Generalized Hyers-Ulam stability of derivations on Lie C^*-algebras

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Abstract: In this paper, we investigate new generalized Hyers-Ulam stability results for (α, β, γ)-derivations on Lie C^*-algebras associated with the following (m,n)-Cauchy-Jensen additive functional equation:

$$\sum_{\substack{1 \le i_1 < \cdots < i_m \le n, 1 \le k_j \le n \\ k_j \ne i_j, \forall j \in \{1; \cdots, m\}}} f\left(\frac{1}{m} \sum_{j=1}^{m} x_{i_j} + \sum_{l=1}^{n-m} x_{k_l} \right) = \frac{n-m+1}{n} \binom{n}{m} \sum_{i=1}^{n} f(x_i)$$

by using the fixed point and direct methods.

Keywords: (α, β, γ)-derivation, (m,n)-Cauchy-Jensen additive functional equation, Lie C^*-algebra, generalized Hyers-Ulam stability.

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1   Introduction

The theory of finite dimensional complex Lie algebras is an important part of Lie theory. Lie algebras have many applications in physics and connections with other parts of mathematics. With an increasing amount of theory and applications concerning Lie algebras of various dimensions, it is becoming necessary to ascertain which tools are applicable for handling them. The miscellaneous characteristics of Lie algebras constitute such tools and have also found applications in Casimir operators [1], derived, lower central and upper central sequences, the Lie algebra of derivations, radicals, nilradicals, ideals, subalgebras [11], [20] and megaideals [19]. These characteristics are particularly crucial when considering possible affinities among Lie algebras. Recently, some authors have studied the stability problems of some functional equations in the setting of Lie algebras.

The stability problem concerning the stability of group homomorphisms of functional equations was originally introduced by Ulam [24], in 1940, as follows:

Let $(G_1, *)$ *be a group and* $(G_2, \Diamond, d)$ *be a metric group with the metric* $d(\cdot, \cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta(\varepsilon) > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality* $d(h(x * y), h(x) \Diamond h(y)) < \varepsilon$ *for all* $x, y \in G_1$, *then there is a homomorphism* $H : G_1 \to G_2$ *with* $d(h(x), H(x)) < \varepsilon$ *for all* $x \in G_1$?

If the answer is affirmative, we would say that the equation of a homomorphism $H(x * y) = H(x) \Diamond H(y)$ is *stable*. The famous Ulam stability problem was partially solved by Hyers [10] for linear functional equation of Banach spaces. Later, the results of Hyers were generalized by Aoki [2], Găvruta [8] and Rassias [23]. Cădariu and Radu [4] applied the fixed point method to investigation of the stability for a Jensen functional equation. They could present a short and a simple proof, which is different from the direct method initiated by Hyers in 1941, for the generalized Hyers-Ulam stability for a Jensen functional equation. In 2008, Novotny and Hrivnak [15] investigated generalizing the concept of Lie derivations via certain complex parameters and obtained various Lie and established the structure and properties of (α, β, γ) -derivations of Lie algebras.

A C^* -algebra A endowed with the Lie product $[x, y] = \dfrac{xy - yx}{2}$ on A is called a *Lie C^* -algebra* (see [16]). Let A be a C^* -algebra. A $\mathbb{C}$ -linear mapping $D : \mathsf{A} \to \mathsf{A}$ is called a (α, β, γ) -*derivation* of A if there exist $\alpha, \beta, \gamma \in C$ such that

$$\alpha D([x, y]) = \beta[D(x), y] + \gamma[x, D(y)]$$

for all $x, y \in \mathsf{A}$ (see [15]).

Let k be a fixed positive integer. We recall that a mapping $\rho : A \to B$ having the domain A and the codomain $(B, \le)$ that are both closed under addition is called a *contractively subadditive mapping* if there exists a constant L with $0 < L < 1$ such that

$$\rho(x + y) \le L(\rho(x) + \rho(y))$$

and an *expansively superadditive mapping* if there exists a constant L with $0 < L < 1$ such that

$$\rho(x + y) \ge \frac{1}{L}(\rho(x) + \rho(y))$$

for all $x, y \in A$. A mapping $\rho : A \to B$ is called a *homogeneous of degree* k if

$$\rho(\lambda x) = \lambda^k \rho(x)$$

for all $x \in A$. Also, if there exists a constant L with $0 < L < 1$ such that a mapping $\rho : A^n \to B$ satisfies

$$\rho(x_1, \ldots, x_{i-1}, \lambda^\ell x, x_{i+1}, \ldots, x_n) \le \lambda^\ell L \rho(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

for all $x, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in A$ and positive integer λ, then we say that ρ is a n -*contractively subhomogeneous mapping* if $\ell = 1$ and ρ is an n -*expansively superhomogeneous mapping* if $\ell = -1$. Note that ρ satisfies the properties

$$\rho(x_1,\ldots,x_{i-1},\lambda^{\ell k}x,x_{i+1},\ldots,x_n) \le (\lambda^{\ell}L)^k \rho(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_n),$$

$$\rho(\lambda^{\ell}x,\ldots,\lambda^{\ell}x) \le (\lambda^{\ell}L)^n \rho(x,\ldots,x)$$

Now, we consider a mapping $f : X \to Y$ satisfying the following functional equation:

$$\sum_{\substack{1 \le i_1 < \cdots < i_m \le n, 1 \le k_j \le n \\ k_l \ne i_j, \forall j \in \{1;\cdots m\}}} f\left(\frac{1}{m}\sum_{j=1}^{m}x_{i_j} + \sum_{l=1}^{n-m}x_{k_l}\right) = \frac{n-m+1}{n}\binom{n}{m}\sum_{i=1}^{n}f(x_i) \qquad (1.1)$$

for all $x_i,\ldots,x_n \in X$ where $n,m \in Z^+$ are fixed integers with $n \ge 2,\ 1 \le m \le n$.

We observe that, in case $m = 1$, the equation (1.1) yields the following Cauchy additive equation:

$$\sum_{1 \le i_l \le n} f(x_{i_l} + \sum_{k_l \ne i_j}^{n}x_{k_l}) = n\sum_{i=1}^{n}f(x_i).$$

Also, we observe that, in case $m = n$, the equation (1.1) yields the following Jensen additive equation:

$$\sum_{1 \le i_1 < \cdots < i_n \le n} f(\frac{1}{n}\sum_{j=1}^{n}x_{i_j}) = \frac{1}{n}\sum_{i=1}^{n}f(x_i).$$

Therefore, the functional equation (1.1) is a generalized form of the Cauchy-Jensen additive equation and thus every solution of the equation (1.1) may be analogously called the *general (m,n)-Cauchy-Jensen additive equation*.

Let X and Y be linear spaces. For each $m \in Z^+$ with $1 \le m \le n$, a mapping $f : X \to Y$ satisfies the equation (1.1) for all $n \ge 2$ if and only if $f(x) - f(0) = A(x)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n-m+1)x) = (n-m+1)f(x)$ and $f(mx) = mf(x)$ for all $x \in X$.

Recently, Asgari et al. [3] established the generalized Hyers-Ulam-Rassias stability of the ternary homomorphisms and ternary derivations between fuzzy ternary Banach algebras associated to the functional equation (1.1). Rassias et al. [21] proved the generalized Hyers-Ulam stability of homomorphisms and derivations in C^*-ternary algebras associated with the functional equation (1.1). Also, Gordji et al. [9] investigated the stability of (α,β,γ)-derivation on Lie C^*-algebras. For more details about the stability for various types of derivations, refer to [6], [12], [13], [14], [18] and [22].

In this paper, using some strategies from [9], [15] and [21], we investigate new stability of (α,β,γ)-derivations on Lie C^*-algebras associated to the general (m,n)-Cauchy-Jensen type additive functional equation (1.1) by using the fixed point and direct methods.

Throughout this paper, let A be a Lie C^*-algebra, $T^1_{1/n_o} = \{e^{i\theta} : 0 \le \theta \le 2\pi/n_o\}$ and $\lambda = n-m+1 \in Z^+$ be a fixed positive integer with $n \ge 2$ and $1 \le m \le n$. For any mapping $f : A \to A$, we define

$$\Delta_{\mu}f(x_1,\ldots,x_n) = \sum_{\substack{1 \le i_1 < \cdots < i_m \le n, 1 \le k_j \le n \\ k_l \ne i_j, \forall j \in \{1;\cdots m\}}} f\left(\frac{1}{m}\sum_{j=1}^{m}\mu x_{i_j} + \sum_{l=1}^{n-m}\mu x_{k_l}\right) - \frac{n-m+1}{n}\binom{n}{m}\sum_{i=1}^{n}f(\mu x_i)$$

for all $x_1,\ldots,x_n,x,y \in A$, $\mu \in T^1_{1/n_o}$ and for some $\alpha,\beta,\gamma \in C$.

2 Stability of (α, β, γ)-derivations on Lie C^*-algebras

In this section, we give some new generalized Hyers-Ulam stability results for (α, β, γ)-derivation on Lie C^*-algebras associated to the equation $\Delta_\mu f(x_1, \ldots, x_n) = 0$ via two methods.

2.1. Fixed point method

Let us recall that a mapping $d : X^2 \to [0, \infty)$ is called a generalized metric on a nonempty set X if (1) $d(x, y) = 0$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$; (3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The following fixed point theorem proved by Diaz and Margolis [7] plays an important role in proving our theorem:

Theorem 2.1 [7] Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with Lipshitz constant L. Then, for each $x \in \Omega$, either $d(T^n x, T^{n+1} x) = \infty$ for all nonnegative integers $n \ge 0$ or there exists a natural number n_o such that

(1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \ge n_o$;

(2) the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T;

(3) y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{n_o} x, y) < \infty\}$;

(4) $d(y, y^*) \le \dfrac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.

Theorem 2.2 Assume that there exist a contractively subadditive mapping $\phi : A^n \to [0, \infty)$ and a 2-contractively subhomogeneous mapping $\psi : A^2 \to [0, \infty)$ with the constant $L < 1$ such that a mapping $f : A \to A$ satisfies the following conditions:

$$\| \Delta_\mu f(x_1, \ldots, x_n) \| \le \phi(x_1, \ldots, x_n), \tag{2.1}$$

$$\| \alpha f([x, y]) - \beta[f(x), y] - \gamma[x, f(y)] \| \le \psi(x, y) \tag{2.2}$$

for all $x_1, \ldots, x_n, x, y \in A$ and $\mu \in T^1_{1/n_o}$ and for some $\alpha, \beta, \gamma \in \mathbb{C}$. Then there exists a unique (α, β, γ)-derivation $D : A \to A$ which satisfies the equation (1.1) and the following inequality:

$$\| f(x) - D(x) \| \le \dfrac{1}{\binom{n}{m}(n - m + 1)(1 - L)} \phi(x, \ldots, x) \tag{2.3}$$

for all $x \in A$.

Proof. Letting $x_1, \ldots, x_n = x$ and $\mu = 1$ in (2.1), we have

$$\left\| \binom{n}{m} f((n - m + 1)x) - \binom{n}{m}(n - m + 1)f(x) \right\| \le \phi(x, \ldots, x), \tag{2.4}$$

which gives

$$\left\| f(x) - \frac{1}{\lambda} f(\lambda x) \right\| \leq \frac{1}{\binom{n}{m}\lambda} \phi(x,\ldots,x) \tag{2.5}$$

for all $x \in A$, where $\lambda = n - m + 1$.

Let Ω be a set of all mappings from A into A and introduce a generalized metric on Ω as follows:

$$d(g,h) = \inf\{C \in [0,\infty) : \| g(x) - h(x) \| \leq C\phi(x,\ldots,x) \text{ for all } x \in A\}.$$

Then (Ω,d) is a generalized complete metric space ([5]). We consider the mapping $T : \Omega \to \Omega$ defined by

$$(Tg)(x) = \frac{1}{\lambda} g(\lambda x)$$

for all $g \in \Omega$ and $x \in A$. Let $g,h \in \Omega$ and $C \in [0,\infty)$ be an arbitrary constant with $d(g,h) < C$. Then we have

$$\left\| (Tg)(x) - (Th)(x) \right\| \leq \frac{C}{\lambda} \phi(\lambda x,\ldots,\lambda x) \leq LC\phi(x,\ldots,x) \tag{2.6}$$

for all $x \in A$, which means that

$$d(Tg,Th) \leq Ld(g,h)$$

for all $g,h \in \Omega$. Then T is a strictly contractive self-mapping on Ω with the Lipschitz constant L. It follows from (2.5) that

$$d(Tf,f) \leq \frac{1}{\binom{n}{m}\lambda}.$$

Thus, from Theorem 2.1, there exists a mapping D, which is a unique fixed point of T in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$D(x) = \lim_{k\to\infty} \frac{1}{\lambda^k} f(\lambda^k x) = \lim_{k\to\infty} \frac{1}{(n-m+1)^k} f((n-m+1)^k x) \tag{2.7}$$

for all $x \in A$ since $\lim_{k\to\infty} d(T^k f, \delta) = 0$. Again, from Theorem 2.1, we have

$$d(f,D) \leq \frac{1}{1-L} d(Tf,f) \leq \frac{1}{\binom{n}{m}\lambda(1-L)} = \frac{1}{\binom{n}{m}(n-m+1)(1-L)} \tag{2.8}$$

and so the inequality (2.3) holds. From (2.1), (2.7) and a contractive subadditive mapping of ϕ, it follows that

$$\left\| \Delta_\mu D(x_1,\ldots,x_n) \right\| \leq \lim_{k\to\infty} \frac{1}{\lambda^k} \left\| \Delta_\mu f(\lambda^k x_1,\ldots,\lambda^k x_n) \right\| \leq \lim_{k\to\infty} L^k \phi(x_1,\ldots,x_n) = 0, \tag{2.9}$$

which gives $\Delta_\mu D(x_1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in A$, $\mu \in T^1_{1/n_o}$. If we put $\mu = 1$ in (2.9), then D is additive. Also, letting $x_1 = x$ and $x_2 = \ldots = x_n = 0$ in (2.9), we have $D(\mu x) = \mu D(x)$. By the same reasoning as is the proof of Theorem 2.1 of [17], the mapping $D \in \Omega$ is $\mathbb{C}$-linear.

It follows from the linearity of D and (2.2) that

$$\left\| \alpha D([x,y]) - \beta[D(x),y] - \gamma[x,D(y)] \right\|$$

$$= \lim_{k\to\infty} \left\| \alpha \frac{1}{\lambda^{2k}} f[\lambda^k x, \lambda^k y] - \beta[\frac{1}{\lambda^k} f(\lambda^k x), y] - \gamma[x, \frac{1}{\lambda^k} f(\lambda^k y)] \right\|$$

$$\leq \lim_{k\to\infty} \frac{1}{\lambda^{2k}} \psi(\lambda^k x, \lambda^k y) \leq \lim_{k\to\infty} L^{2k} \psi(x,y) = 0$$

for all $x,y \in A$ and for some $\alpha,\beta,\gamma \in C$. Then we have

$$\alpha D([x,y]) = \beta[D(x),y] + \gamma[x,D(y)]$$

for all $x,y \in A$ and for some $\alpha,\beta,\gamma \in C$. Therefore, D is a unique (α,β,γ)-derivation on a Lie C^*-algebra A satisfying (2.3). This complete the proof.

Theorem 2.3 Assume that there exist an expansively superadditive mapping $\phi: A^n \to [0,\infty)$ and a 2-expansively superhomogeneous mapping $\psi: A^2 \to [0,\infty)$ with the constant $L < 1$ such that a mapping $f: A \to A$ satisfies (2.1) and (2.2). Then there exists a unique (α,β,γ)-derivation $D: A \to A$ which satisfies the equation (1.1) and the following inequality:

$$\| f(x) - D(x) \| \leq \frac{L}{\binom{n}{m}(n-m+1)(1-L)} \phi(x,\ldots,x) \qquad (2.10)$$

for all $x \in A$.

Proof. It follows from (2.4) and an expansively superadditive mapping of ϕ that

$$\left\| \lambda f(\frac{x}{\lambda}) - f(x) \right\| \leq \frac{L}{\binom{n}{m}\lambda} \phi(x,\ldots,x) \qquad (2.11)$$

for all $x \in A$.

Let Ω and d be same as in the proof of Theorem 2.2. Then (Ω, d) is a generalized complete metric space. We consider the mapping $T: \Omega \to \Omega$ defined by

$$(Tg)(x) = \lambda g(\frac{x}{\lambda})$$

for all $g \in \Omega$, $x \in A$. Thus we have $d(Tg, Th) \leq L d(g,h)$ for all $g,h \in \Omega$ and, by (2.11),

$$d(Tf, f) \leq \frac{L}{\binom{n}{m}\lambda} = \frac{L}{\binom{n}{m}(n-m+1)}.$$

It follows from Theorem 2.1 that there exists a mapping D, which is a unique fixed point of T in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$D(x) = \lim_{k \to \infty} \lambda^k f\left(\frac{x}{\lambda^k}\right)$$

for all $x \in A$. Then we have

$$d(f,D) \le \frac{1}{1-L} d(Tf,f) \le \frac{L}{\binom{n}{m}\lambda(1-L)} = \frac{L}{\binom{n}{m}(n-m+1)(1-L)}$$

and so the inequality (2.10) holds. The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.2. This complete the proof.

From Theorems 2.2 and 2.3, we have the following:

Corollary 2.4 Let $r, s \in \mathbb{R}$ with $r \ne 1, s \ne 2$ and θ be positive real numbers. Suppose that a mapping $f : A \to A$ satisfies the following conditions:

$$\| \Delta_1 f(x_1, \ldots x_n) \| \le \theta \sum_{i=1}^{n} \| x_i \|^r, \tag{2.12}$$

$$\| \alpha f([x,y]) - \beta[f(x),y] - \gamma[x,f(y)] \| \le \theta(\| x \|^s + \| y \|^s) \tag{2.13}$$

for all $x_1, \ldots, x_n, x, y \in A, \ \mu \in T^1_{1/n_o}$ and for some $\alpha, \beta, \gamma \in \mathbb{C}$. Then there exists a unique (α, β, γ) -derivation $D : A \to A$ which satisfies the equation (1.1) and the following inequality:

$$\| f(x) - D(x) \| \le \begin{cases} \dfrac{n\theta \| x \|^r}{\binom{n}{m}((n-m+1) - (n-m+1)^r)}, & r < 1, s < 2; \\[20pt] \dfrac{n\theta \| x \|^r}{\binom{n}{m}((n-m+1)^r - (n-m+1))}, & r > 2, s > 2; \end{cases} \tag{2.14}$$

for all $x \in A$.

Proof. The proof follows from Theorems 2.2 and 2.3 by taking $\phi(x_1, \ldots, x_n) = \theta \sum_{i=1}^{n} \| x_i \|^r$ and $\psi(x,y) = \theta(\| x \|^s + \| y \|^s)$ for all $x_1, \ldots, x_n, x, y \in A$. Then we can choose $L = (n-m+1)^{r-1}$ if $r < 1$ and $L = (n-m+1)^{1-r}$ if $r > 2$, respectively, and so we can obtain the desired result.

Corollary 2.5 Let $r_i \in \mathbb{R}$ with $r = \sum_{i=1}^{n} r_i \ne 1$, s with $s \ne 2$ and θ be positive real numbers. Suppose that a mapping $f : A \to A$ satisfies the following conditions:

$$\| \Delta_1 f(x_1, \ldots x_n) \| \le \theta \prod_{i=1}^{n} \| x_i \|^{r_i}, \tag{2.15}$$

$$\| \alpha f([x,y]) - \beta[f(x),y] - \gamma[x,f(y)] \| \le \theta \| x \|^{\frac{s}{2}} \cdot \| y \|^{\frac{s}{2}} \tag{2.16}$$

for all $x_1,\ldots,x_n,x,y \in A$ and for some $\alpha,\beta,\gamma \in C$. Then there exists a unique (α,β,γ)-derivation $D:A \to A$ which satisfies the equation (1.1) and the following inequality:

$$\| f(x) - D(x) \| \le \begin{cases} \dfrac{\theta \| x \|^r}{\dbinom{n}{m}((n-m+1)-(n-m+1)^r)}, & r < 1, s < 2; \\[4ex] \dfrac{\theta \| x \|^r}{\dbinom{n}{m}((n-m+1)^r-(n-m+1))}, & r > 2, s > 2; \end{cases}$$

(2.17)

for all $x \in A$.

Proof. Putting $\phi(x_1,\ldots,x_n) = \theta \prod_{i=1}^{n} \| x_i \|^{r_i}$ and $\psi(x,y) = \theta(\| x \|^{\frac{s}{2}} \cdot \| y \|^{\frac{s}{2}})$ for all $x_1,\ldots,x_n,x,y \in A$ and choosing $L = (n-m+1)^{r-1}$ if $r < 1$ and $L = (n-m+1)^{1-r}$ if $r > 1$, respectively, we obtain the desired result by Theorems 2.2 and 2.3.

2.2. Direct method

In this subsection, we apply the direct method to investigate the new generalized Hyers-Ulam stability results for (α,β,γ)-derivations on Lie C^*-algebras associated with the equation (1.1).

Theorem 2.6 Assume that there exist a contractively subadditive mapping $\phi:A^n \to [0,\infty)$ and a 2-contractively subhomogeneous mapping $\psi:A^2 \to [0,\infty)$ with the constant $L < 1$ such that a mapping $f:A \to A$ satisfies (2.1) and (2.2). Then there exists a unique (α,β,γ)-derivation $D:A \to A$ which satisfies the equation (1.1) and the following inequality:

$$\| f(x) - D(x) \| \le \frac{1}{\dbinom{n}{m}(n-m+1)(1-L)} \phi(x,\ldots,x)$$

(2.18)

for all $x \in A$.

Proof. If we replace x by $\lambda^j x$ and divide λ^j both sides of (2.5), then we have

$$\left\| \frac{f(\lambda^j x)}{\lambda^j} - \frac{f(\lambda^{j+1} x)}{\lambda^{j+1}} \right\| \le \frac{1}{\dbinom{n}{m}\lambda} \phi(\lambda^j x,\ldots,\lambda^j x)$$

for all $x \in A$. Thus we have

$$\left\| \frac{f(\lambda^k x)}{\lambda^k} - \frac{f(\lambda^l x)}{\lambda^l} \right\| \le \frac{1}{\dbinom{n}{m}\lambda} \sum_{j=k}^{l-1} \frac{1}{\lambda^j} \phi(\lambda^j x,\ldots,\lambda^j x) \le \frac{1}{\dbinom{n}{m}\lambda} \sum_{j=k}^{l-1} L^j \phi(x,\ldots,x)$$

(2.19)

for all $x \in A$ and $l,k \in Z^+$ with $l > k \ge 0$, which implies that the sequence $\left\{ \dfrac{f(\lambda^l x)}{\lambda^l} \right\}$ is a Cauchy

sequence for all $x \in A$ and so it converges. Thus we can define a mapping $D : A \to A$ by $D(x) = \lim_{l \to \infty} \dfrac{f(\lambda^l x)}{\lambda^l}$ for all $x \in A$. Then we have

$$
\begin{aligned}
\| \Delta_\mu D(x_1, \ldots, x_n) \| &\leq \lim_{l \to \infty} \frac{1}{\lambda^l} \| \Delta_\mu f(\lambda^l x_1, \ldots, \lambda^l x_n) \| \\
&\leq \lim_{l \to \infty} \frac{1}{\lambda^l} \phi(\lambda^l x_1, \ldots, \lambda^l x_n) = 0,
\end{aligned}
\tag{2.20}
$$

which gives $\Delta_\mu D(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in A$ and $\mu \in T^1_{1/n_o}$. If we put $\mu = 1$ in (2.20), then the mapping D is additive. Letting $x_1 = x$ and $x_2 = \cdots = x_n = 0$, we have $D(\mu x) = \mu D(x)$. By the same reasoning as is the proof of Theorem 2.1 of [17], D is C-linear. Further, we have

$$
\begin{aligned}
\| \alpha D[x, y] &- \beta[Dx, y] - \gamma[x, D(y)]) \| \\
&= \lim_{l \to \infty} \| \alpha \frac{1}{\lambda^{2l}} f[\lambda^l x, \lambda^l y] - \beta[\frac{1}{\lambda^l} f(\lambda^l x), y] - \gamma[x, \frac{1}{\lambda^l} f(\lambda^l y)] \| \\
&\leq \lim_{l \to \infty} \frac{1}{\lambda^{2l}} \psi(\lambda^l x, \lambda^l y) \\
&= 0
\end{aligned}
$$

for all $x, y \in A$. So, we have $\alpha D[x, y] = \beta[Dx, y] + \gamma[x, D(y)]$ for all $x, y \in A$ and for some $\alpha, \beta, \gamma \in C$. Letting $l \to \infty$ in (2.19) with $k = 0$, we can find that the mapping D is a (α, β, γ)-derivation on a Lie C^*-algebra A satisfying (2.18).

Next, let $D' : A \to B$ be another (α, β, γ)-derivation on A satisfying (2.18). Then we have

$$
\begin{aligned}
\| D(x) &- D'(x) \| \\
&\leq \lim_{l \to \infty} \frac{1}{\lambda^l} \left(\| D(\lambda^l x) - f(\lambda^l x) \| + \| D'(\lambda^l x) - f(\lambda^x) \| \right) \\
&\leq \frac{2}{\binom{n}{m} \lambda} \lim_{l \to \infty} L^l \phi(x, \ldots, x) = 0
\end{aligned}
$$

for all $x \in A$. Thus we can conclude that $D(x) = D'(x)$ for all $x \in A$. This complete the proof.

Theorem 2.7 Assume that there exist an expansively superadditive mapping $\phi : A^n \to [0, \infty)$ and a 2-expansively superhomogeneous mapping $\psi : A^2 \to [0, \infty)$ with the constant $L < 1$ such that a mapping $f : A \to A$ satisfies (2.1) and (2.2). Then there exists a unique (α, β, γ)-derivation $D : A \to A$ which satisfies the equation (1.1) and the following inequality:

$$
\| f(x) - D(x) \| \leq \frac{L}{\binom{n}{m}(n - m + 1)(1 - L)} \phi(x, \ldots, x)
\tag{2.21}
$$

for all $x \in A$.

Proof. The proof is similar to that of Theorem 2.6.

Corollary 2.8 Let θ_1, θ_2 be positive real numbers. Suppose that a mapping $f : A \to A$ satisfies

$$\| \Delta_1 f(x_1, \ldots x_n) \| \le \theta_1, \tag{2.22}$$

$$\| \alpha f([x,y]) - \beta[f(x),y] - \gamma[x,f(y)] \| \le \theta_2 \tag{2.23}$$

for all $x_1, \ldots, x_n, x, y \in A$ and for some $\alpha, \beta, \gamma \in C$. Then there exists a unique (α, β, γ)-derivation $D : A \to A$ which satisfies

$$\| f(x) - D(x) \| \le \frac{\theta_1}{\binom{n}{m}(n-m)}, \tag{2.24}$$

for all $x \in A$.

Remark. Let $\phi : A^n \to [0,\infty)$ and $\psi : A^2 \to [0,\infty)$ be mappings such that

$$\sum_{j=0}^{\infty} \frac{1}{\binom{n}{m} \lambda^{j+1}} \phi(\lambda^j x, \cdots, \lambda^j x) < \infty$$

and

$$\lim_{j \to \infty} \frac{1}{\lambda^j} \phi(\lambda^j x_1, \cdots, \lambda^j x_n) = 0, \qquad \lim_{j \to \infty} \frac{1}{\lambda^{2j}} \psi(\lambda^j x, \lambda^j y) = 0$$

for all $x_1, \ldots, x_n, x, y \in A$, where $\lambda = n - m + 1$. Suppose that $f : A \to A$ is a mapping which satisfies (2.1) and (2.2). By similar method to the proof Theorem 2.6, we can show that there exists a unique (α, β, γ)-derivation $D : A \to A$ which satisfies (1.1) and

$$\| f(x) - D(x) \| \le \frac{1}{\binom{n}{m} \lambda} \sum_{j=0}^{\infty} \frac{1}{\lambda^j} \phi(\lambda^j x, \cdots, \lambda^j x)$$

for all $x \in A$.

For the case $\phi(x_1, \cdots, x_n) = \varepsilon + \theta \sum_{i=1}^{n} \| x_i \|^r$ and $\psi(x,y) = \varepsilon + \theta(\| x \|^s + \| y \|^s)$ (where ε, θ are positive real numbers and $r, s \in R$ with $r < 1, s < 2$), there exists a unique (α, β, γ)-derivation $D : A \to A$ satisfying

$$\| f(x) - D(x) \| \le \frac{\varepsilon}{\binom{n}{m}(n-m)} + \frac{n\theta \| x \|^r}{\binom{n}{m}((n-m+1) - (n-m+1)^r)}$$

for all $x \in A$.

References

[1]   Abellanas, L. and Alonso, L. 1975. A general setting for Casimir invariants, J. Math. Phys. 16, 1580-1584.

[2]   Aoki, T. 1950. On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan. 2, 64-66.

[3]   Asgari, Gh., Cho, Y.J., Lee, Y.W. and Gordji, M.E. 2013. Fixed points and stability of functional equations in fuzzy ternary Banach algebras, J. Inq. Appl. 2013:166 doi:10,1186/1029-242X-2013-166.

[4]   C$\breve{a}$dariu, L. and Radu, V. 2003. Fixed points and the stability of Jensen's functional equation, J. Ineq. Pure Appl. Math. 4, No.1, Art. 4.

[5]   C$\breve{a}$dariu, L. and Radu, V. 2004. On the stability of the Cauchy functional equation: a fixed point approach, Grazer Math. Berichte 346, 43-52.

[6]   Cho, Y.J., Saadati, R. and Vahidi, J. 2012. Approximation of homomorphisms and derivations on non-Archimedean Lie C^*-algebras via fixed point method, Dis. Dyn. Nat. Soc. 2012, Article ID 373904, pp 9.

[7]   Dias, J.B. and Margolis, B. 1968. A fixed point theorem of the alternative for contrations on a generalized complete metric space, Bull. Amer. Math. Soc. 74, 305-309.

[8]   G$\breve{a}$vruta, P. 1994. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184(1994), 431-436.

[9]   Gordji, M.E. and Ghobadipour, N. 2010. Stability of (α, β, γ)-derivation on Lie C^*-algebras, Int. J. Geom. Methods in Modern Phys. 7, 1093-1102.

[10]   Hyers, D.H. 1941. On the stability of the linear functional equation, Proc. Natl. Acad. Sci. 27, 222-224.

[11]   Jacobson, N. 1979. Lie Algebras, Dover, New York.

[12]   Kim, S.S., Rassias, J.M., Cho, Y.J. and Kim, S.H. 2013. Stability of n-Lie homomorphisms and Jordan n-Lie homomorphisms on n-Lie algebras, J. Math. Phys. 54, 53501.

[13]   Mirmostaface, A.K. 2012. Pertubation of generalized derivations in fuzzy Menger normed algebras, Fuzzy sets and systems, 195, 109-117.

[14]   Najati, A., Park, C. and Lee, J.R. 2009. Homomorphisms and derivations in C^*-ternary algebras, Abstract Appl. Anal. 2009, Artical ID 612392.

[15]   Novotny, P. and Hrivnak, J. 2008. On (α, β, γ)-derivations of Lie algebras and corresponding invariant functions, J. Geom. Phys. 58, 208-217.

[16]   Park, C. 2005. Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations, J. Lie Theory 15, 393-414.

[17]   Park, C. 2005. Homomorphisms between Poissin JC^*-algebras, Bull. Brazil. Math. Soc. 36, 79-97.

[18]   Park, C. and Rassias, J.M. 2009. Stability of the Jensen-type functional equation in C*-algebras: a fixed point approach, Abs. Appl. Anal. Art. ID 360432.

[19]   Popovych, R., Boyko, V., Nesterenko, M. and Lutfullin, M. 2003. Realizations of real low-dimensional Lie algebras, J. Phys. A36. 7337-7360.

[20]   Rand, D., Winternitz, P. and Zassenhaus, H. 1988. On the identification of Lie algebra given by its structure constants I. Direct decompositions, Levi decompositions and nil radicals, Linear Algebra Appl. 109, 197-246.

[21]   Rassias, J.M. and Kim, H.M. 2008. Approximate homomorphisms and derivations between C^*-ternary algebras, J. Math. Phys. 49. 063507.

[22]   Rassias, J.M., Jun, K.W. and Kim, H.M. 2011. Approximate (m,n)-Cauchy-Jensen additive mappings in C^*-algebras, Acta Math. Sinica, 27, 1907-1922.

[23]   Rassias, Th.M. 1978. On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72, 297-300.

[24]   Ulam, S.M. 1940. Problems in Modern Mathematics, Science Editions, John Wiley & Sons, New York, USA.