



The Yukawa Potential Function and Reciprocal Univalent Function

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ABSTRACT

This paper presents a mathematical model that provides analytic connection between four fundamental forces (interactions), by using modified reciprocal theorem, derived in the paper, as a convenient template. The essential premise of this work is to demonstrate that if we obtain with a form of the Yukawa potential function [as a meromorphic univalent function], we may eventually obtain the Coloumb Potential as a univalent function outside of the unit disk. Finally, we introduce the new problem statement about assigning Meijer's G-functions to Yukawa and Coloumb potentials as an open problem.

Keywords: Strong and weak nuclear force; Univalent functions; Meromorphic univalent functions; reciprocal theorem; Meijer's G-function.



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INTRODUCTION

The concepts of preservation and the unit disk $E : |z| < 1$, as the domain, are essential in the theory of univalent functions, see for instance [1, 2, 3, 4]. In some transformations, the univalent functions are preserved. Such functions remain univalent after transformations such as translation, rotation, dilation, inversion, etc. Generally, these transformations increase the number of members in a family of univalent functions. Another important concept in handling univalent functions is this fact that the exterior of the unit disk can be as important as the unit disk itself. It is often useful to replace the disk E by the exterior of the unit disk, and the function $f(z)$ by its reciprocal $1/f(z)$ [1, 5].

Similarly, the concepts of invariance of some properties in nucleus play critical roles in the underlying dynamics of (nuclear) physics, and even lead to conservation laws that are universal, see for example [6, 7]. Translational invariance leads to the conservation of linear momentum, and rotational invariance leads to the conservation of angular momentum. Meanwhile, it is also acknowledged in nuclear physics that weak interactions are associated with the exchange of very heavy particles such as W and Z bosons, which have ranges that are of order $R_{W,Z} \approx 2 \times 10^{-13}$ m. On the other hand, the strong nuclear force (inside the nucleus) has a much shorter range of approximately $(1-2) \times 10^{-15}$ m. In contrast, the electromagnetic interaction has an infinite range, (exterior of nucleus), because the exchanged particle is a massless photon [8]. In light of these concepts of invariance and nuclear/ gravitational forces, this paper aims to offer the novel possibility to visualize [relevant] geometric connections and to justify [relevant] analytic connections at the same time wherein the paper provides a mathematical framework to link Gravitational and Electrostatic forces [with long (infinity) range (exterior of the unit disk)] with Strong and Weak nuclear forces with short range (in the unit disk)]. In essence, the mathematical framework in this paper begins with a single theorem of the univalent functions theory (the nuclear interaction described by the Yukawa potential function), and then develops a unique/ holistic presentation of the unification [through their inter-analytic connections] of all forces in nature!

Thereto, and firstly, the static solution of Klein-Gordon equation satisfies:

$$\nabla^2 \phi(r) = \frac{M_x^2 c^2}{\hbar^2} \phi(r),$$

where $\phi(r)$ is interpreted as a static potential.

where $r = |\vec{r}|$ and ϵ_0 represent the dielectric constant. The corresponding solution in the case where $M_x^2 \neq 0$ is easily rendered [by substitution] to be:

$$V_C(r) = -e\phi(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}, \quad (1.1)$$

$$V_Y(r) = -\frac{g^2}{4\pi} \frac{e^{-\frac{r}{R}}}{r}, \quad (1.2)$$

where R is the [Electrostatic/ nuclear force] range defined earlier, and g (the so-called coupling contrast) is a parameter associated with each vertex of a Feynman diagram, and represents the basic strength of the interaction. The form of $V_Y(r)$ in (1.2) is called the Yukawa potential, named after the physicist who, in 1935, first introduced the idea of forces resulting from the exchange of massive particles. As $M_x \rightarrow 0, R \rightarrow \infty$ and the Coulomb potential is recovered from the Yukawa potential, wherein, for very large masses, the interaction is approximately point-like (zero range). Here, it is also conventional to introduce a dimensionless parameter $\alpha_x = \frac{g^2}{4\pi\hbar c}$ that characterizes the strength of the interaction at short distances $r \leq R$. For electromagnetic interactions, however, this parameter becomes the fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$ that governs the splitting of atom energy levels [8].

This paper introduces a convenient mathematical framework to relate Yukawa potential which is used for describing strong and weak nuclear interactions with Electrostatic (Gravitational) potential. Herein, a modified form of the reciprocal theorem (a very useful theorem in univalent functions theory) is used to relate the inside of the punctured unit disk to exterior of it. Finally, we introduce an open problem: how do we mathematically and physically result analytic forms of the Yukawa and electrostatic (gravitational) potential functions from existing different distribution of zeros and poles on the complex plane (appeared in inverse Mellin transform and Meijer's G -functions)? These are Meijer's G -functions which is defined in terms of Mellin-Barnes path integrals.

Definition 1.1 [5] A function

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.3)$$

is said to be normalized univalent function. The class of all normalized functions that are regular, and univalent in unit disk $E : |z| < 1$, is denoted by S .

Also, it is often useful to replace the disk E by the exterior of the unit circle, and the function $f(z)$ by its reciprocal $1=f(z)$.



Further, let E^\times denotes the domain $1 < |\xi| < \infty$.

Definition 1.2 [5] A function $f(z)$ is said to omit γ in a domain D if the equation $f(z) = \gamma$ has no solution with z in D .

Theorem 1.1 [5] Suppose that $f(z)$ is in S and $f(z)$ omits γ in E . Then

$$g(z) = \frac{f(z)}{1 - f(z)/\gamma} = z + (a_2 + 1/\gamma)z^2 + \dots \tag{1.4}$$

is also in S .

Definition 1.3 [5] Let Σ be the class of all functions of the form:

$$\phi(\xi) = \xi + c_0 + \frac{c_1}{\xi} + \frac{c_2}{\xi^2} + \dots = \xi + \sum_{n=0}^{\infty} \frac{c_n}{\xi^n} \tag{1.5}$$

which are regular and univalent in E^\times . Thence, the subclasses of those functions that omit $\gamma = 0$ is denoted by Σ_0 .

Theorem 1.2 [5] If $f(z)$, given by (1.1), is in S , then

$$\phi(\xi) \equiv \frac{1}{f(1/\xi)} = \xi - a_2 + (a_2^2 - a_3)\frac{1}{\xi} + \dots \tag{1.6}$$

is in Σ_0 . Conversely, if $\phi(\xi)$, given by (1.3), is in Σ_0 , then

$$f(z) = \frac{1}{\phi(1/z)} = z - c_0z^2 + (c_0^2 - c_1)z^3 + (2c_0c_1 - c_2 - c_0^3)z^4 + \dots \tag{1.7}$$

is in S .

Gamma function

Through all analytic functions, Gamma function $\Gamma(z)$ has infinity poles at $n = 0, 1, 2, \dots$. However, it does not have any zeros.

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{z(z+1)\dots(z+n)} \tag{1.8}$$

It is meromorphic function, analytic except for isolated singularities which are poles. Vice versa, $1/\Gamma(z)$ has not any pole, instead it has infinity zeroes. In fact, $1/\Gamma(z)$ is an entire function, analytic on all of C . Its Weierstrass product gives

$$1/\Gamma(z) = ze^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n})e^{-z/n}, \tag{1.9}$$

where γ is known as the Euler constant. Gamma function with different argument has infinity poles in another places. In integral definition of Meijer's G-functions, we face with the ratio of products of Gamma functions in numerator and denominator. These functions are defined as follows:

Definition 1.4 A definition of the Meijer's G-function is given by the following path integral in the complex plane [9, 10]:

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \times \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \times \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds. \tag{1.10}$$

This integral is included in the so-called Mellin-Barnes type, and may be viewed as an inverse Mellin transform. In this case, an empty product means unity, and the integers, $m; n; p$ and q are called the orders of the G-function, or the components of the order $(m; n; p; q)$. Here, both a_p and b_q are called "parameters" which are generally complex numbers. The definition holds under the following assumptions: $0 \leq m \leq q$ and $0 \leq n \leq p$, where $m; n; p$ and q are integer numbers. $a_j - b_k \neq 1, 2, 3, \dots$ for $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ imply that no pole of any $\Gamma(b_j - s), j = 1, \dots, m$ coincides with any pole of any $\Gamma(1 - a_k + s), k = 1, \dots, n$. Evidence for the importance of the G-function is given by the fact that the basic elementary functions and most of the special functions of mathematical physics, including the generalized hypergeometric functions, follow as its particular cases. Each result concerning the G-function becomes a key leading to numerous particular results for the Bessel functions, confluent hypergeometric functions, classical orthogonal polynomials, etc.



2 Discussion and Results

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and let $w = 1/z$, so we can define a function $h(z) = f(1/z) = 1/z + \sum_{n=2}^{\infty} a_n (1/z)^n$ for $z \neq 0$. The function h has an isolated singularity (not essential) at $z = 0$.

Definition 2.1 A function

$$f(z) = \frac{1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \dots = \frac{1}{z} + \sum_{n=2}^{\infty} \frac{a_n}{z^n} \tag{2.1}$$

is said to be a complement of normalized univalent function. Further, the class of all normalized functions that are regular, and univalent in E^* , is denoted by Σ .

Definition 2.2 Let Σ be the class of all functions of the form:

$$\phi(z) = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \dots = \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n \tag{2.2}$$

which is said to be a complement of the Meromorphic univalent function. Also, the class of all functions that are regular and univalent in punctured unit disk E^* is denoted by Σ^* . Further, the subclasses of these functions that omit $\gamma = 0$ are then denoted by Σ_0^* .

Theorem 2.1 (Modified Reciprocal Theorem) If $f(z)$, given by (2.1), is in Σ , then

$$\phi(z) \equiv \frac{1}{f(1/z)} = \frac{1}{z} - a_2 + (a_2^2 - a_3)z + \dots \tag{2.3}$$

is in Σ_0^* . Conversely, if $\phi(z)$, given by (2.2), is in Σ_0^* , then

$$f(z) = \frac{1}{\phi(1/z)} = \frac{1}{z} - \frac{c_0}{z^2} + \frac{(c_0^2 - c_1)}{z^3} + \frac{(2c_0c_1 - c_2 - c_0^3)}{z^4} + \dots \tag{2.4}$$

is in Σ .

The atomic nucleus is often approximated to be a homogeneously charged sphere. Consequently, the radius of this sphere, R , is referred to as the nuclear radius. Quantitatively, thus, we have:

$$R = 1.21A^{\frac{1}{3}} \text{ fm}, \tag{2.5}$$

where A is a mass number, and fm (Fermi) is the common unit for length in nuclear and elementary particle physics, called femtometre (fm). The Fermi is a standard SI-unit, defined as $10^{-15}m$, and corresponds approximately to the size of a proton. In order to apply the reciprocal theorem we must first choose a convenient physical unit for the radius of the unit disk. Incidentally, the radius of a physical unit disk may be $1 m$; $1 mm$; $1 \mu m$; $1 fm$ or even each a convenient physical unit. For our framework, we choose the nuclear radius as the physical unit because nuclear forces at this length become zero [or different values, according to (3.1), for each nucleus with mass number A]. To use the reciprocal theorem, we must first write the series representation for the exponential factor in the normalized Yukawa potential function (because of coefficient $-4\pi/g^2$) thus:

$$-\frac{4\pi}{g^2} V_Y(r) = \frac{e^{-\frac{r}{R}}}{r} = \frac{1}{r} \sum_{n=0}^{\infty} \frac{(-\frac{r}{R})^n}{n!} = \frac{1}{r} - \frac{1}{R} + \left(\frac{1}{2R^2}\right)r + \left(\frac{-1}{6R^3}\right)r^2 + \dots \tag{2.6}$$

In other words, $c_0 = -\frac{1}{R}$; $c_1 = \left(\frac{1}{2R^2}\right)$; $c_2 = \left(\frac{-1}{6R^3}\right)$; ...

Then, we apply Theorem 2.1 to obtain the normalized columb potential (because of coefficient $-4\pi\epsilon_0/e^2$) outside of the physical unit disk :

$$\begin{aligned} -\frac{4\pi\epsilon_0}{e^2} V_C(r) &= 1/r - \frac{c_0}{r^2} + \frac{(c_0^2 - c_1)}{r^3} + \frac{(2c_0c_1 - c_2 - c_0^3)}{r^4} + \dots \\ -\frac{4\pi\epsilon_0}{e^2} V_C(r) &= 1/r + \left(\frac{1}{R}\right)\frac{1}{r^2} + \left(\frac{1}{2R^2}\right)\frac{1}{r^3} - \left(\frac{1}{6R^3}\right)\frac{1}{r^4} + \dots \end{aligned} \tag{2.7}$$

Since the range, R , for electrostatic and gravitational force is infinity, only the first term remains on the right side of the equation. This implies that if the range R had not been infinity,



we would have had an infinite number of terms. On this basis, it may be concluded that the existing link (in the reciprocal theorem) between the inside and outside of the unit disk offers an effective working template/ model to relate all [nuclear, gravitational and electrostatic] interactions.

Open problem

Using equation (1.10) for two cases of Meijer's G-function, namely: $G_{1,1}^{1,0}; G_{0,1}^{1,0}$; the following are obtained:

$$G_{1,1}^{1,0} \left[\begin{matrix} a_1 \\ b_1 \end{matrix} \middle| z \right] = \frac{1}{2\pi i} \int \frac{\Gamma(b_1 - s)}{\Gamma(a_1 - s)} z^s ds. \tag{2.8}$$

Position of poles: $b_1 - s = -n; n = 0; 1; 2; \dots$

Position of zeros: $a_1 - s = -n; n = 0; 1; 2; \dots$

We also have [9] $G_{1,1}^{1,0} \left[\begin{matrix} \gamma+1 \\ \gamma \end{matrix} \middle| z \right] = z^\gamma H(1-z)$ for $|z| < 1$ with $H(z)$ standing for the Heaviside unit function. we obtain

$$\frac{1}{z} = G_{1,1}^{1,0} \left[\begin{matrix} 0 \\ -1 \end{matrix} \middle| z \right]. \tag{2.9}$$

which is normalized Coulomb potential.

$$G_{0,1}^{1,0} \left[\begin{matrix} - \\ b_1 \end{matrix} \right] = \frac{1}{2\pi i} \int \Gamma(b_1 - s) z^s ds. \tag{2.10}$$

Position of poles: $b_1 - s = -n; n = 0; 1; 2; \dots$

Position of zeros: there are no zeros.

We also have [9] $G_{0,1}^{1,0} \left[\begin{matrix} - \\ \frac{\beta}{\alpha} \end{matrix} \middle| \eta z^\alpha \right] = \eta^{\frac{\beta}{\alpha}} z^\beta e^{-\eta z^\alpha}$, then we get

$$G_{0,1}^{1,0} \left[\begin{matrix} z \\ R \end{matrix} \middle| -1 \right] = R \frac{e^{-\frac{z}{R}}}{z}, \tag{2.11}$$

which is the Yukawa potential function.

New idea: these two G-functions are from two different family of G-function, see [11] so finding relationship between them (changes in their orders and parameters) and circumstance of changing one of them to the other one is new way. By using different properties of G-functions we can also use to relate them conveniently. Now, how do we mathematically and physically result analytic forms of the Yukawa and electrostatic (gravitational) potential functions from existing different distribution of zeros and poles on the complex plane belong to $G_{1,1}^{1,0} \left[\begin{matrix} - \\ -1 \end{matrix} \middle| z \right]$ and $G_{0,1}^{1,0} \left[\begin{matrix} z \\ R \end{matrix} \middle| -1 \right]$? and how do we explain the physical role of zeroes and poles in changing $G_{1,1}^{1,0} \left[\begin{matrix} - \\ -1 \end{matrix} \middle| z \right]$ to $G_{0,1}^{1,0} \left[\begin{matrix} z \\ R \end{matrix} \middle| -1 \right]$ (and vice versa) when we become near or far from the nucleus?

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