ISSN 2347-3487

A new modification of the HPM for the Duffing equation with high nonlinearity

Ahmed A. Khidir
Faculty of Technology of Mathematical Sciences and Statistics
Alneelain University, Algamhoria Street, P.O. Box 12702, Khartoum - Sudan
E-mail: ahmed.khidir@yahoo.com

ABSTRACT

In this work we introduce a new modification of the homotopy perturbation method for solving nonlinear ordinary differential equations. The technique is based on the blending of the Chebyshev pseudo-spectral methods and the Homotopy Perturbation Method (HPM). The method is tested by solving the strongly nonlinear Duffing equation for undamped oscillators. Comparison is made between the proposed technique, the standard HPM, an earlier modification of the HPM and the numerical solutions to demonstrate the high accuracy, applicability and validity of the present approach.

Keywords: Homotopy perturbation method ; Chebyshev spectral method; Duffing equation.

Council for Innovative Research
Peer Review Research Publishing System
Journal of Advances in Physics
Vol 2, No.2
editor@cirworld.com
www.cirworld.com, member.cirworld.com

ISSN 2347-3487

1. Introduction

Finding accurate and efficient methods for solving nonlinear differential equations in bounded or unbounded domains is an ongoing challenge in engineering and science. For most nonlinear equations with no closed form solutions, recourse is usual made to numerical methods such as the iterative shooting method, the Runge-Kutta schemes and the Keller-box method. These numerical solutions often give very little insight into the structure of the solutions or the effects of the various parameters embedded in the governing equations. Non-numerical approaches include the classical power-series method and its variants such as the homotopy perturbation method for systems of nonlinear differential equations with small or large embedded parameters. For equations with neither small nor large parameters, non-perturbation techniques such as the Adomian decomposition method and the homotopy analysis method (Adomian [1], Liao [14]). He [9, 10, 11,12] proposed the homotopy perturbation method and he successfully used this method to solve many types of linear and nonlinear differential equations. This method provides a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different scientific fields by continuously deforming the difficult problem into a set of simple linear problems that are easy to solve. Modifications of the HPM to improve its accuracy and convergence rate have been reported by, among others, Beléndez et al. [2, 3, 4], Odibat [17], Odibat and Moman [18] and more recently, by Ganji et al. [7]. The HPM has been used, for example, to solve the Lighthill equation [9] and the Duffing equation [11]. In Beléndez et al. [3] the HPM was modified by truncating the infinite series corresponding to the first-order approximate solution before introducing this solution in the second-order linear differential equation and so. Ganji et al. [7] reported some differences between their implementation and the standard HPM in the choice of the linear operator with all the other processes identical. The nonlinear Duffing harmonic oscillation differential equation (see Mickens [15, 16]) has been solved using the homotopy perturbation method by Beléndez et al. [5], the variational iteration method (He [13] and Ramos [20]), the Adomian decomposition method (Ghosh et al. [8]), the artificial parameter-decomposition (Ramos [19]) and He's parameter-expanding method (Xu [22]). A comparison between the ADM and exact solution was given in [8] where it was shown that the ADM is only valid for a small region of the domain.

The purpose of the present paper to introduce a new alternative and improved of the HPM called Spectral Homotopy Perturbation Method (SHPM) in order to address some of the perceived limitations of the HPM uses the Chebyshev pseudospectral method. We use the method to find approximate solutions of the equation of motion of Duffing's undamped oscillator. We show that this hybrid method gives rapid convergence and good accuracy. This study proposes a standard way of choosing the linear operators and initial approximations for the SHPM. The obtained results suggest that this newly improvement technique introduces a powerful for solving non-linear differential equations. The new modification demonstrates an accurate solution compared with the numerical solution.

2. Homotopy perturbation method (HPM)

For the convenience of the reader, we first present a brief review of the standard HPM. This is then followed by a description of the algorithm of the SHPM solving nonlinear ordinary differential equations. To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{2}$$

where A is a general operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω.

The operator A can, in generally, be divided into two parts L and part N so that equation (1) can be written as

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

where L is a simple part which is easy to handle and N contains the remaining parts of A. By the homotopy technique [23, 24] , we construct a homotopy $v(r, p) : \Omega \times [0,1] \to \Re$ which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \ r \in \Omega \tag{4}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of equation (1), wich satisfies the boundary conditions. Obviously, from equation (4) we have

ISSN 2347-3487

$$H(v,0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{7}$$

The changing process of p from zero to unity is equivalent to the deformation of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v)-L(u_0)$, $A(v)-f(r)$ are homotopic. We can assume that the solution of equation (4) can be written as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{8}$$

setting $p = 1$, results in the approximation to the solution of equation (1)

$$u = \lim_{p \to 1} v = v_0 + u_1 + v_2 + \ldots \tag{9}$$

The series u is convergent for most cases. However, the convergence rate depends on the nonlinear operator of differential equation (3). The following opinions are suggested and proved by He [9, 10]

1. The second derivative of $N(u)$ with respect to u must be small because the parameter p may be relatively large, i.e $p \to 1$.
2. The norm of $L^{-1}\dfrac{\partial N}{\partial u}$ must be smaller than one so that the series converges.

The coupling of the perturbation method and the homotopy method gives the homotopy perturbation method (HPM), which has eliminated limitations of the traditional perturbation methods.

3. The Duffing oscillator equation

The nonlinear differential equation for the cubic free undamped Duffing oscillator is given by (Ganji et al. [7]);

$$\frac{d^2u(t)}{dt} + \alpha u(t) + \beta\psi(u) = 0 \tag{10}$$

subject to the initial conditions

$$u(0) = a \quad and \quad u'(0) = 0, \tag{11}$$

where u and t are generalized dimensionless displacement and time variables, respectively, α, β and a are arbitrary constants, ψ is a known non-linear function. There are several studies on this nonlinear equation with different value of α, β and $\psi(u)$. Various types of this equation studied by means of Adomian decomposition artificial parameter-decomposition, variational iteration method, He's parameter-expanding and homotopy perturbation method. It is convenient to study the solution of (10) in a region $[0,T]$ which is then translated to the domain [-1,1] on which the Chebyshev spectral method can be applied by using the transformation

$$y = \frac{2t}{T} - 1. \tag{12}$$

It is also convenient to make the boundary conditions homogeneous by making use of the transformation

$$u(t) = f(y) + u_0(t), \tag{13}$$

where $u_0(t) = a\cos(\lambda t)$ is chosen to satisfy the initial conditions (11) and λ is a spatial scaling parameter.

4. Applications and discussions

In order to assess the advantages and the accuracy of our method for solving nonlinear Duffing equation, we have applied it to a variety of initial-value problems arising in nonlinear dynamics.

Example 1:

Consider the governing equations of motion of free undamped Duffing's oscillator as follows

$$u''(t)+\alpha u(t)+\beta u^3=0, \tag{14}$$

and then substitute (12) and (13) in (14) gives

$$f''(y)+a_1(t)f(y)+a_2(t)f^3(y)+a_3(t)f^2(y)=\phi(t) \tag{15}$$

where

$$a_1(t)=\frac{T^2}{4}\ \alpha+3\beta,u_0^3\ ,\ a_2(t)=\frac{\beta T^2}{4},\ a_3(t)=\frac{3\beta T^2u_0}{4},\ \phi(t)=-\frac{T^2}{4}\ u_0''+\alpha u_0+\beta u_0^3 \tag{16}$$

Equation (15) is solved subject to the initial conditions

$$f(-1)=\ f'(-1)=0. \tag{17}$$

We thus construct the homotopy:

$$H(u;p)=L[u]-L[f_0]+pL[f_0]+pa_2(t)u^3+pa_3(t)u^2-p\phi(t)=0, \tag{18}$$

where $p\in[0,1]$ is an embedding parameter and v is an approximate series solution of (15) given by

$$u=\sum_{i=0}^{m}f_ip^i, \tag{19}$$

where m is the order of the method. The linear operator L is taken to be

$$L=\frac{d^2}{dy^2}+a_1(y) \tag{20}$$

and f_0 is the initial approximation for the solution of (18) and it is obtained from the solution to the nonhomogeneous linear part of (18) which is

$$f_0''(y)+a_1f_0(y)=\phi(t), \tag{21}$$

with the boundary conditions

$$f_0(-1)=\ f_0'(-1)=0 \tag{22}$$

Equation (18) can be written as

$$f''+a_1f+(p-1)\ \ f_0''+a_1f_0\ +p\ \ a_2f^3+a_3f^2-\phi(t)\ =0 \tag{23}$$

The higher order approximations of solution for (18) are evaluated by equating terms with power of $p^i,(i=1,2,3,...)$ on both sides of (23) gives a linear system of equations of the form

$$Af_i=B_i \tag{24}$$

subject the initial conditions

$$f_i(-1)=\ f_i'(-1)=0 \tag{25}$$

where

$$A=\left[D^2+diag[a_1]\right], \tag{26}$$

$$B_i=\chi\phi-(1-\chi)\left(a_3\sum_{n=0}^{i-1}f_nf_{i-1-n}+a_2\sum_{n=0}^{i-1}f_{i-1-n}\sum_{i=0}^{n}f_if_{n-i}\right),\ i=0,1,2,3,... \tag{27}$$

$$a_k=\ a_k(t_0),a_k(t_1),...,a_k(t_N)^{\ T},\ \ k=1,2 \tag{28}$$

$$f_0=\ f_0(y_0),\ f_0(y_1),...,\ f_0(y_N)^{\ T} \tag{29}$$

ISSN 2347-3487

$$\phi = \left[\phi(t_0), \phi(t_1), ..., \phi(t_N)\right]^T \tag{30}$$

$$\chi = \begin{cases} 1, & i = 0 \\ 0, & i > 0 \end{cases} \tag{31}$$

and D is the Chebyshev spectral differentiation matrix whose entries see [6, 21] are given by

$$\left.\begin{aligned} \mathrm{D}_{jk} &= \frac{c_j}{c_k}\frac{(-1)^{j+k}}{x_j - x_k} \qquad j \neq k;\ j,k = 0,1,\ldots,N, \\ \mathrm{D}_{kk} &= -\frac{x_k}{2(1-x_k^2)} \qquad k = 1,2,\ldots,N-1, \\ \mathrm{D}_{00} &= \frac{2N^2+1}{6} = -\mathrm{D}_{NN}. \end{aligned}\right\} \tag{32}$$

Here $c_0 = c_N = 2$ and $c_j = 1$ with $1 \le j \le N-1$ and x_j are the Chebyshev collocation points (see[21]) defined by

$$x_j = \cos\frac{j\pi}{N}, \quad j = 0,1,2...,N. \tag{33}$$

To implement the boundary conditions (25) to the systems (24), we delete the last row and column of the matrix A and delete the last row of f_i and $B\,i$, also we replace the resulting of last row of the modified matrix A and setting the resulting of last rows of the modified matrices $B\,i$ to be zero. The solution of (15) is obtained by substituting the series f_i in the series (19) after setting $p = 1$. This gives

$$f_i = A^{-1}B_i \tag{34}$$

Finally, the solution of the Duffing oscillator differential equation (14) is obtained by substitute the series f_i in (13). This problem has been solved by Ganji et al. [7] using a different modification of the homotopy perturbation method. Their algorithm adds and subtracts a term to the main governing equation. One of the additional terms is then added to the linear operator. The only difference between the scheme used by Ganji et al. [7] and the standard HPM is in the choice of the linear operator with other procedures the same. They found that the first two terms in the solution series are

$$u_0 = a\cos\sqrt{\beta a^2 + \alpha}\,t\ , \tag{35}$$

$$\begin{aligned} u_1 = {} & \frac{1}{32(\beta a^2+\alpha)^{\frac{3}{2}}} 4t\beta a^2 \sin\sqrt{\beta a^2+\alpha}\,t - \cos\sqrt{\beta a^2+\alpha}\,t\ \sqrt{\beta a^2+\alpha} \\ & + 4t\alpha\sin\sqrt{\beta a^2+\alpha}\,t + \beta a^3\sqrt{\beta a^2\alpha}\ \cos\ 3\sqrt{\beta a^2+\alpha}\,t \end{aligned} \tag{36}$$

The current method is convergent if $\Phi(\lambda) = norm\ L^{-1}\ \partial N / \partial u$ is smaller than one, this is the same strategy that used in the standard HPM approach, it is observed that from Figure 1(a) the solution of (14) is convergent when $\lambda = 1.4$ (the correspond minimum value of $\Phi(\lambda)$). If we consider $A = 1.5, \alpha = 1, \beta = 0.5$, the optimum value for λ is 1.35 as shown in Figure 1(b).

Figures 2 - 3 give a comparison between numerical solution, standard homotopy perturbation method (HPM), the Ganji et al. [7] scheme (MHPM) and the present results (SHPM) for different values of a, α and β. We considered only the first two terms in the approximations series for the all the methods. It is clear that for larger values of t, the present method (SHPM) gives better convergence to the numerical solution than both the standard HPM and MHPM. The accuracy of both the standard HPM and the MHPM improves with increasing values of a although not sufficiently to match the current method. To improve the accuracy of the Ganji et al. [7] scheme it is necessary to increase the number of terms in the solution series. This shows that for the same

number of terms, the proposed spectral modification of the HPM gives superior accuracy and convergence to the numerical solution.

Table 1 gives a further comparison between the standard HPM, MHPM, the present SHPM and the numerical results for different values of a, α, β, λ. Here we considered six decimal places of the results after two iterations for all methods. It is clear that the results obtained by the present method are more convergence to the numerical solution compared to the other methods. The accuracy of the SHPM solution is up to 4 decimal places at the 2nd order. It should be noted that better accuracy can be achieved by taking more turnings in the solution series.

Example 2:

We also solved the nonlinear undamped Duffing oscillator equation for fifth nonlinearity which it takes the form:

$$u'' + \alpha u + \beta u^5 = 0, \tag{37}$$

Figures 3 and 3 displayed the λ-curves of the current method against $\Phi(\lambda)$ of *Example 2* for different values of a, α and β. It is noted that the optimum value of the scaler λ is 0.75 when $a = 0.5, \alpha = 0.5, \beta = 1.5$ and $\lambda = 1.145$ when $a = 1, \alpha = 1, \beta = 0.5$

Figure 5 gives a comparison between numerical solution, standard homotopy perturbation method (HPM) and the present results (SHPM) for different values of a, α and β. These figures show validity of our new modification of HPM and a good agreement between the numerical and the present method results and it noted that for higher order of nonlinearity of Duffing equation, the homotopy perturbation method failed to give a solution even for large region.

We made a comparison between HPM, SHPM and numerical results of *Example 2* in Table 2 to validate the SHPM procedure, it seen it is evident that our results are in excellent agreement with that tnumerical solution.

5. Conclusion

In this work we have proposed a novel spectral modification of the standard homotopy perturbation method (SHPM) for solving nonlinear ordinary differential equations. The method has been used to solve the strongly nonlinear Duffing oscillator equation. Comparison with the standard HAM, the modified homotopy perturbation method of Ganji et al. [7] and numerical solutions show that the SHPM is highly accurate, efficient and converges rapidly with only a few iterations required to achieve the accuracy of the numerical results. The main difference between the HPM and the SHPM is that the solutions are obtained by solving a system of higher order ordinary differential equations in the HPM while for the SHPM solutions are obtained by solving a system of linear algebraic equations that are easier to solve. The advantages of this new method is that it converges much faster than the standard HPM and other recent modifications of the HPM such as the recent scheme by Ganji et al. [7]. The method can be used as alternative to the traditional Runge-Kutta, finite difference, finite element and Keller-Box methods.

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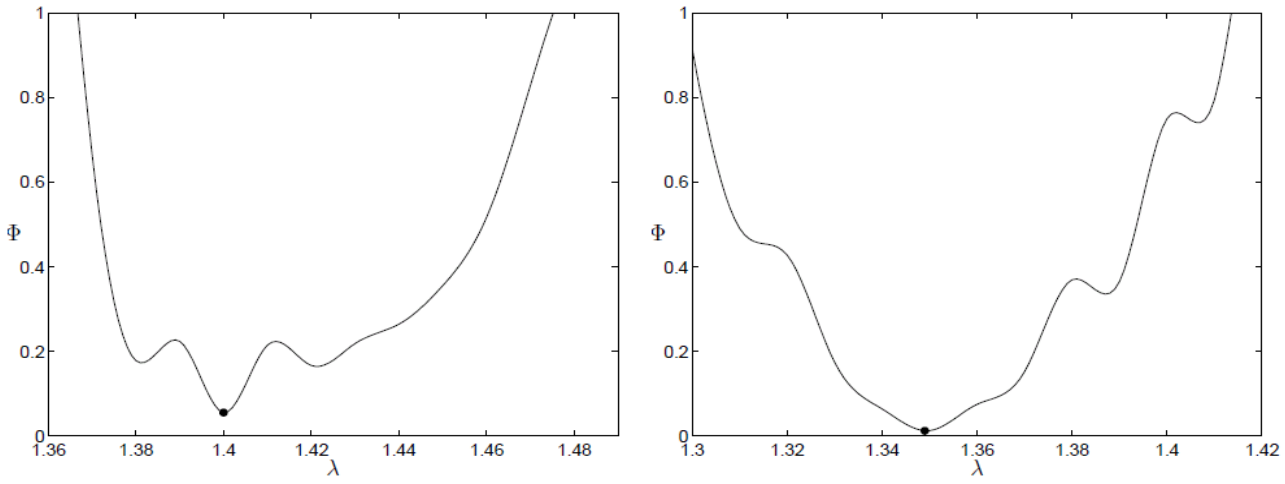
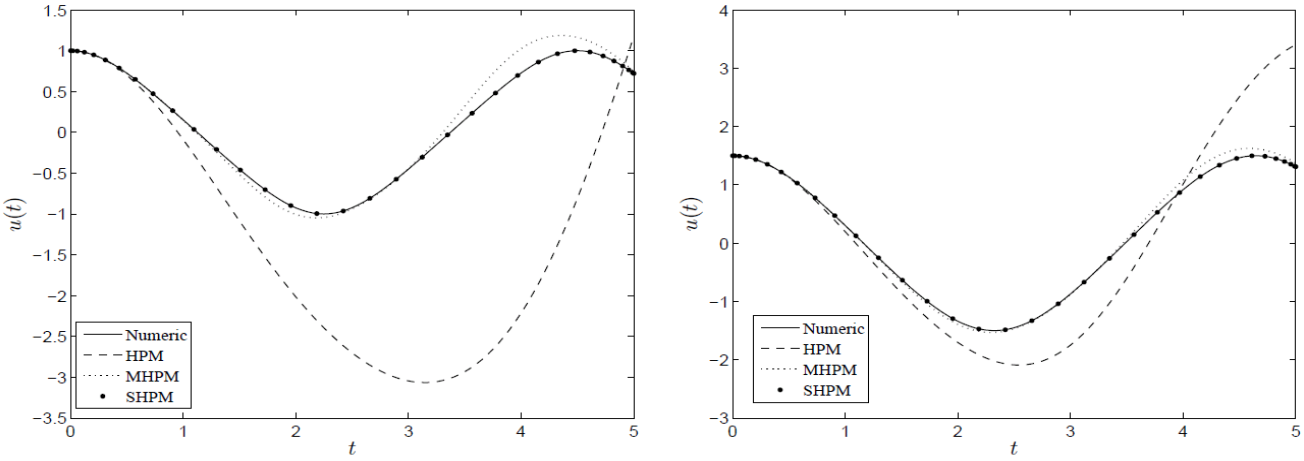
ISSN 2347-3487



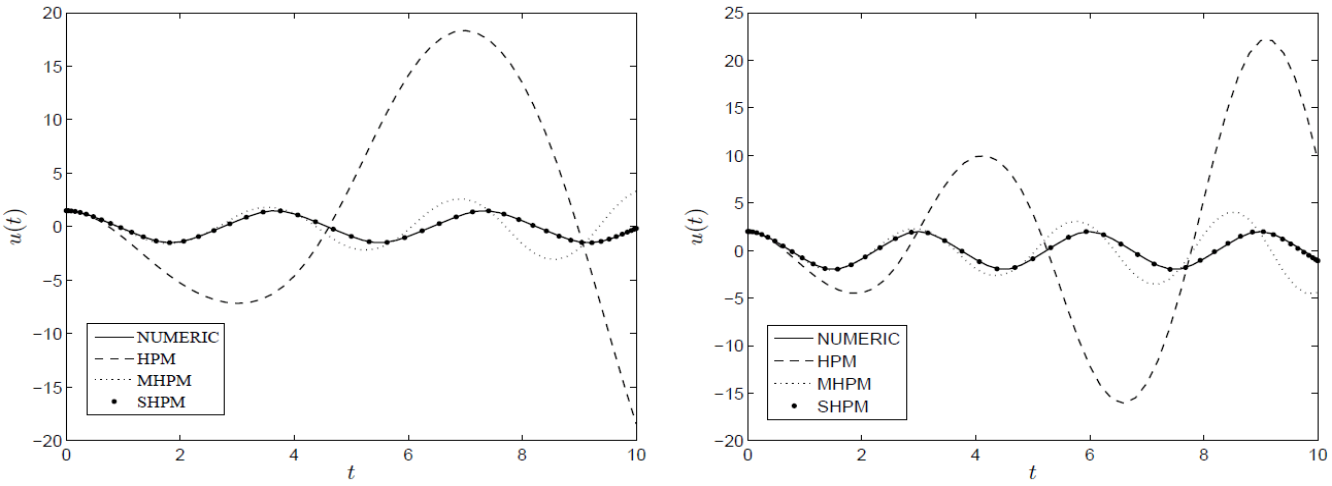
(a) (b)

Figure 1: 2nd order λ curve of SHPM against $\Phi(\lambda)$ of *Example 1* when (a) $a=1,\alpha=0.5,\beta=2$ (b) $a=1.5,\alpha=1,\beta=0.5$



(a) (b)

Figure 2: Comparison between the standard HPM, the Ganji et al. [7] MHPM, the current method (SHPM) and numerical solution of nonlinear undamped oscillator for *Example 1* when (a) $a=1,\alpha=0.5,\beta=2,\lambda=1.4$ (b) $a=1.5\ \alpha=1\ \beta=0.5,\lambda=1.35$

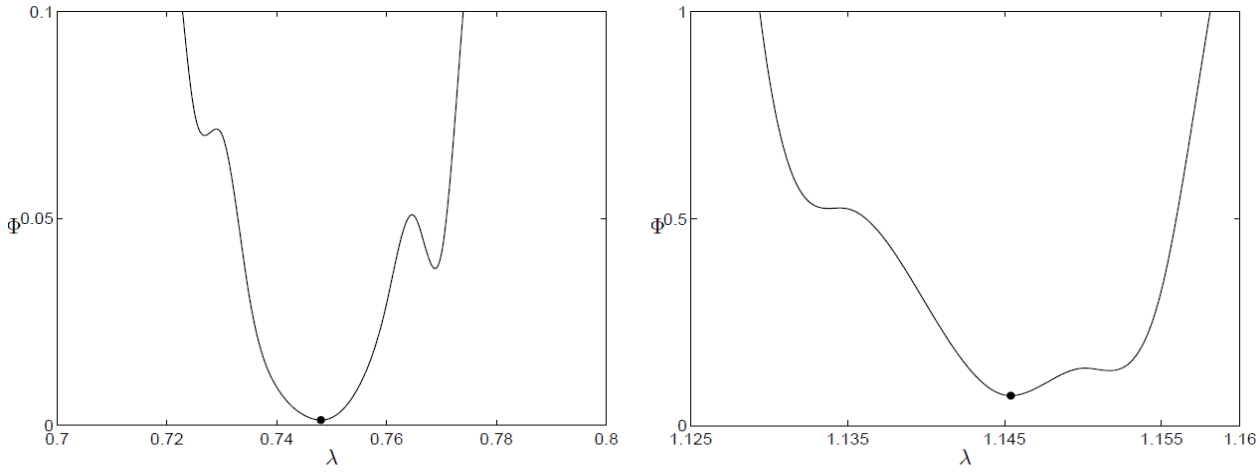


(a) (b)

Figure 3: Comparison between the standard HPM, the Ganji et al. [7] MHPM, the current method (SHPM) and the numerical solution of a nonlinear undamped oscillator for *Example 1* when (a) $a=1.5,\alpha=0.5,\beta=1.5,\lambda=1.72$ and (b)

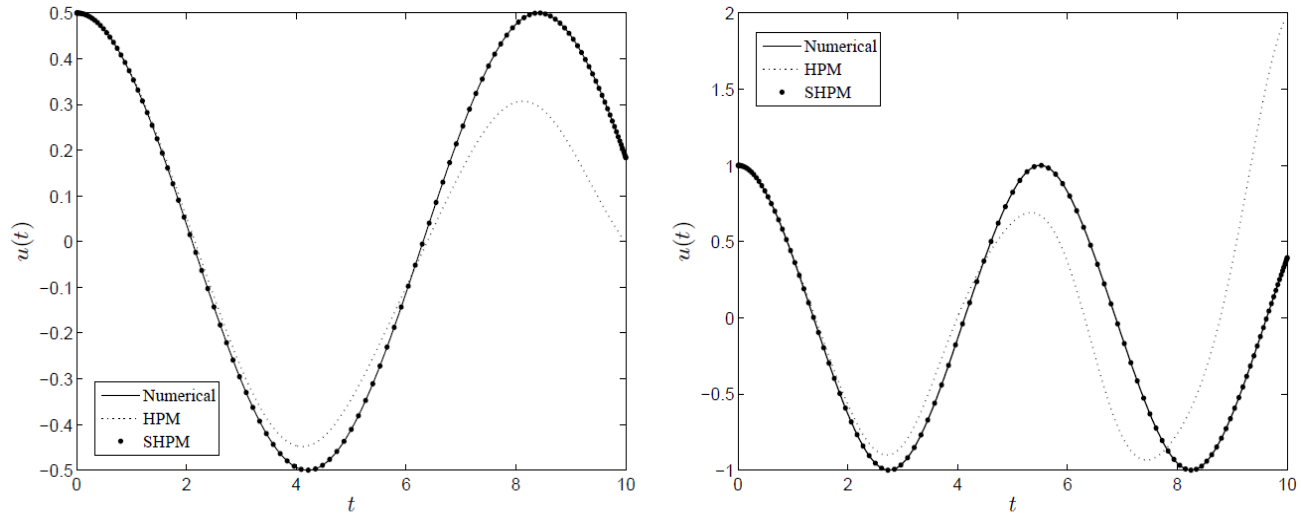
$a=2.\alpha=1.5.\ \beta=1.\lambda=2.11$

ISSN 2347-3487



(a) (b)

Figure 4: 2nd order λ curve of SHPM against $\Phi(\lambda)$ of *Example 2* when (a) $a=0.5, \alpha=0.5, \beta=1.5$ and (b) $a=1, \alpha=1, \beta=0.5$



(a) (b)

Figure 5: Comparison between HPM, SHPM and numerical solution of *Example 2* when (a) $a=0.5, \alpha=0.5, \beta=1.5, \lambda=0.75$ and (b) $a=1, \alpha=1, \beta=0.5, \lambda=1.15$

Table 1: Comparison between standard HPM, MHPM, current method (SHPM) and the numerical solution of a nonlinear undamped oscillator of first two iterations at time t values for *Example 1*

$a=1,\ \alpha=1,\ \beta=1,\ \lambda=1.32$					$a=0.75,\ \alpha=1.5,\ \beta=1.5,\ \lambda=1.47$				
t	Standard HPM	Modified HPM	Present results	Numerical results	t	Standard HPM	Modified HPM	Present result	Numerical results
.0.5	0.762476	0.768902	0.768766	0.768802	1	0.056288	0.080176	0.080519	0.080527
1	0.176929	0.233741	0.233680	0.233692	2	-0.808192	-0.739174	-0.729000	-0.729018
2	-1.055110	-0.891260	-0.859323	-0.859349	3	-0.339208	-0.239413	-0.238620	-0.238626
3.5	-0.461650	-0.079433	-0.093034	-0.093013	4	0.891267	0.706827	0.667953	0.668022
5	2.049041	0.996472	0.947107	0.947130	5	0.893003	0.395315	0.387550	0.387551

ISSN 2347-3487

Table 2: Comparison between standard HPM , current method (SHPM) and the numerical solution of a nonlinear undamped oscillator at time t values for *Example 2* when $a = 0.5, \alpha = 0.5, \beta = 1.5$ and $\lambda = 0.75$

	Standard HPM	Present results		Numerical results
t	2nd order	1st order	2nd order	
2	0.053397	0.037493	0.037485	0.037485
4	-0.446575	-0.493748	-0.493728	-0.493728
6	-0.100078	-0.111702	-0.111662	-0.111662
8	0.305004	0.475231	0.475165	0.475165