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On Use of Meijer's G-functions In The Theory of Univalent Functions

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ABSTRACT

Using some properties of Meijer's G-functions and univalent functions, in this paper some definitions, transformations and theorems in univalent function theory are discussed and then reformulated in the language of Meijer's G-functions. The starting point is to consider the Koebe function as a Meijer's G-function.

Mathematics Subject Classification

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Keywords

Meijer's G-function; Univalent functions; Koebe function; starlike function; convex function; close-to-convex function.

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INTRODUCTION

In recent decades Meijer's G-function (MGF) has found various applications in different areasclose to applied mathematics like: mathematical physics, theoretical physics, mathematical statistics, queuingtheory, optimization theory, sinusoidal signals, generalized birth and death processes,etc. Due to the interesting and important properties of MGFs, it becamepossible to represent the solutions of many problems in these fields in terms of thesefunctions. Stated in this way, the problems gain a much more general character, due tothe great freedom of choice of the orders *m; n; p; q* and parameters of MGFs, incomparison with the other special functions [6]. Simultaneously, the calculations becomesimpler and more unified. Evidence for the importance of MGF is givenby the fact that the basic elementary functions and most of the special functions ofmathematical physics, including the generalized hyper-geometric functions, follow asitsparticular cases. Therefore, each result concerning MGF becomes a key leading to numerous particular results for the Bessel functions, confluent hypergeometric functions, classical orthogonal polynomials [13].

Many papers in the theory of univalent functions are devoted to linear integral or integro-differential operators which map these functions and its subclasses into themselves. One of the important problems in the theory of univalent functions is the construction of linear operators preserving the class S and some of its subclasses. Recently the operators have been defined by means of single integrals (differ-integrals) involving MGFs as kernels. Kiryakova et.al [1] proposed sufficient conditions that guarantee mappings related to the operators of generalized fractional calculus involving Meijer's G-functions as kernels in univalent functions. In addition Kiryakova et.al [2] considered some mappings, distortion and characterization properties of the operators of the generalized fractional calculus involving Meijer's G-functions.

This paper is organized into the following sections. In the first section, we recall the definition of the Meijer's G-function. This definition gives us to obtain the path integral form for the *Koebe* function by determining its orders and parameters. In the second section we give two important properties of MGFs and the multiplication theorem. The third section discusses on the classification of hypergeometric functions [6]. This classification helps us to specify the position of the Koebe function among the hypergeometric functions. In the fourth section some well-known definitions, transformations and theorems in univalent function theory are recalled. Finally, we obtain main resultsas reformulation of previous sections in the format and language of MGFs.

MEIJER'S G-FUNCTION

During the past century, many attempts led to introduce a very general function, which in special cases the results were known as special functions. We may name hypergeometric function, Macrobert'sE-function and Meijer's G-function (MGF).

However, the Meijer's G-function is the best ones, since this function includes even those very general functions as a particular case as well.

Definition 1.1 A general definition of the Meijer's G-function is given by the following path integral in the complex plane [4,5,13,14,15]:

$$G_{p,q}^{m,n}\left({a_1 \ldots a_p \atop b_1 \ldots b_q}\middle| z\right) = \frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds, \qquad (1.1)$$

Here, the integers *m; n; p; q* are called "orders" of the G-function, or the components of the order *(m; n; p; q); a_j* and b_j are called "parameters" and in general, they are complex numbers. The definition holds under the following assumptions: $0 \le m \le q$ and $0 \le n \le p$, where *m; n; p;* and *q* are integer numbers. Subtracting parameters $a_j - b_k \neq 1, 2, 3, \ldots$ for $k = 1, \ldots, n$ and $j = 1, 2, \ldots, m$ imply that no pole of any $\Gamma(b_j - s), j = 1, \ldots, m$ coincides with any pole of any $\Gamma(1 - a_k + s), k = 1, \ldots, n$.

Starting from the definition, it can be easily proved the following properties:

$$z^{\alpha} G_{p,q}^{m,n}\left({\mathbf{a_p} \atop \mathbf{b_q}}\middle| z\right) = G_{p,q}^{m,n}\left({\mathbf{a_p}+\alpha \atop \mathbf{b_q}+\alpha}\middle| z\right). \qquad (2.1)$$

Moreover, to obtain derivatives of arbitrary order k, one can use

$$z^k \frac{d^k}{dz^k} G_{p,q}^{m,n}\left({\mathbf{a_p} \atop \mathbf{b_q}}\middle| z\right) = G_{p+1,q+1}^{m,n+1}\left({0,\mathbf{a_p} \atop \mathbf{b_q},\mathbf{k}}\middle| z\right). \qquad (2.2)$$

Theorem 2.1 Multiplication theorem is proved for integer parameters p and q, with conditions $q \geq 1; p \leq q$and provided that $z \neq 0$ [4,5,14,15] :

$$G_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|\omega z\right) = \sum_{k=0}^{\infty} \frac{(\omega-1)^k}{k!} G_{p+1,q+1}^{m,n+1}\left({}_{b_q,k}^{0,a_p}\middle|z\right). \quad (2.3)$$

This theorem is used in the dilation transformation of univalent functions and to calculate the Jackson derivative of MGFs.

REPRESENTATIONS OF $_PF_Q$

In [6] Kiryakovaclassifies hypergeometric functions into 3 classes. She develops aunified approach to the generalized hypergeometric functions based on a generalized fractional calculus [6,7]. This approach deals with differ-integral operators involving Meijer's G- and Fox's H functions as kernel functions. Depending on whether p < q; p = q or p = q + 1 the pFq- functionshave been separated into three classes and the functions of each class representedas generalized fractional integrals or derivatives of three basic elementary functions:

$$\text{1- } \cos_{q-p+1}(x)(\text{if } p < q) \text{ , 2- } x^{\alpha} e^{x}(\text{if } p = q) \text{ , 3- } x^{\alpha}(1-x)^{\beta}(\text{if } p = q+1)$$

The generalized hypergeometric functions (GHFs) pFqfor p = q + 1 are said to be GHFs of Gauss type [6,7] and are considered for |x|< 1.

The relation between $G_{p,q}^{m,n}$ and $_pF_q$

The hypergeometric functions play an important role in univalent functions theoryand in order to give this role to the MGFs, we mention the relation between two families of functions. Any generalized hypergeometric functioncan readily be expressed in terms of the MGF:

$$_pF_q\left({}_{b_q}^{a_p}\middle|z\right) = \frac{\Gamma(a_p)}{\Gamma(b_q)} G_{p,q+1}^{1,p}\left({}_{0,1-b_q}^{1-a_p}\middle|-z\right) \quad (3.1)$$

, where the vector notation has been used such that:$\Gamma(\boldsymbol{a_p}) = \prod_{j=1}^{p} \Gamma(a_j)$.

This relationship is valid whenever the generalized hypergeometric series pFq(z) converges, for $|z| < 1$ when p = q + 1.

Notice that working with the $_pF_q$- functions is equivalent to working with the MGFs of type $G_{p,q+1}^{1,p}$.

For instance, for p = 1; q = 0 we have:

$$_1F_0\left({}_{1}^{a}\middle|z\right) = \Gamma(a) G_{1,1}^{1,1}\left({}_{1}^{1-a}\middle|-z\right).$$

SOME DEFINITIONS IN UNIVALENT FUNCTION THEORY

Let A denote the class of functions [3]:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (4.1)$$

analytic in the unit disk *U*. We denote by *S*the subclass of univalent functions in *A*and by *C*, $S^*$and *K*the subclass of S whose members are close-to-convex, starlike (with respect to the origin) and convex in *U*, respectively.

Some transformations and theorems in univalent function theory

In any class of functions, it can be shown transformations or operations that take a function of the set into another function of the same set. If f is in S then f IS under the rotation, dilation and conjugation transformations also in S:

Taylor's theorem: Let *f(z)* be analytic in domain *D*. let *z* = *a*be any point in *D*, then there exist a Taylor's series with center at "*a*" which represents *f(z)*

$$f(z)=\sum_{k=0}^{\infty} b_k(z-a)^k, b_k=\frac{1}{k!}\frac{d^k}{dz^k}f(a), k=0,1,2,... \qquad (4.2)$$

Residue theorem: Let *f(z)* be analytic and single-valued inside and on a rectifiable Jordan curve C, except in a finite number of singular points $z_1, z_2, ..., z_n$ inside *C*, where has residues $\alpha_1, \alpha_2, ..., \alpha_n$ then

$$\oint_C f(z)dz = 2\pi i(\alpha_1+\alpha_2+...+\alpha_n) \qquad (4.3)$$

MAIN RESULTS

The Koebe function is the*key* function in the theory of univalent functions. Moreover, this function is also MGF.Choosing suitable values of parameters and orders, we have the following interesting relation for the Koebe function:

$$f(z)=\frac{z}{(1-z)^2}=G_{1,1}^{1,1}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\middle|z\right)=\frac{1}{2\pi i}\int_L \Gamma(s+1)\Gamma(1-s)z^s ds \qquad (5.1)$$

On the other hand, this elementary function is also a hypergeometric function [6-7].

Comparison of branch cuts in the Koebe and the Meijer's G-function

$$K(z)\equiv\frac{z}{(1-z)^2}=G_{1,1}^{1,1}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\middle|z\right)=z+2z^2+3z^3+...+nz^n=\sum_{n=1}^{\infty} nz^n \qquad (5.2)$$

The Koebe function is in *S*and maps the unit disk *U*in one-to-one and onto the domain D that covers the entire complex plane except the branch cut for a slit alongthe negative real axis from *w* =-∞*to w* =$\frac{1}{4}$. On the other hand, the parametersof MGF determine the existence of the branch cut. If the branch cutexists, it will be along $(-\infty,0)$ or sometimes $(-\infty,-1) or (-1,0)$.

For example the MGF

$$G_{1,1}^{1,1}\left(\begin{smallmatrix}a\\b\end{smallmatrix}\middle|z\right)=\frac{1}{2\pi i}\int_L \Gamma(s+b)\Gamma(1-a-s)z^s ds=\Gamma(1-a+b)z^b(1-z)^{a-b-1}$$

has branch cut along $(-\infty,0)$ if *b*and *b*-*a*do not belong to *Z*; in the case $b\in Z$, itgenerally has branch cut along $(-\infty,-1)$.

Meijer G-functions in theory of univalent functions

Let $f(z)=G_{p,q}^{m,n}\left(\begin{smallmatrix}a_p\\b_q\end{smallmatrix}\middle|z\right)$be analytic and MGF. By definition for starlike functions, we have$Re\frac{zf'(z)}{f(z)}>0$, and

$$Re\frac{z\frac{dG_{p,q}^{m,n}\left(\begin{smallmatrix}a_p\\b_q\end{smallmatrix}\middle|z\right)}{dz}}{G_{p,q}^{m,n}\left(\begin{smallmatrix}a_p\\b_q\end{smallmatrix}\middle|z\right)}>0\equiv Re\frac{G_{p+1,q+1}^{m,n+1}\left(\begin{smallmatrix}0,a_p\\b_q,1\end{smallmatrix}\middle|z\right)}{G_{p,q}^{m,n}\left(\begin{smallmatrix}a_p\\b_q\end{smallmatrix}\middle|z\right)}>0 \qquad (5.3)$$

This inequality for the Koebe function is

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$$Re\frac{G_{2,2}^{1,2}\left({0,0 \atop 1,1}|z\right)}{G_{1,1}^{1,1}\left({0 \atop 1}|z\right)} > 0.$$

And also by definition for convex functions, $Re\ (1+\frac{zf'(z)}{f(z)}) > 0$ so this definition for MGF can be written as

$$Re(1+\frac{G_{p+1,q+1}^{m,n+1}\left({0,a_p \atop b_q,2}|z\right)}{G_{p+1,q+1}^{m,n+1}\left({0,a_p \atop b_q,1}|z\right)}) > 0 \qquad (5.4)$$

For the Koebe function, we have

$$Re(1+\frac{G_{2,2}^{1,2}\left({0,0 \atop 1,2}|z\right)}{G_{2,2}^{1,2}\left({1,2 \atop 2,2}|z\right)} > 0.$$

The definition of close-to-convex functions, we have

$$f(z) = G_{p_1,q_1}^{m_1,n_1}\left({a_{p1} \atop b_{q_1}}|z\right), g(z) = G_{p_2,q_2}^{m_2,n_2}\left({a_{p2} \atop b_{q_2}}|z\right)$$

and

$$Re(\frac{f'(z)}{g'(z)}) = Re(\frac{zf'(z)}{zg'(z)}) > 0.$$

The first condition gives

$$Re(\frac{G_{p_1+1,q_1+1}^{m_1,n_1+1}\left({0,a_{p1} \atop b_{q_1},1}|z\right)}{G_{p_2+1,q_2+1}^{m_2,n_2+1}\left({0,a_{p2} \atop b_{q_2},1}|z\right)}) > 0 \qquad (5.5)$$

While for the second condition we have

$$Re(1+\frac{G_{p_2+1,q_2+1}^{m_1,n_2+1}\left({0,a_{p_2} \atop b_{q_2},1}|z\right)}{G_{p_2+1,q_2+1}^{m_2,n_2+1}\left({0,a_{p_2} \atop b_{q_2},1}|z\right)}) > 0. \qquad (5.6)$$

Some operations on Univalent functions

For the rotation transformation in the language of MGF we have

$$e^{-i\theta}f(e^{i\theta}z) = e^{-i\theta}G_{p,q}^{m,n}\left({a_p \atop b_q}|e^{i\theta}z\right) = \sum_{k=0}^{\infty}\frac{(e^{i\theta}-1)^k}{(e^{i\theta})k!}G_{p+1,q+1}^{m,n+1}\left({0,a_p \atop b_q,k}|z\right), \qquad (5.7)$$

The Eq. (5.7) means that if

$$f(z) = G_{p,q}^{m,n}\left({a_1 \ldots a_p \atop b_1 \ldots b_q}|z\right) = \frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\times\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\times\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s ds$$

is univalent, then the following path integral

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$$e^{-i\theta}G_{p,q}^{m,n}\left({}_{b_1\dots b_q}^{a_1\dots a_p}\middle|e^{i\theta}z\right)=\frac{1}{2\Pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\times\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\times\prod_{j=n+1}^{p}\Gamma(a_j-s)}e^{i\theta(s-1)}z^s ds \tag{5.8}$$

is also univalent.

For the rotated Koebe function $e^{-i\theta}f(ze^{i\theta})=\frac{z}{(1-ze^{i\theta})^2}$, the path integral $G_{1,1}^{1,1}\left({}_1^0\middle|ze^{i\theta}\right)=\frac{1}{2\pi i}\int_L\Gamma(s+1)\Gamma(1-s)e^{is\theta}z^s ds$ is also univalent.

For the dilation transformation

$$\frac{1}{t}f(tz)=\frac{1}{t}G_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|tz\right)=\sum_{k=0}^{\infty}\frac{(t-1)^k}{(t)k!}G_{p+1,q+1}^{m,n+1}\left({}_{b_q,k}^{0,a_p}\middle|z\right), \tag{5.9}$$

Multiplication's theorem can be used for the dilated Koebe function. Then we have

$$\frac{z}{(1-tz)^2}=\sum_{k=0}^{\infty}\frac{(t-1)^k}{(t)k!}G_{2,2}^{1,2}\left({}_{1,k}^{0,0}\middle|z\right)$$

For the conjugation transformation, we have

$$f(\overline{z})=G_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|\overline{z}\right)=\frac{-1}{2\Pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\times\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\times\prod_{j=n+1}^{p}\Gamma(a_j-s)}\overline{z}^s ds$$

$$\overline{f(\overline{z})}=G_{p,q}^{m,n}\left({}_{\overline{b_q}}^{\overline{a_p}}\middle|z\right)=\frac{1}{2\Pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(\overline{b_j}-s)\times\prod_{j=1}^{n}\Gamma(1-\overline{a_j}+s)}{\prod_{j=m+1}^{q}\Gamma(1-\overline{b_j}+s)\times\prod_{j=n+1}^{p}\Gamma(\overline{a_j}-s)}z^s ds. \tag{5.10}$$

The commutation relations between two properties of MGFs

The commutation between two operators is defined as [A,B]=AB-BA. To apply the two consecutive operations, we note that the operators are $A=z^{\alpha}, B=z^k\frac{d^k}{dz^k}$ and thus the effect of operator BA on $G_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)$ is

$$AG_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)=z^{\alpha}G_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)=G_{p,q}^{m,n}\left({}_{b_q+\alpha}^{a_p+\alpha}\middle|z\right),$$

and

$$BAG_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)=z^k\frac{d^k}{dz^k}G_{p,q}^{m,n}\left({}_{b_q+\alpha}^{a_p+\alpha}\middle|z\right)=G_{p+1,q+1}^{m,n+1}\left({}_{b_q+\alpha,k}^{0,a_p+\alpha}\middle|z\right).$$

But the effect of operator AB on $G_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)$ is

$$BG_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)=z^k\frac{d^k}{dz^k}G_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)=G_{p+1,q+1}^{m,n+1}\left({}_{b_q,k}^{0,a_p}\middle|z\right),$$

and

$$ABG_{p,q}^{m,n}\left({}_{b_q}^{a_p}\middle|z\right)=AG_{p+1,q+1}^{m,n+1}\left({}_{b_q,k}^{0,a_p}\middle|z\right)=G_{p+1,q+1}^{m,n+1}\left({}_{b_q+\alpha,k+\alpha}^{\alpha,a_p+\alpha}\middle|z\right).$$

Thus for [A,B]=AB-BA, we have

$$[A,B]G_{p,q}^{m,n}\left({}^{a_p}_{b_q}|z\right)=G_{p+1,q+1}^{m,n+1}\left({}^{\alpha,a_p+\alpha}_{b_q+\alpha,k+\alpha}|z\right)-G_{p+1,q+1}^{m,n+1}\left({}^{0,a_p}_{b_q+\alpha,k}|z\right). \quad (5.11)$$

Choosingk =α= 1, using G-function form of the Koebe function gives

$$[A,B](G_{1,1}^{1,1}({}^0_1|z))=[z,z\frac{d}{dz}](G_{1,1}^{1,1}({}^0_1|z))=G_{2,2}^{1,2}({}^{1,1}_{2,2}|z)-G_{2,2}^{1,2}({}^{0,1}_{2,1}|z).$$

Taylor's series for G-functions

If $f(z)=G_{p,q}^{m,n}\left({}^{a_p}_{b_q}|z\right)$is analytic at point z = a, then to obtain Taylor's series for MGF at z = $a$we use (2.1) and (2.2) such that

$$\frac{1}{k!}\frac{d^k}{dz^k}G_{p,q}^{m,n}\left({}^{a_p}_{b_q}|z\right)=\frac{z^{-k}z^k}{k!}\frac{d^k}{dz^k}G_{p,q}^{m,n}\left({}^{a_p}_{b_q}|z\right)=\frac{z^{-k}}{k!}G_{p+1,q+1}^{m,n+1}\left({}^{0,a_p}_{b_q,k}|z\right)=\frac{1}{k!}G_{p+1,q+1}^{m,n+1}\left({}^{-k,a_p-k}_{b_q-k,0}|z\right),$$

For which

$$b_k=\frac{1}{k!}G_{p+1,q+1}^{m,n+1}\left({}^{-k,a_p-k}_{b_q-k,0}|z\right),G_{p,q}^{m,n}\left({}^{a_p}_{b_q}|z\right)=\sum_{k=0}^{\infty}\frac{1}{k!}G_{p+1,q+1}^{m,n+1}\left({}^{-k,a_p-k}_{b_q-k,0}|z\right)(z-a)^k=$$

$$=G_{p+1,q+1}^{m,n+1}\left({}^{0,a_p}_{b_q,0}|a\right)+G_{p+1,q+1}^{m,n+1}\left({}^{-1,a_p-1}_{b_q-1,0}|a\right)(z-a)+\frac{1}{2!}G_{p+1,q+1}^{m,n+1}\left({}^{-2,a_p-2}_{b_q-2,0}|a\right)(z-a)^2+...$$

So

$$G_{p,q}^{m,n}\left({}^{a_p}_{b_q}|z\right)=G_{p,q}^{m,n}\left({}^{a_p}_{b_q}|a\right)+G_{p+1,q+1}^{m,n+1}\left({}^{-1,a_p-1}_{b_q-1,0}|a\right)(z-a)+\frac{1}{2!}G_{p+1,q+1}^{m,n+1}\left({}^{-2,a_p-2}_{b_q-2,0}|a\right)(z-a)^2+.... \quad (5.12)$$

Thus Taylor's series for the Koebe function, $G_{1,1}^{1,1}({}^0_1|z)$z = a is

$$G_{1,1}^{1,1}({}^0_1|z)=G_{1,1}^{1,1}({}^0_1|a)+G_{2,2}^{1,2}({}^{-1,-1}_{0,0}|a)(z-a)+\frac{1}{2!}G_{2,2}^{1,2}({}^{-2,-2}_{-1,0}|a)(z-a)^2+...$$

Residue theorem for Meijer's G-function

The residue theorem for MGF is represented by

$$\oint_C G_{p,q}^{m,n}\left({}^{a_1...a_p}_{b_1...b_q}|z\right)dz=\oint_C\left(\frac{1}{2\Pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-s)\times\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\times\prod_{j=n+1}^p\Gamma(a_j-s)}z^s ds\right)dz$$

$$=\frac{1}{2\Pi i}\int_L ds\frac{\prod_{j=1}^m\Gamma(b_j-s)\times\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\times\prod_{j=n+1}^p\Gamma(a_j-s)}\oint_C z^s dz$$

Note that $z=0$ is the only singular point in the open unit disk, and by choosing closed curve to include this singular point we obtain

$$\oint_C G^{m,n}_{p,q}\left({a_1\dots a_p \atop b_1\dots b_q}\middle|z\right)dz = \int_L ds \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\times\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\times\prod_{j=n+1}^{p}\Gamma(a_j-s)} \qquad (5.13)$$

The Jackson derivative of MGFs

The Jackson derivative originally was introduced by Jackson [12] in the study of the basic hypergeometric series. Here we calculate the Jackson derivative of Meijer's G-functions, first we refer to the ordinary derivative that measures the rate of change of the function in terms of an incremental translation of its argument. In deformed calculus, it is used from this derivative, see for instance [8-11]. In contrast to the usual derivative, Jackson derivative (JD) measures its rate of change with respect to a dilation of its argument by a factor q, as follows**:**

$$D_z^{(q)} f(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}$$

$$f(z) = G^{m,n}_{p,q}\left({a_p \atop b_q}\middle|z\right).$$

Using the multiplication theorem Eq. (2.3), we have

$$D_z^{(q)} G^{m,n}_{p,q}\left({a_p \atop b_q}\middle|z\right) = \frac{G^{m,n}_{p,q}\left({a_p \atop b_q}\middle|qz\right) - G^{m,n}_{p,q}\left({a_p \atop b_q}\middle|q^{-1}z\right)}{(q-q^{-1})z}$$

And finally

$$D_z^{(q)} G^{m,n}_{p,q}\left({a_p \atop b_q}\middle|z\right) = \frac{\sum_{k=0}^{\infty}\frac{(q-1)^k}{k!}G^{m,n+1}_{p+1,q+1}\left({0,a_p \atop b_q,k}\middle|z\right) - \sum_{k=0}^{\infty}\frac{(q^{-1}-1)^k}{k!}G^{m,n+1}_{p+1,q+1}\left({0,a_p \atop b_q,k}\middle|z\right)}{(q-q^{-1})z}. \qquad (5.14)$$

In special case Jackson's derivative(J.D.) for the Koebe function can be deduced as follows:

$$D_z^{(q)} G^{1,1}_{1,1}\left({0 \atop 1}\middle|z\right) = \frac{\sum_{k=0}^{\infty}\frac{(q-1)^k}{k!}G^{1,2}_{2,2}\left({0,0 \atop 1,k}\middle|z\right) - \sum_{k=0}^{\infty}\frac{(q^{-1}-1)^k}{k!}G^{1,2}_{2,2}\left({0,0 \atop 1,k}\middle|z\right)}{(q-q^{-1})z}.$$

CONCLUSIONS

In this paper we first represent the Koebe function as Meijer's G-function by giving value of parameters and orders. Interesting properties of Meijer G-functions are applied in the univalent functions theory to obtain new notation for some definitions, transformations and theorems in this area. This work can be convenient for unification purposes in univalent function theory and other related fields.

REFERENCES

[1]V. Kiryakova, M. Saigo,H.M. Srivastava, Some criteria for univalence of analytic Functions involving generalized fractional calculus operators,*Frac. Calc. Appl. Anal.*, Vol. 1(1998),79-104.

[2]V. Kiryakova , M. Saigo , S.Owa , Distortion and characterization theorems for starlike and convex functions related to generelized fractional calculus, *Res. Inst. Math.Sci.*,Kyoto Univ.,Proc. Kyoto Seminar on Unival. F-s,March 24-27 1997,(1997), 25-46.

[3] P. L. Duren, Univalent Functions, Springer-Verlag, New York,1983.

[4] L. C. Andrews, Special Functions for Engineers and Applied Mathematicians,New York,: MacMillan, 1985.

[5] Y. L. Luke, The Special Functions and Their Approximations, Complex Variables, Volume I , New York : Academic Press, 1969.

[6] V. Kiryakova, All the special functions are fractional differ-integrals of Elementary functions, *J. Phys. A: Math. Gen.*, 30 (1997), 5085-5103.

[7]V. Kiryakova, The special functions of fractional calculus as generalized fractional Calculus operators of some basic functions, *Computers &Mathematics with Applications*,59 (2010), 1128-1141.

[8]M. Darus, R. W. Ibrahim, On new Laguere-type of functions involving new deformedcalculus, *Adv. Studies Theor. Phys.*, Vol. 3 (2009), no. 9, 313-322.

[9]M. Darus, R. W. Ibrahim, A note on Generalized E-Function and its Applications, *Adv.Studies Theor. Phys.*, Vol.3, (2009), 421-427.

[10]M. Darus, R. W. Ibrahim, Deformed Fox's H-function and its applications, Pacific Jornalof Applied Mathematics, 4 Vol.2,(2009), 1-10.

[11]Jing, S., W. Yang, A new kind of deformed Hermite Polynomials and its applications,http:// arxiv.org/abs/math-ph/0212011.

[12] Jackson F. H. ,Am. J. Math.38 26, (1909),38-57.

[13] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman, Harlow(UK), 1994.

[14] A. U. Klimyik, Meijer G-function, in Hazewinkel, Michiel, Encyclopaedia of Mathematics,Springer, ISBN 978-1556080104, 2001.