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ON THE STABILITY OF NONIC FUNCTIONAL EQUATION IN QUASI-NORMED SPACES

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ABSTRACT

In this paper, we extend normed spaces to quasi-normed spaces and prove the generalized Hyers-Ulam stability of a nonic functional equation:

$$\boldsymbol{f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)+126f(x+y)-126f(x)}$$
$$\boldsymbol{+84f(x-y)-36f(x-2y)+9f(x-3y)-f(x-4y)=9!\,f(y)},$$

where $9!=362880$, in quasi-normed spaces.

KEYWORDS

Generalized Hyers-Ulam stability, Nonic functional equations, Quasi-normed spaces

SUBJECT CLASSIFICATION

2010 Mathematics Subject Classification: 39B52, 39B82, 46S50, 47B48

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [22] concerning the stability of group homomorphisms. Hyers [9] gave the first affirmative answer to the question of Ulam for Banach spaces. Hyers's theorem was generalized by Aoki [1] for additive mappings. In 1978, Rassias [19] generalized Hyers theorem by obtaining a unique linear mapping near an approximate additive mapping. The paper of Rassias has provided a lot of influence in the development of what we called the generalized Hyers-Ulam-Rassias stability of functional equations. 1994, Găvruta generalized the Rassias' result in [7] for unbounded Cauchy difference. Different with the direct proof used before, Căudariu and Radu [3] proposed a novel method for studying the stability of the Cauchy functional equation based on a fixed point result in generalized metric spaces. The stability problems of several functional equations have been extensively investigated by a number of [2, 5, 8, 10, 11, 12, 15, 16, 21, 23] and references therein for more detailed information.

The mathematical analysis one of the most important functional equations. In 1821, Cauchy noted that every continuous solution of the additive Cauchy functional equation proved the functional equation:

$$f(x+y)=f(x)+f(y)$$

for all $x,y\in\mathbb{R}$, is linear. Every solution of the additive Cauchy equation is called an additive function.

In 1984, Cholewa [4] initiated the study of the stability of the following quadratic functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

A quadratic functional equation was used to characterize inner product spaces.

In [17], [18], Rassias proposed the cubic and quartic functional equations:

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y)$$

and

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+f(x-y)+24f(y),$$

respectively, and considered the solution and the stability problem of these equations in normed spaces.

In [24], Xu et al. investigate the general solutions of the quintic and sextic functional equations:

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=120f(y)$$

and

$$f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y)$$

$$-6(x-2y)+f(x-3y)=720f(y)$$

respectively, and then proved the stability of these two types of equations in quasi-β-normed spaces.

In 2015, Shen and Chen [20] proved the general solutions and investigated the stability of the septic and octic functional equations in normed linear spaces:

$$f(x+4y)-7f(x+3y)+21f(x+2y)-35f(x+y)-21f(x-y)$$
$$+7f(x-2y)-f(x-3y)+35f(x)=5040f(y),$$

and

$$f(x+4y)-8f(x+3y)+28f(x+2y)-56f(x+y)-56f(x-y)$$
$$+28f(x-2y)-8f(x-3y)+f(x-4y)+70f(x)=40320f(y),$$

respectively.

Before we present our results, we introduce some basic facts concerning quasi-normed space. Let X be a real linear space. A quasi-norm $||\cdot||$ is a real-valued function on satisfying the following:

(i) $||x||\geq 0$ for all $x\in X$ and $||x||=0$ if and only if $x\geq 0$;

(ii) $||\lambda x||=|\lambda|\,||x||$ for all $\lambda\in\mathbb{R}$ and $x\in X$;

(iii) there is a constant $K\geq 1$ such that $||x+y||\leq K(||x||+||y||)$ for all $x,y\in X$.

The pair $(X,||\cdot||)$ is called a quasi-normed space if $||\cdot||$ is a quasi-norm. The smallest possible K is called the modulus of concavity of $||\cdot||$. A quasi-Banach space is a complete quasi-normed space.

Now, we consider a mapping $f:X\to Y$ satisfying the following functional equation, which introduced in [13], in 2016, as follows:

$$f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)+126f(x+y)-126f(x)$$
$$+84f(x-y)-36f(x-2y)+9f(x-3y)-f(x-4y))=9!\,f(y)$$

for all $x,y\in X$. It is easy to see that the function is a solution of the functional equation (1.1). Every solution of the functional equation (1.1) is said to be a nonic mapping. Note that the functional equation (1.1) have the properties (i) $f(0)=0$, (ii) $f(-x)=-f(x)$ and $f(2x)=2^9f(x)$.

2. MAIN RESULTS

For a given mapping $f:X\to Y$, we defined the difference operator

$$Df(x,y)=f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)+126f(x+y)-126f(x)$$
$$+84f(x-y)-36f(x-2y)+9f(x-3y)-f(x-4y)-9!\,f(y)$$

for all $x,y\in X$.

Firstly, we investigate the generalized Hyers-Ulam stability of the functional equation (1.1) in quasi-normed spaces in the spirit of Hyers, Ulam, and Găvruta.

Theorem 2.1. Let X be a quasi-normed space and Y be a quasi-Banach space. Suppose that there exist a mapping $\varphi: X\times X\to[0,\infty)$ for which a mapping $f:X\to Y$ with $f(0)=0$ satisfies

$$||Df(x,y)||\leq\varphi(x,y) \qquad (2.1)$$

for all $x,y\in X$, and the series

$$\sum_{i=1}^{\infty}\left(\frac{K}{2^9}\right)^i\Phi(2^ix)<\infty \qquad (2.2)$$

for all $x\in X$, where $\Phi(x)=\frac{K}{9!}\left[K^5\left(\varphi(0,2x)+\varphi(5x,x)\right)+9K^4\varphi(4x,x)+37K^3\varphi(3x,x)\right.$

$$\left.+93K^2\varphi(2x,x)+162K\varphi(x,x)+210\varphi(0,x)\right]$$

and $\lim_{n\to\infty}\frac{1}{2^{9n}}\varphi(2^nx,2^ny)=0$ for all $x,y\in X$. Then there exists a unique nonic mapping $N:X\to Y$ which satisfies (1.1) and

$$||f(x)-N(x)||\leq\frac{K}{2^9}\sum_{i=1}^{\infty}\left(\frac{K}{2^9}\right)^i\Phi(2^ix) \qquad (2.3)$$

for all $x\in X$.

Proof. Replacing (x,y) with $(0,2x)$ in (2.1), we get

$$||f(10x)-8f(8x)+27f(6x)-48f(4x)-362838f(2x)|| \leq \varphi(0,2x) \qquad (2.4)$$

for all $x \in X$. Replacing (x,y) with $(5x,x)$ in (2.1), we get

$$||f(10x)-9f(9x)+36f(8x)-84f(7x)+126f(6x)-126f(5x)$$
$$+84f(4x)-36f(3x)+9f(2x)-362881f(x)|| \leq \varphi(5x,x) \qquad (2.5)$$

for all $x \in X$. Subtracting equations (2.4) and (2.5), we get

$$||9f(9x)-44f(8x)+84f(7x)-99f(6x)+126f(5x)-132f(4x)$$
$$+36f(3x)-362847f(2x)+362881f(x)|| \leq K\big(\varphi(0,2x)+\varphi(5x,x)\big) \qquad (2.6)$$

for all $x \in X$. Replacing (x,y) with $(4x,x)$ in (2.1), and multiplying the resulting equation by 9, we have

$$||9f(9x)-81f(8x)+324f(7x)-756f(6x)+1134f(5x)-1134f(4x)$$
$$+756f(3x)-324f(2x)-3265839f(x)|| \leq 9\,\varphi(4x,x) \qquad (2.7)$$

for all $x \in X$. Subtracting equations (2.6) and (2.7), we get

$$||37f(8x)-240f(7x)+657f(6x)-1008f(5x)+1002f(4x)$$
$$-720f(3x)-362523f(2x)+3628720f(x)||$$
$$\leq K^2\big(\varphi(0,2x)+\varphi(5x,x)\big)+9\,K\varphi(4x,x) \qquad (2.8)$$

for all $x \in X$. Replacing (x,y) with $(3x,x)$ in (2.1), and multiplying the resulting equation by 37, we have

$$||37f(8x)-333f(7x)+1332f(6x)-3108f(5x)+4662f(4x)$$
$$-4662f(3x)+3108f(2x)-13427855f(x)|| \leq 37\,\varphi(3x,x) \qquad (2.9)$$

for all $x \in X$. Subtracting equations (2.8) and (2.9), we arrive at

$$||93f(7x)-675f(6x)+2100f(5x)-3660f(4x)$$
$$+3942f(3x)-365631f(2x)+17056575f(x)||$$
$$\leq K^3\big(\varphi(0,2x)+\varphi(5x,x)\big)+9\,K^2\varphi(4x,x)+37\,K\varphi(3x,x) \qquad (2.10)$$

for all $x \in X$. Replacing (x,y) with $(2x,x)$ in (2.1), and multiplying the resulting equation by 93, we get

$$||93f(7x)-837f(6x)+3348f(5x)-7812f(4x)+11718f(3x)$$
$$-11625f(2x)-33740865f(x)|| \leq 93\,\varphi(2x,x) \qquad (2.11)$$

for all $x \in X$. Subtracting equations (2.10) and (2.11), we arrive at

$$||162f(6x)-1248f(5x)+4152f(4x)-7776f(3x)-354006f(2x)+5079440f(x)||$$
$$\leq K^4\big(\varphi(0,2x)+\varphi(5x,x)\big)+9\,K^3\varphi(4x,x)+37\,K^2\varphi(3x,x)+93\,K\varphi(2x,x) \qquad (2.12)$$

for all $x \in X$. Replacing (x,y) with (x,x) in (2.1), and multiplying the resulting equation by 162, we get

$$||162f(6x)-1458f(5x)+5832f(4x)-13446f(3x)$$
$$+18954\,f(2x)-58801140f(x)|| \leq 162\,\varphi(x,x) \qquad (2.13)$$

for all $x \in X$. Subtracting equations (2.12) and (2.13), we arrive at

$$||210f(5x)-1680f(4x)+5670f(3x)-372960f(2x)+109598580f(x)||$$
$$\leq K^5\big(\varphi(0,2x)+\varphi(5x,x)\big)+9\,K^4\varphi(4x,x)$$
$$+37\,K^3\varphi(3x,x)+93\,K^2\varphi(2x,x)+162\,K\varphi(x,x) \qquad (2.14)$$

For all $x \in X$. Replacing (x,y) with $(0,x)$ in (2.1), and multiplying the resulting equation by 210, we get

$$\big||210f(5x)-1680f(4x)+5670f(3x)-10080f(2x)-76195980f(x)|\big| \leq 210\,\varphi(0,x) \qquad (2.15)$$

for all $x \in X$. Subtracting (2.14) and (2.15), we get

$$\big||-362880f(2x)+185794560f(x)|\big|$$
$$\leq K^6(\varphi(0,2x)+\varphi(5x,x))+9\,K^5\varphi(4x,x)+37\,K^4\varphi(3x,x)$$
$$+93\,K^3\varphi(2x,x)+162\,K^2\varphi(x,x)+210\,K\varphi(0,x)$$

for all $x \in X$. Thus, we can deduce that

$$\|f(2x) - 2^9 f(x)\| \leq \frac{K}{9!}\,[K^5\left(\varphi(0,2x) + \varphi(5x,x)\right) + 9\,K^4\varphi(4x,x) + 37\,K^3\varphi(3x,x)$$
$$+93\,K^2\varphi(2x,x) + 162\,K\varphi(x,x) + 210\,\varphi(0,x)]$$
$$\equiv \Phi(x) \qquad (2.16)$$

for all $x \in X$. It follows from (2.16) that

$$\left\|f(x) - \frac{f(2x)}{2^9}\right\| \leq \frac{1}{2^9}\Phi(x) \qquad (2.17)$$

for all $x \in X$. Replacing x by $2^i x$ in (2.17) and dividing by 2^{9i}, we get

$$\left\|\frac{f(2^i x)}{2^{9i}} - \frac{f(2^{i+1} x)}{2^{9(i+1)}}\right\| \leq \frac{1}{2^{9(i+1)}}\Phi(2^i x) \qquad (2.18)$$

for all $x \in X$. Using induction on $n \in Z^+$ and summing up the resulting inequality for $i = 0, \dots, n-1$, we have

$$\left\|f(x) - \frac{f(2^n x)}{2^{9n}}\right\| \leq \frac{K}{2^9}\sum_{i=1}^{n-2}\left(\frac{K}{2^9}\right)^i \Phi(2^i x) + \frac{1}{2^9}\left(\frac{K}{2^9}\right)^{n-1}\Phi(2^{n-1}x) \qquad (2.19)$$

for all $x \in X$ and $n \in Z^+$. Putting x by $2^m x$ and dividing by 2^{9m} in (2.17), we get

$$\left\|\frac{f(2^{n+m} x)}{2^{9(n+m)}} - \frac{f(2^m x)}{2^{9m}}\right\|$$
$$\leq \frac{K}{2^{9(m+1)}}\sum_{i=1}^{n-2}\left(\frac{K}{2^9}\right)^i \Phi(2^{m+i}x) + \frac{1}{2^{9(m+1)}}\left(\frac{K}{2^9}\right)^{n-1}\Phi(2^{n+m-1}x)$$
$$\leq \frac{K}{2^9 K^m}\sum_{i=1}^{n-2}\left(\frac{K}{2^9}\right)^{m+i}\Phi(2^{m+i}x) + \frac{1}{2^9 K^m}\left(\frac{K}{2^9}\right)^{n+m-1}\Phi(2^{n+m-1}x) \qquad (2.20)$$

Since the right-hand side of (2.20) tends to 0 as $m \to \infty$, the sequence $\left\{\frac{f(2^n x)}{2^{9n}}\right\}$ is a Cauchy sequence in the quasi-Banach space Y. Thus, we may define a mapping $N: X \to Y$ by

$$N(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^{9n}}$$

for all $x \in X$. Taking the limit as $n \to \infty$ in (2.19), we obtain that the mapping N satisfies (2.3).

Replacing (x,y) by $(2^n x, 2^n y)$ in (2.1) and dividing it by 2^{9n}, we have

$$\|DN(x,y)\| = \lim_{n\to\infty}\frac{1}{2^{9n}}\|Df(2^n x, 2^n y)\| = \lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{2^{9n}} = 0$$

for all $x, y \in X$. Thus, the mapping $N : X \to Y$ satisfies (1.1). This implies that the mapping N is nonic.

Now, let $N' : X \to Y$ be another nonic mapping satisfies (1.1) and (2.3). Fix $x \in X$. Then $N(2^n x) = 2^{9n} N(x)$ and $N'(2^n x) = 2^{9n} N'(x)$ for all $n \in Z^+$. It follows from (2.3) that

$$\|N(x) - N'(x)\| \leq \left\|N(x) - \frac{f(2^n x)}{2^{9n}}\right\| + \left\|\frac{f(2^n x)}{2^{9n}} - N'(x)\right\|$$
$$= \frac{2}{2^9 K^{n-1}}\sum_{i=0}^{\infty}\left(\frac{K}{2^9}\right)^{n+i}\Phi(2^{n+i}x) \qquad (2.21)$$

for all $x \in X$. Letting the limit as $n \to \infty$ in (2.21), we have $N = N'$. Thus, the nonic mapping N is unique. This completes the proof.

Theorem 2.2. Let X be a quasi-normed space and Y be a quasi-Banach space. Suppose that there exist a mapping $\varphi : X \times X \to [0,\infty)$ for which a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (2.1) and the series $\sum_{i=0}^{\infty}(2^9 K)^i\,\Phi\left(\frac{x}{2^i}\right) < \infty$ and $\lim_{n\to\infty} 2^{9n}\,\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}\right) = 0$ for all $x, y \in X$, where

$$\Phi(x) = \frac{K}{9!}[\,K^5\left(\varphi(0,2x) + \varphi(5x,x)\right) + 9\,K^4\varphi(4x,x) + 37\,K^3\varphi(3x,x)$$
$$+ 93\,K^2\varphi(2x,x) + 162\,K\varphi(x,x) + 210\,\varphi(0,x)].$$

Then, there exists a unique nonic mapping $N : X \to Y$ which satisfies (1.1) and

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$$\left|\left|f(x)-N(x)\right|\right| \le \frac{1}{2^9}\sum_{i=0}^{\infty}(2^9K)^i\,\Phi\left(\frac{x}{2^i}\right) \qquad (2.22)$$

for all $x \in X$.

Proof. The proof is similarly proved by the following inequality due to (2.17)

$$\left|\left|f(x)-2^{9n}f\left(\frac{x}{2^n}\right)\right|\right| \le \frac{1}{2^9}\sum_{i=1}^{n-1}(2^9K)^i\,\Phi\left(\frac{x}{2^i}\right)+\frac{1}{2^9K}\,(2^9K)^n\,\Phi\left(\frac{x}{2^n}\right) \qquad (2.23)$$

for all $x \in X$ and $n \in Z^+$. Then the sequence $\left\{2^{9n}\,f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in the quasi-Banach space Y. So, we can define a mapping $N : X \to Y$ by $N(x)=\lim_{n\to\infty}2^{9n}\,f\left(\frac{x}{2^n}\right)$ for all $x \in X$. Letting the limit as $n \to \infty$ in (2.23), we obtain (2.22). The rest of proof is similar method to the corresponding part of Theorem 2.1. This complete the proof.

Corollary 2.3. Let p and θ be positive real numbers. Suppose that a mapping $f : X \to Y$ with $f(0)=0$ satisfies the inequality

$$\left|\left|Df(x,y)\right|\right| \le \theta(\left|\left|x\right|\right|^p+\left|\left|y\right|\right|^p)$$

for all $x, y \in X$. Then there exists a unique nonic mapping $N : X \to Y$ which satisfies (1.1) and

$$\left|\left|f(x)-N(x)\right|\right| \le \begin{cases} \dfrac{KM_{\theta,p}}{2^9-2^pK}\left|\left|x\right|\right|^p, & if \quad p<9-\log_2 K \\[2ex] \dfrac{KM_{\theta,p}}{2^p-2^9K}\left|\left|x\right|\right|^p, & if \quad p>9+\log_2 K \end{cases}$$

for all $x \in X$, where $M_{\theta,p}=\frac{\theta K}{9!}[(2^p+5^p+1)K^5+9(4^p+1)K^4+37(3^p+1)K^3+93(2^p+1)K^2+324K+210]$.

Corollary 2.4. Let $p,\ q,\ \theta$ be positive real numbers and $f : X \to Y$ be a mapping with $f(0)=0$ satisfying

$$\left|\left|Df(x,y)\right|\right| \le \theta(\left|\left|x\right|\right|^p\left|\left|y\right|\right|^q)$$

for all $x, y \in X$. Then there exists a unique nonic mapping $N : X \to Y$ which satisfies (1.1) and

$$\left|\left|f(x)-N(x)\right|\right| \le \begin{cases} \dfrac{KM_{\theta,p}}{2^9-2^pK}\left|\left|x\right|\right|^{p+q}, & if \quad p+q<9-\log_2 K \\[2ex] \dfrac{KM_{\theta,p}}{2^p-2^9K}\left|\left|x\right|\right|^{p+q}, & if \quad p+q>9+\log_2 K \end{cases}$$

for all $x \in X$, where $M_{\theta,p}=\frac{\theta K^2}{9!}[5^pK^4+9\cdot4^pK^3+37\cdot3^pK^2+93\cdot2^pK+162]$.

Next, we investigate the generalized Hyers-Ulam stability of the functional equation (1.1) in quasi-normed spaces by the fixed point alternative. We will deal with the fixed point theorem which was proved by Diaz and Margolis.

Theorem 2.5. [6] Let (Ω, d) be a complete generalized metric space and $T : \Omega \to \Omega$ be a strictly contractive mapping with Lipshitz constant L. Then, for any $x \in \Omega$, either $d(T^nx,\ T^{n+1}x)=\infty$ for all nonnegative integers $n \ge 0$ or other exists a natural number n_0 such that

(i) $d(T^nx,\ T^{n+1}x)<\infty$ for all $\mathrm{n} \ge n_0$;

(ii) the sequence $\{T^nx\}$ is convergent to a fixed point y^* of T;

(iii) y^* is the unique fixed point of T in the set $\Lambda=\{y \in \Omega \mid d(T^{n_0}x,y)<\infty\}$;

(iv) $d(y,\ y^*) \le \frac{1}{1-L}\,d(y,Ty)$ for all $\mathrm{y} \in \Lambda$.

Theorem 2.6. Let X be a quasi-normed space, Y be a quasi-Banach space and $\varphi : X \times X \to [0,\infty)$ be a mapping such that there exists a constant $L<1$ with

$$\varphi(2^j\,x, 2^j\,y) \le 2^{9j}\,L\,\varphi(x,y) \qquad (2.24)$$

for all $x, y \in X$, where $j=\pm1$. If $f : X \to Y$ is a mapping with $f(0)=0$ satisfying

$$\left|\left|Df(x,y)\right|\right| \le \varphi(x,y) \qquad (2.25)$$

for all $x, y \in X$, then there exists a unique nonic mapping $N : X \to Y$ such that

$$\left|\left|f(x)-N(x)\right|\right| \le \frac{1}{2^{9j}\,|1-L^j|}\,\Phi(x) \qquad (2.26)$$

for all $x \in X$, where $\Phi(x)=\frac{K}{9!}\,[\,K^5\left(\varphi(0,2x)+\varphi(5x,x)\right)+9\,K^4\varphi(4x,x)+37\,K^3\varphi(3x,x)$

$$+ 93K^2\varphi(2x,x) + 162\, K\varphi(x,x) + 210\varphi(0,x)] \,.$$

Proof. It follows from (2.16) that

$$||f(2x) - 2^9 f(x)|| \leq \Phi(x) \qquad (2.27)$$

for all $x \in X$. Let $\Omega = \{f \mid f : X \to Y\}$ and introduce a generalized metric d on Ω as follows:

$$d(g,h) = \inf\{c \in [0,\infty) \mid ||g(x) - h(x)|| \leq c\Phi(x) \ \text{ for all } \ x \in X\}$$

where, as usual, $\inf \phi = +\infty$. Then (Ω, d) is a generalized complete metric space [14]. We consider the mapping $T : \Omega \to \Omega$ is defined by

$$Tg(x) = \frac{g(2^j x)}{2^{9j}}$$

for all $x \in X$. Let $g,h \in \Omega$ and $c \in [0,\infty)$ be an arbitrary constant with $d(g,h) < c$. It follows from the definition of d, T and (2.24) that

$$||Tg(x) - Th(x)|| \leq \frac{c}{2^{9j}} \Phi(2^j x) \leq cL\Phi(x)$$

for all $x \in X$, which gives $d(Tg,Th) \leq Ld(g,h)$ for all $g,h \in \Omega$. This means that T is a strictly contractive self-mapping on Ω with Lipschitz constant L.

It follows from (2.27) that

$$d(f,Tf) = \begin{cases} \frac{1}{2^9}, & if \quad j = 1 \\ \frac{L}{2^9}, & if \quad j = -1 \end{cases}$$

for all $x \in X$. It follows from the conditions (2) and (3) of Theorem 2.5 that there exists a mapping N which is a unique fixed point of T in the set $\Omega_1 = \{g \in \Omega \mid d(f,g) < \infty\}$ such that

$$N(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{9nj}}$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n f, N) = 0$. Again, from the condition (4) of Theorem 3.1, we have

$$d(f,N) \leq \frac{1}{1-L}\, d(f,Tf) \leq \frac{1}{2^{9j}\,|1-L^j|}.$$

Then, we conclude that the inequality (2.26) holds for all $x \in X$.

If we replace x by $2^{nj}x$ and y by $2^{nj}y$ in (2.25), then we obtain

$$||\frac{Df(2^{nj}x, 2^{nj}y)}{2^{9nj}}|| \leq \frac{||Df(2^{nj}x, 2^{nj}y\ )||}{2^{9nj}} \leq \frac{\varphi(2^{nj}x,\ 2^{nj}y)}{2^{9nj}}$$

for all $x \in X$. Taking the limit as $n \to \infty$ in the above inequality, we deduce that $DN(x,y) = 0$ for all $x, y \in X$. Therefore, the mapping $N : X \to Y$ is nonic, as desired. This completes the proof.

Corollary 2.7. Let θ be positive real number and be a mapping $f : X \to Y$ with $f(0) = 0$ satisfying

$$||Df(x,y)|| \leq \theta$$

for all $x, y \in X$. Then there exists a unique nonic mapping $N : X \to Y$ such that

$$||f(x) - N(x)|| \leq \frac{513}{9!\,(2^9 - 2^p)}$$

for all $x \in X$.

Corrllary 2.8. Let p, θ be positive real numbers with $p \neq 9$ and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying

$$||Df(x,y)|| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then there exists a unique nonic mapping $N : X \to Y$ such that

$$||f(x) - N(x)|| \leq \frac{\delta_{\theta,p}}{|2^9 - 2^p|} ||x||^p$$

for all $x \in X$, where $\delta_{\theta,p} = \frac{\theta}{9!}(5^p + 9 \cdot 4^p + 37 \cdot 3^p + 93 \cdot 2^p + 674)$.

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