2bv^3+4n^2bkv^4=0. \tag{3.5}$$

Balancing vv'' with v^4 in (3.5) we get $N+N+2=4N\Rightarrow N=1$. Consequently, Eq. (3.5) has the formal solution:

$$v(\xi)=a_0+a_1Y(\xi)+a_{-1}Y^{-1}(\xi), \tag{3.6}$$

where a_0, a_1 and a_{-1} are constants to be determined later satisfying $a_1^2+a_{-1}^2\neq 0$.

Now, substituting (3.6) along with Eqs. (2.5) and (2.6) into (3.5), collecting the coefficients of powers of $Y(\xi)$ and setting them to zero, we obtain the following system of algebraic equations:

$$Y^4:\ am^2\mu^2a_1^2+2amn\mu^2a_1^2+4bkn^2a_1^4=0,$$

$$Y^3:\ 4bn^2a_1^3+16bkn^2a_0a_1^3+4amn\mu^2a_0a_1=0,$$

$$Y^2:\ -4am^2n^2a_1^2k_1^2-2am^2\mu^2a_1^2-2aa_{-1}m^2\mu^2a_1-4\omega mn^2a_1^2+8aa_{-1}mn\mu^2a_1$$
$$+24bkn^2a_0^2a_1^2+12bn^2a_0a_1^2+16bka_{-1}n^2a_1^3=0,$$

$$Y:\ -8am^2n^2a_0a_1k_1^2-8\omega mn^2a_0a_1-4amn\mu^2a_0a_1+16bkn^2a_0^3a_1+12bn^2a_0^2a_1$$
$$+48bka_{-1}n^2a_0a_1^2+12ba_{-1}n^2a_1^2=0,$$

$$Y^0:\ -4am^2n^2a_0^2k_1^2-8am^2n^2a_1k_1^2a_{-1}+am^2\mu^2a_1^2+4am^2\mu^2a_1a_{-1}+am^2\mu^2a_{-1}^2-4\omega mn^2a_0^2$$
$$-8\omega mn^2a_1a_{-1}-2amn\mu^2a_1^2-16amn\mu^2a_1a_{-1}-2amn\mu^2a_{-1}^2+4bkn^2a_0^4+4bn^2a_0^3$$
$$+48bkn^2a_0^2a_1a_{-1}+24bn^2a_0a_1a_{-1}+24bkn^2a_1^2a_{-1}^2=0,$$

$$Y^{-1}: -8am^2n^2a_0k_1^2a_{-1} - 8\omega mn^2a_0a_{-1} - 4amn\mu^2a_0a_{-1} + 16bkn^2a_0^3a_{-1} + 12bn^2a_0^2a_{-1}$$
$$+48bka_1n^2a_0a_{-1}^2 + 12ba_1n^2a_{-1}^2 = 0,$$
$$Y^{-2}: -4am^2n^2k_1^2a_{-1}^2 - 2am^2\mu^2a_{-1}^2 - 2aa_1m^2\mu^2a_{-1} - 4\omega mn^2a_{-1}^2 + 8aa_1mn\mu^2a_{-1}$$
$$+24bkn^2a_0^2a_{-1}^2 + 12bn^2a_0a_{-1}^2 + 16bka_1n^2a_{-1}^3 = 0,$$
$$Y^{-3}: 4bn^2a_{-1}^3 + 16bkn^2a_0a_{-1}^3 + 4amn\mu^2a_0a_{-1} = 0,$$
$$Y^{-4}: 4bkn^2a_{-1}^4 + am^2\mu^2a_{-1}^2 + 2amn\mu^2a_{-1}^2 = 0.$$

On solving the above algebraic equations with the aid of Maple 14, we have the following results:

Case 1. We have

$$a_0 = -\frac{2n+m}{4k(n+m)},\ a_1 = \pm\frac{2n+m}{4k(n+m)},\ a_{-1}=0, \mu = \sqrt{-\frac{bn^2(2n+m)}{4amk(n+m)^2}},$$
$$\omega = -\left(amk_1^2 + \frac{b(2n+m)}{4k(n+m)^2}\right). \tag{3.7}$$

Form (3.1), (3.2), (3.4), (3.6) and (3.7), we deduce the exact traveling wave solutions of Eq. (1.1) as follows:

$$q(x,t) = \left[-\frac{2n+m}{4k(n+m)}\left(1\pm\tanh\left(\sqrt{-\frac{bn^2(2n+m)}{4amk(n+m)^2}}(x+2mak_1)\right)\right)\right]^{\frac{1}{2n}}$$
$$\times e^{i\left(-k_1x-\left(amk_1^2+\frac{b(2n+m)}{4k(n+m)^2}\right)t+\theta\right)}, \tag{3.8}$$

Where $k<0$ and $ab>0$.

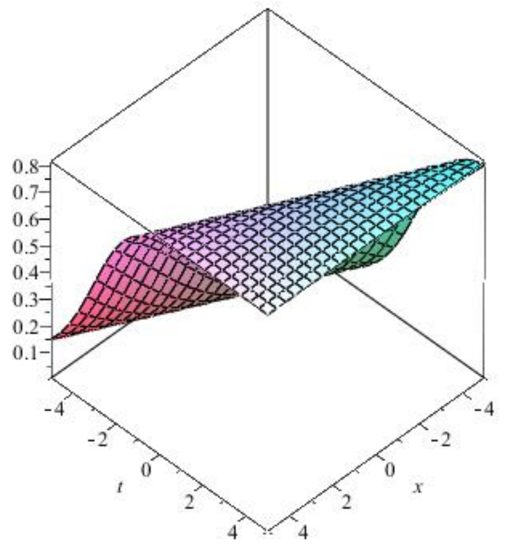

Fig1: The plot of the $|q(x,t)|$ of (3.8) when $n=1, m=2, b=2, k=-1, a=1, k_1=\frac{1}{2}$.

Case 2. We have

$$a_0=-\frac{2n+m}{4k(n+m)},\; a_1=0,\; a_{-1}=-\frac{2n+m}{4k(n+m)},\; \mu=\sqrt{-\frac{bn^2(2n+m)}{4amk(n+m)^2}},$$
$$\omega=-\left(amk_1^2+\frac{b(2n+m)}{4k(n+m)^2}\right). \tag{3.9}$$

In this case, we deduce the exact traveling wave solutions of Eq. (1.1) as follows:

$$q(x,t)=\left[-\frac{2n+m}{4k(n+m)}\left(1+\coth\left(\sqrt{-\frac{bn^2(2n+m)}{4amk(n+m)^2}}(x+2mak_1)\right)\right)\right]^{\frac{1}{2n}}$$
$$\times e^{i\left(-k_1x-\left(amk_1^2+\frac{b(2n+m)}{4k(n+m)^2}\right)t+\theta\right)}, \tag{3.10}$$

Where $k<0$ and $ab>0$.

Case 3. We have

$$a_0=-\frac{2n+m}{4k(n+m)},\; a_1=-\frac{2n+m}{8k(n+m)},\; a_{-1}=-\frac{2n+m}{8k(n+m)},$$
$$\mu=\sqrt{-\frac{bn^2(2n+m)}{16amk(n+m)^2}},\; \omega=-\left(amk_1^2+\frac{b(2n+m)}{4k(n+m)^2}\right). \tag{3.11}$$

In this case, we deduce the exact traveling wave solutions of Eq. (1.1) as follows:

$$q(x,t)=\left\{-\frac{2n+m}{4k(n+m)}\left[1+\frac{1}{2}\left(\tanh\left(\sqrt{-\frac{bn^2(2n+m)}{16amk(n+m)^2}}(x+2mak_1)\right)\right.\right.\right.$$
$$\left.\left.\left.+\coth\left(\sqrt{-\frac{bn^2(2n+m)}{16amk(n+m)^2}}(x+2mak_1)\right)\right)\right]\right\}^{\frac{1}{2n}} e^{i\left(-k_1x-\left(amk_1^2+\frac{b(2n+m)}{4k(n+m)^2}\right)t+\theta\right)}, \tag{3.12}$$

Where $k<0$ and $ab>0$.

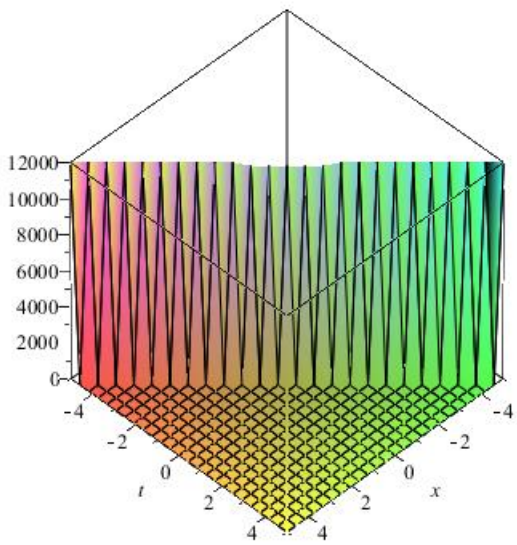

Fig 2: The plot of the $|q(x,t)|$ of (3.12) when $n=1, m=1, b=1, k=-1, a=1, k_1=\frac{1}{2}$.

3.2. Exact traveling wave solutions of Eq. (1.2)

In this subsection, we consider the exact traveling wave solutions of Biswas-Milovic equation (1.2) with variable coefficients. To this end, we assume that the solution of Eq. (1.2) can be written as:

$$q(x,t)=u(\xi)e^{i(-k_1x+\omega(t)t)},\quad \xi=x-\lambda(t)t, \tag{3.13}$$

where $u(\xi)$ is a real function of ξ, $\lambda(t)$ is the soliton velocity, k_1 is the wave number of the soliton, while $\omega(t)$ is the frequency of the soliton velocity. Substituting (3.13) into Eq. (1.2) and separating the real and imaginary parts, we obtain

$$t\frac{d\lambda(t)}{dt}+\lambda(t)+2mk_1a(t)=0, \tag{3.14}$$

and

$$a(t)\left(u^m\right)''-\left(mt\frac{d\omega(t)}{dt}+m\omega(t)+m^2k_1{}^2a(t)\right)u^m+b(t)u^{2n+m}+b(t)k(t)u^{4n+m}=0. \tag{3.15}$$

Integrating Eq. (3.14) with respect to t with vanishing the constant of integration we get

$$\lambda(t)=-\frac{2mk_1}{t}\int a(t)dt. \tag{3.16}$$

By balancing between $\left(u^m\right)''$ with u^{4n+m} in (3.15) we get $mN+2=N\left(4n+m\right)\Rightarrow N=\dfrac{1}{2n}$. According to step 3, we use the transformation

$$u(\xi)=v^{\frac{1}{2n}}(\xi), \tag{3.17}$$

where $v(\xi)$ is a new function of ξ. Substituting (3.17) into (3.15), we get the new ODE

$$\begin{aligned}&2nma(t)vv''+m(m-2n)a(t)(v')^2-4n^2m\left(\omega(t)+t\frac{d\omega(t)}{dt}+mk_1{}^2a(t)\right)v^2\\&+4n^2b(t)v^3+4n^2b(t)k(t)v^4=0.\end{aligned} \tag{3.18}$$

Balancing vv'' with v^4 in (3.18) we get $N+N+2=4N\Rightarrow N=1$. Consequently, Eq. (3.18) has the formal solution:

$$v(\xi)=a_0+a_1Y(\xi)+a_{-1}Y^{-1}(\xi), \tag{3.19}$$

where a_0, a_1 and a_{-1} are constants to be determined later satisfying $a_1^2+a_{-1}^2\neq 0$.

Now, substituting (3.19) along with Eqs. (2.5) and (2.6) into (3.18), collecting the coefficients of powers of $Y(\xi)$ and setting them to zero, we obtain the following system of algebraic equations:

$$Y^4:\ a(t)m^2\mu^2a_1^2+2a(t)mn\mu^2a_1^2+4b(t)k(t)n^2a_1^4=0,$$

$$Y^3:\ 4n^2b(t)a_1^3+16n^2b(t)k(t)a_0a_1^3+4mn\mu^2a(t)a_0a_1=0,$$

$$\begin{aligned}Y^2:\ &12n^2b(t)a_0a_1^2-4mn^2\omega(t)a_1^2-2m^2\mu^2a(t)a_1^2-2m^2\mu^2a(t)a_1a_{-1}+16n^2b(t)k(t)a_1^3a_{-1}\\&-4mn^2t\tfrac{d\omega(t)}{dt}a_1^2+24n^2b(t)k(t)a_0^2a_1^2-4m^2n^2a(t)a_1^2k_1^2+8mn\mu^2a(t)a_1a_{-1}=0,\end{aligned}$$

$$\begin{aligned}Y:\ &12n^2b(t)a_0^2a_1+12n^2b(t)a_1^2a_{-1}+16n^2b(t)k(t)a_0^3a_1-8mn^2\omega(t)a_0a_1\\&-8mn^2t\tfrac{d\omega(t)}{dt}a_0a_1+48n^2b(t)k(t)a_0a_1^2a_{-1}-4mn\mu^2a(t)a_0a_1-8m^2n^2a(t)a_0a_1k_1^2=0,\end{aligned}$$

$$Y^0 : 4n^2 b(t)a_0^3 + m^2\mu^2 a(t)a_1^2 + m^2\mu^2 a(t)a_{-1}^2 - 4mn^2\omega(t)a_0^2 + 4n^2 b(t)k(t)a_0^4$$

$$+4m^2\mu^2 a(t)a_1 a_{-1} - 8mn^2\omega(t)a_1 a_{-1} - 4mn^2 t\frac{d\omega(t)}{dt}a_0^2 + 24n^2 b(t)k(t)a_1^2 a_{-1}^2$$

$$-2mn\mu^2 a(t)a_1^2 + 24n^2 b(t)a_0 a_1 a_{-1} - 4m^2n^2 a(t)a_0^2 k_1^2 - 2mn\mu^2 a(t)a_{-1}^2 - 8mn^2 t\frac{d\omega(t)}{dt}a_1 a_{-1}$$

$$+48n^2 b(t)k(t)a_0^2 a_1 a_{-1} - 16mn\mu^2 a(t)a_1 a_{-1} - 8m^2n^2 a(t)a_1 k_1^2 a_{-1} = 0,$$

$$Y^{-1} : 12n^2 b(t)a_0^2 a_{-1} + 12n^2 b(t)a_1 a_{-1}^2 + 16n^2 b(t)k(t)a_0^3 a_{-1} - 8mn^2\omega(t)a_0 a_{-1}$$

$$-8mn^2 t\frac{d\omega(t)}{dt}a_0 a_{-1} + 48n^2 b(t)k(t)a_0 a_1 a_{-1}^2 - 4mn\mu^2 a(t)a_0 a_{-1} - 8m^2n^2 a(t)a_0 k_1^2 a_{-1} = 0,$$

$$Y^{-2} : 12n^2 b(t)a_0 a_{-1}^2 - 4mn^2\omega(t)a_{-1}^2 - 2m^2\mu^2 a(t)a_{-1}^2 - 2m^2\mu^2 a(t)a_1 a_{-1} + 16n^2 b(t)k(t)a_1 a_{-1}^3$$

$$-4mn^2 t\frac{d\omega(t)}{dt}a_{-1}^2 + 24n^2 b(t)k(t)a_0^2 a_{-1}^2 - 4m^2n^2 a(t)k_1^2 a_{-1}^2 + 8mn\mu^2 a(t)a_1 a_{-1} = 0,$$

$$Y^{-3} : 4n^2 b(t)a_{-1}^3 + 16n^2 b(t)k(t)a_0 a_{-1}^3 + 4mn\mu^2 a(t)a_0 a_{-1} = 0,$$

$$Y^{-4} : m^2\mu^2 a(t)a_{-1}^2 + 4n^2 b(t)k(t)a_{-1}^4 + 2mn\mu^2 a(t)a_{-1}^2 = 0.$$

On solving the above algebraic equations with the aid of Maple 14, we have the following results:

Case 1. We have

$$a_0 = -\frac{2n+m}{4(n+m)k(t)},\ a_1 = \pm\frac{2n+m}{4(n+m)k(t)},\ a_{-1} = 0,$$
$$\mu = \sqrt{-\frac{n^2(2n+m)b(t)}{4m(n+m)^2 a(t)k(t)}},\ \omega(t) = -\frac{1}{t}\int\left\{mk_1^2 a(t) + \frac{(2n+m)b(t)}{4(n+m)^2 k(t)}\right\}dt. \tag{3.20}$$

Form (3.13), (3.16), (3.17), (3.19) and (3.20), we deduce the exact traveling wave solutions of Eq. (1.2) as follows:

$$q(x,t) = \left\{-\frac{2n+m}{4(n+m)k(t)}\left[1\pm\tanh\left(\sqrt{-\frac{n^2(2n+m)b(t)}{4m(n+m)^2 a(t)k(t)}}(x+2mk_1\int a(t)\,\mathrm{d}t)\right)\right]\right\}^{\frac{1}{2n}}$$
$$\times e^{i\left(-k_1 x - \int\left\{mk_1^2 a(t) + \frac{(2n+m)b(t)}{4(n+m)^2 k(t)}\right\}dt + \theta\right)}, \tag{3.21}$$

Where $k(t) < 0$ and $a(t)b(t) > 0$.

Remark 1. Our result (3.8) for Eq. (1.1) and the result (3.21) for Eq. (1.2) have the same expressions as the results (18) and (31) of [52] respectively. But the authors [52] have derived the result (18) if $abk < 0$. From their analysis and the values of parameters of figure 1 of [52], it seems to us that the authors have chosen $ab < 0$ and $k > 0$. This yields the function $u(\xi)$ is complex. Therefore, the result (18) of [52] does not exist if $ab < 0$ and $k > 0$. The same discussion is applied for the result (31) of [52].

Case 2. We have

$$a_0 = -\frac{2n+m}{4(n+m)k(t)},\ a_1 = 0,\ a_{-1} = -\frac{2n+m}{4(n+m)k(t)},$$
$$\mu = \sqrt{-\frac{n^2(2n+m)b(t)}{4m(n+m)^2 a(t)k(t)}},\ \omega(t) = -\frac{1}{t}\int\left\{mk_1^2 a(t) + \frac{(2n+m)b(t)}{4(n+m)^2 k(t)}\right\}dt. \tag{3.22}$$

In this case, we deduce the exact traveling wave solutions of Eq. (1.2) as follows:

$$q(x,t)=\left\{-\frac{2n+m}{4(n+m)k(t)}\left[1+\coth\left(\sqrt{-\frac{n^2(2n+m)b(t)}{4m(n+m)^2a(t)k(t)}}(x+2mk_1\int a(t)\,\mathrm{d}t)\right)\right]\right\}^{\frac{1}{2n}} \tag{3.23}$$

$$\times e^{i\left(-k_1x-\int\left\{mk_1^2a(t)+\frac{(2n+m)b(\mathrm{t})}{4(n+m)^2k(t)}\right\}dt+\theta\right)},$$

Where $k(t)<0$ and $a(t)b(t)>0$.

Case 3. We have

$$a_0=-\frac{2n+m}{4(n+m)k(t)},\ a_1=-\frac{2n+m}{8(n+m)k(t)},\ a_{-1}=-\frac{2n+m}{8(n+m)k(t)},$$
$$\mu=\sqrt{-\frac{n^2(2n+m)b(t)}{16m(n+m)^2a(t)k(t)}},\ \omega(t)=-\frac{1}{t}\int\left\{mk_1^2a(t)+\frac{(2n+m)b(\mathrm{t})}{4(n+m)^2k(t)}\right\}dt. \tag{3.24}$$

In this case, we deduce the exact traveling wave solutions of Eq. (1.2) as follows:

$$q(x,t)=\left\{-\frac{2n+m}{4(n+m)k(t)}\left[1+\frac{1}{2}\left(\tanh\left(\sqrt{-\frac{n^2(2n+m)b(t)}{16m(n+m)^2a(t)k(t)}}(x+2mk_1\int a(t)\,\mathrm{d}t)\right)\right.\right.\right.$$
$$\left.\left.\left.+\coth\left(\sqrt{-\frac{n^2(2n+m)b(t)}{16m(n+m)^2a(t)k(t)}}(x+2mk_1\int a(t)\,\mathrm{d}t)\right)\right)\right]\right\}^{\frac{1}{2n}} \tag{3.25}$$

$$\times e^{i\left(-k_1x-\int\left\{mk_1^2a(t)+\frac{(2n+m)b(\mathrm{t})}{4(n+m)^2k(t)}\right\}dt+\theta\right)},$$

Where $k(t)<0$ and $a(t)b(t)>0$.

Remark 2. Our results (3.10), (3.12) for Eq. (1.1) and the results (3.23), (3.25) for Eq. (1.2) are new and not found in [52] or elsewhere. This shows that the extended tanh-function method is more general and effective than the $\left(\frac{G'}{G}\right)$-expansion method used in [52].

4. Conclusions

The extended tanh-function method is used in this article to obtain some new exact traveling wave solutions of the the Biswas-Milovic equation with dual-power law nonlinearity, which describes the propagation of solitons through optical fibers for trans-continental and trans-oceanic distances. From our results, we deduce that the solutions (3.8), (3.21) are kink shaped soliton solutions, the solutions (3.10), (3.23) are singular kink shaped soliton solutions and the solutions (3.12), (3.25) are kink-singular kink shaped soliton solutions. Note that all solutions obtained in this article are new and not reported elsewhere which have been checked with the Maple 14 by putting them back into the original equations (1.1) and (1.2).

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