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Extended Jacobian Elliptic Function Expansion Method and Its Applications in Mathematical Physics

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Abstract

In this work, an extended Jacobian elliptic function expansion method is proposed for constructing the exact solutions of nonlinear evolution equations. The validity and reliability of the method are tested by its applications to some nonlinear evolution equations which play an important role in mathematical physics.

Keywords

Extended Jacobian elliptic function expansion method; The nonlinear Phi- Four equation; Traveling wave solutions.

1- Introduction

No one can deny the important role which played by the nonlinear partial differential equations in the description of many and a wide variety of phenomena not only in physical phenomena, but also in plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry phenomena. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. Such methods are tanh - sech method [1]-[3], extended tanh - method [4]-[6], sine - cosine method [7]-[9], homogeneous balance method [10, 11], F-expansion method [12]-[14], exp-function method [15, 16], trigonometric function series method [17], $\left(\frac{G'}{G}\right)$ -expansion method [18]-[21], the exp (-φ (ξ))-expansion method [22]-[24],extended Jacobi elliptic function method [25]-[28], the exp (-φ (ξ))-expansion method [26]-[28] and so on. The objective of this article is to apply an extended Jacobian elliptic function expansion for finding the exact traveling wave solution of the nonlinear Phi- Four equation which play an important role in mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of an extended Jacobian elliptic function expansion. In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 5, we give the physical interpretations of the solutions. In Section 5, conclusions are given.

2- Description of method

Consider the following nonlinear evolution equation

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

since, P is a polynomial in $u(x,t)$ and its partial derivatives. In the following, we give the main steps of this method

Step 1. We use the traveling wave solution in the form

$$u(x,t) = u(\xi), \xi = x - ct, \tag{2.2}$$

where c is a positive constant, to reduce Eq. (2.1) to the following ODE:

$$p(u, u', u'', u''', \ldots) = 0, \tag{2.3}$$

where P is a polynomial in u (ξ) and its total derivatives, while $\left(u' = \frac{du}{d\xi}\right)$.

Step 2. Making good use of ten Jacobian elliptic functions we assume that (2.3) have the solutions in these forms:

$$u(\xi) = a_0 + \sum_{j=1}^{N} f_i^{\,j-1}(\xi)\left[a_j f_i(\xi) + b_j g(\xi)\right], \; i = 1,2,3,\ldots \tag{2.4}$$

With

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$$
\begin{aligned}
f_1(\xi) &= sn\xi,\ g_1(\xi) = cn\xi,\\
f_2(\xi) &= sn\xi,\ g_2(\xi) = dn\xi,\\
f_3(\xi) &= ns\xi,\ g_3(\xi) = cs\xi,\\
f_4(\xi) &= ns\xi,\ g_4(\xi) = ds\xi,\\
f_5(\xi) &= sc\xi,\ g_5(\xi) = nc\xi,\\
f_6(\xi) &= sd\xi,\ g_6(\xi) = nd\xi,
\end{aligned}
\tag{2.5}
$$

Where$sn\xi, cn\xi, dn$are the Jacobian elliptic sine function, thejacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobianfunctions which are denoted by Glaisher's symbols and are generated by these three kinds of functions, namely:

$$
\begin{aligned}
&ns\xi = \frac{1}{sn\xi}, nc\xi = \frac{1}{cn\xi}, nd\xi = \frac{1}{dn\xi}, sc\xi = \frac{cn\xi}{sn\xi},\\
&cs\xi = \frac{sn\xi}{cn\xi}, ds\xi = \frac{dn\xi}{sn\xi}, sd\xi = \frac{sn\xi}{dn\xi},
\end{aligned}
\tag{2.6}
$$

those have the relations

$$
\begin{aligned}
&sn^2 + cn^2 = 1,\ dn^2\xi + m^2 sn^2\xi = 1,\ ns^2\xi = 1 + cs^2\xi,\\
&ns^2\xi = m^2 + ds^2\xi,\ sc^2\xi + 1 = nc^2\xi,\ m^2 sd^2 + 1 = nd^2\xi,
\end{aligned}
\tag{2.7}
$$

with the modulus m (0 < m < 1). In addition we know that

$$
\frac{d}{d\xi} sn\xi = cn\xi\, dn\xi, \frac{d}{d\xi} cn\xi = -sn\xi\, dn\xi, \frac{d}{d\xi} dn\xi = -m^2 sn\xi\, cn\xi.
\tag{2.8}
$$

The derivatives of other Jacobian elliptic functions are obtained by using Eq. (1.8). Tobalance the highest order linear term with nonlinear term we define the degree of u asD[u] = n which gives rise to the degrees of other expressions as

$$
D\left[\frac{d^q u}{d\xi^q}\right] = n + q,\ D\left[u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right] = np + s(n+q).
\tag{2.9}
$$

According the rules, we can balance the highest order linear term and nonlinear term in Eq. (2.3) so that n in Eq.(2.4) can be determined.In addition we see that when$(m \to 1), (sn\xi, cn\xi, dn\xi)$ degenerate as$(tanh\xi, sech\xi, sech\xi)$respectively, while when therefore Eq.(2.5) degenerate as the following forms

$$
\begin{aligned}
u(\xi) &= a_0 + \sum_{j=1}^{N} \tanh^{j-1}(\xi)\left[a_j \tanh(\xi) + b_j \operatorname{sech}(\xi)\right],\\
u(\xi) &= a_0 + \sum_{j=1}^{N} \coth^{j-1}(\xi)\left[a_j \coth(\xi) + b_j \operatorname{csch}(\xi)\right],\\
u(\xi) &= a_0 + \sum_{j=1}^{N} \tanh^{j-1}(\xi)\left[a_j \tan(\xi) + b_j \sec(\xi)\right],\\
u(\xi) &= a_0 + \sum_{j=1}^{N} \coth^{j-1}(\xi)\left[a_j \cot(\xi) + b_j \csc(\xi)\right],
\end{aligned}
\tag{2.10}
$$

Therefore the extended Jacobian elliptic function expansion method is more general than sine-cosine method, the tan-function method and Jacobian elliptic function expansion method.

Application

Here, we will apply, an extended Jacobian elliptic function expansion method described in Sec.2 to find the exact traveling wave solutions and the solitary wave solutions of the nonlinear Phi- Four equation[32, 33]. We consider the nonlinear Phi-Four equation**.**

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$$u_{tt} - \alpha u_{xx} - u + u^3 = 0, \tag{3.1}$$

where α and β are real constants. By using the wave transformation $u(\xi) = u(x,t)$, since

$(\xi = x - t)$,we get:

$$(1-\alpha)u'' - u + u^3 = 0. \tag{3.2}$$

Balancing between the highest order derivatives and nonlinear terms appearing in Eq. (3.2)$(u^3 and\ u'' \Rightarrow m = 1)$. So that, by using Eq. (2.4) we get the formal solution of Eq. (3.2)

$$v(\xi) = a_0 + a_1 sn\xi + b_1 cn\xi. \tag{3.3}$$

Substituting Eq. (3.5) and its derivative into Eq. (3.2) and collecting all term with the same power of $sn^3, sn^2cn, sn^2, sncn, sn, cn, sn^0$ we get:

$$\begin{cases} (1-\alpha)(2a_1m^2) + (a_1^3 - 3a_1b_1^2) = 0, \\ (1-\alpha)(2m^2b_1) + (3a_1^2b_1 - b_1^3) = 0, \\ (3a_0a_1^2 - 3a_0b_1^2) = 0, \\ (6a_0a_1b_1) = 0, \\ (1-\alpha)(-m^2a_1 - a_1) - (a_1) + (3a_0^2a_1 + 3a_1b_1^2) = 0, \\ (1-\alpha)(-b_1) - (b_1) + (3a_0^2b_1 + b_1^3) = 0, \\ -(a_0) + (a_0^3 + 3a_0b_1^2) = 0, \end{cases} \tag{3.4}$$

Solving above system by using maple 16, we get:

Case1:

$$\alpha = \frac{2(m^2-1)}{2m^2-1}, a_0 = a_1 = 0, b_1 = \pm m\sqrt{\frac{2}{2m^2-1}}.$$

Thus the exact traveling wave solution is

$$u(\xi) = \pm m\sqrt{\frac{2}{2m^2-1}}\, cn(\xi). \tag{3.5}$$

When $(m \longrightarrow 1)$ we get:

$$u(\xi) = \pm\sqrt{2}\, sech(\xi). \tag{3.6}$$

Case2:

$$\alpha = \frac{m^2-4}{m^2-2}, a_0 = 0, a_1 = \pm m\sqrt{\frac{1}{2-m^2}}, b_1 = \pm m\sqrt{\frac{2}{2m^2-1}}.$$

Thus the exact traveling wave solution is

$$u(\xi) = \pm m\sqrt{\frac{1}{2-m^2}}sn(\xi) \pm m\sqrt{\frac{2}{2m^2-1}}\, cn(\xi). \tag{3.7}$$

When $(m \longrightarrow 1)$ we get:

$$u(\xi) = \pm tanh(\xi) \pm \sqrt{2}\, sech(\xi). \tag{3.8}$$

Case3:

$$\alpha = \frac{2+m^2}{1+m^2}, a_0 = 0, a_1 = \pm m\sqrt{\frac{2}{m^2+1}}, b_1 = 0.$$

Thus the exact traveling wave solution is

$$u(\xi) = \pm m\sqrt{\frac{2}{m^2+1}}sn(\xi). \tag{3.9}$$

When $(m \longrightarrow 1)$ we get:

$$u(\xi) == \pm tanh(\xi). \tag{3.10}$$

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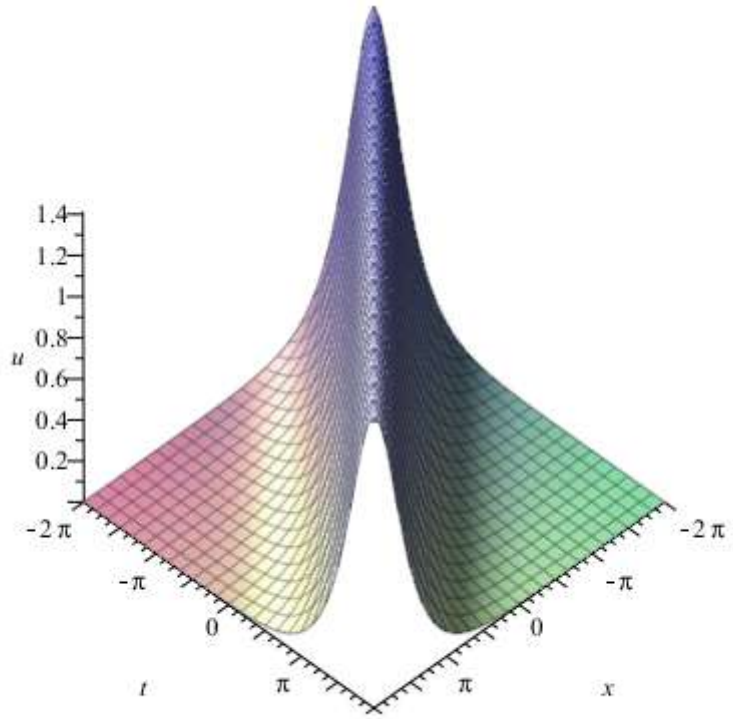
- **Note That**:

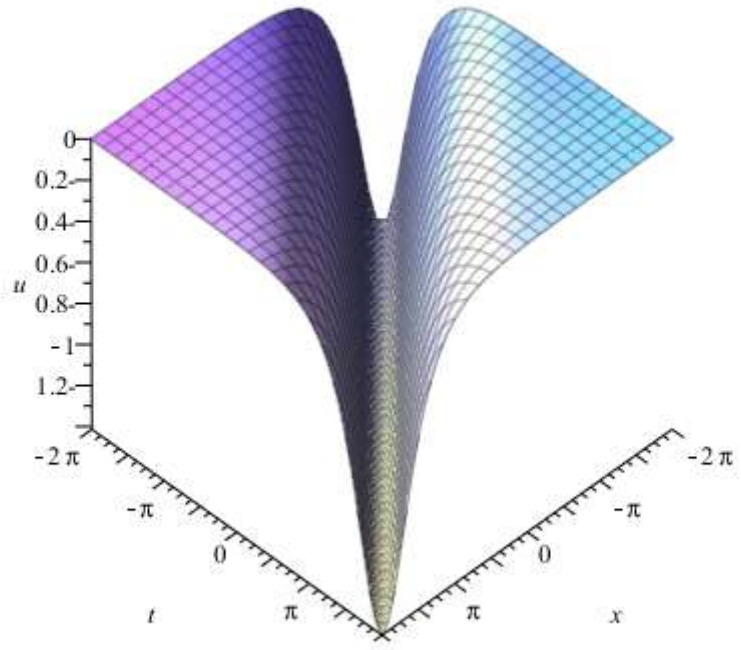
All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.

Figures:

When $(x = -2\pi..2\pi, t = -2\pi..2\pi)$



Plot of (3. 6)



Plot of (3. 6)

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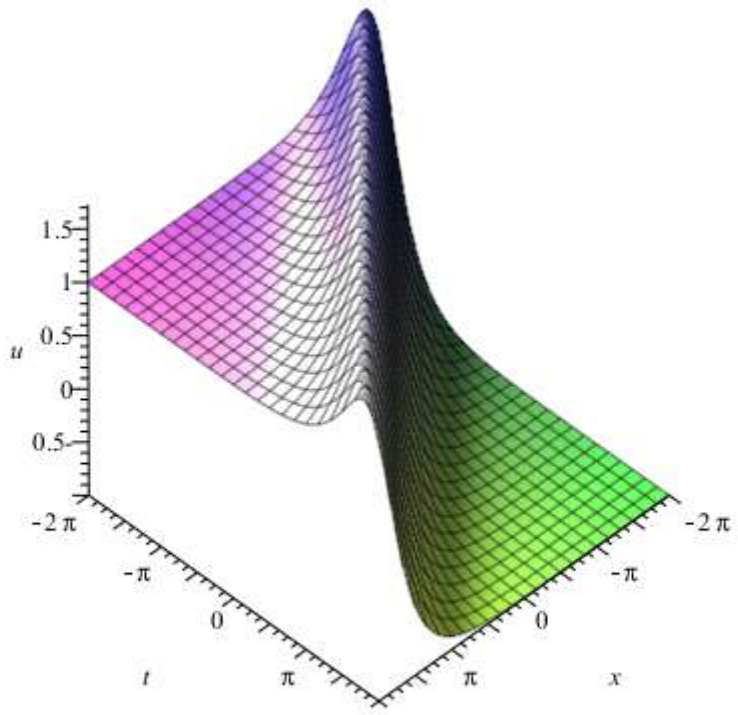
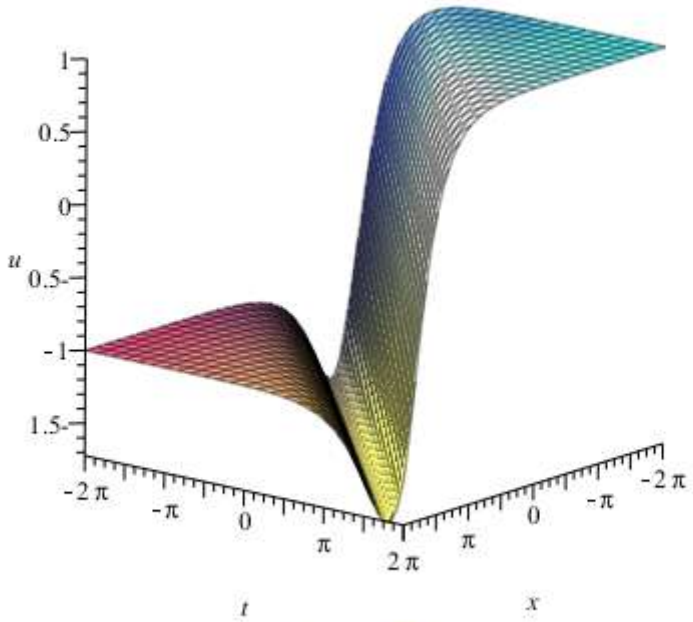


Plot of (3.8)



Plot of (3.8)

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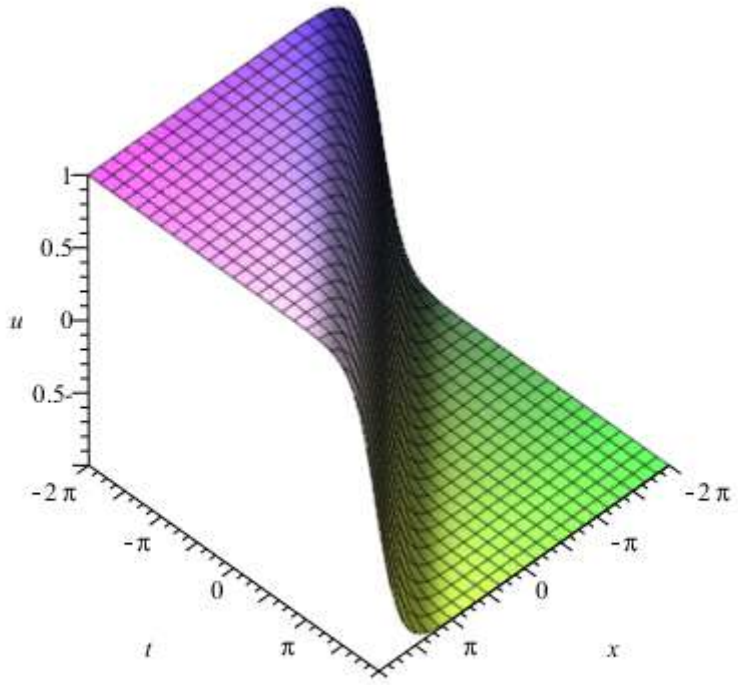
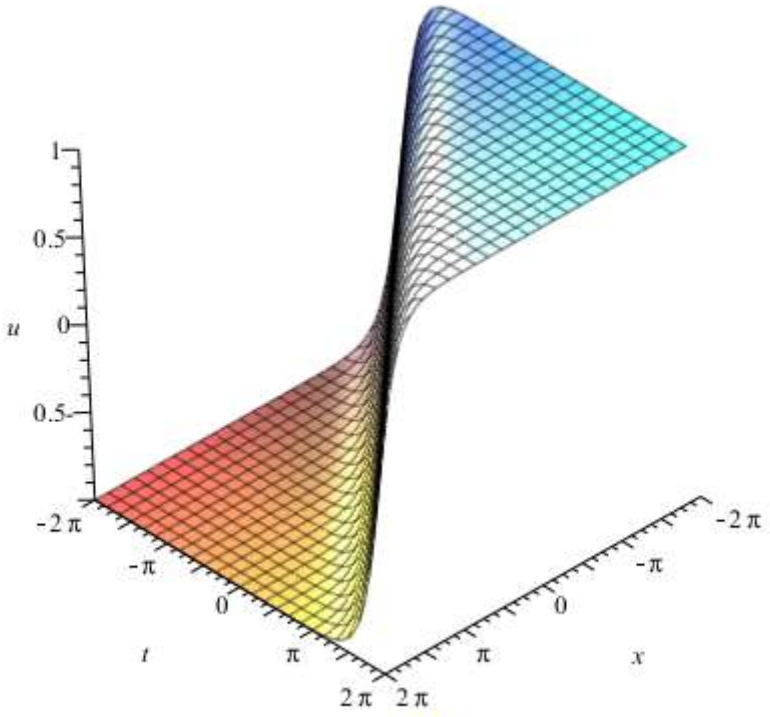


Plot of (3. 10)



Plot of (3. 10)

3- Conclusion

An extended Jacobian elliptic function expansion method has been applied in this paper to find the exact traveling wave solutions and then the solitary wave solutions of the nonlinear Phi- Four equation. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the nonlinear Phi- Four equationare new and different from those obtained in [32,33], and figures show the solitary traveling wave solution ofthe nonlinear Phi- Four equation. We can conclude that an extended Jacobian elliptic

function expansion method is a very powerful and efficient technique in finding exact solutions for wide classes of nonlinear problems and can be applied to many other nonlinear evolution equations in mathematical physics. Another possible merit is that the reliability of the method and the reduction in the size of computational domain give this method a wider applicability

4- Competing interests

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

5- Author's contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied

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(Corresponding author: Mostafa M. A. Khater) I would like to dedicate this article to my mother and the soul of my father, he was there for the beginning of this degree, and did not make it to the end. His love, support, and constant care will never be forgotten. He is very much missed.

References

[1] W. Maliet, Solitary wave solutions of nonlinear wave equation, Am. J. Phys., 60 (1992) 650-654.

[2] W. Maliet, W. Hereman, The tanh method: Exact solutions of nonlinear evolution and wave equations, Phys.Scr., 54 (1996) 563-568.

[3] A. M. Wazwaz, The tanh method for travelling wave solutions of nonlinear equations, Appl. Math. Comput., 154 (2004) 714-723.

[4] S. A. EL-Wakil, M.A.Abdou, New exact travelling wave solutions using modified extended tanh-function method, Chaos Solitons Fractals, 31 (2007) 840-852.

[5] Mahmoud A.E. Abdelrahman, Emad H. M. ZahranMostafa M.A. Khater, Exact Traveling Wave Solutions for Modified Liouville Equation Arising in Mathematical Physics and Biology, (International Journal of Computer Applications (0975 8887) Volume 112 - No. 12, February 2015).

[6] Mostafa M.A. Khater and Emad H. M. Zahran, Modified extended tanh function method and its applications to the Bogoyavlenskii equation, Applied Mathematical Modelling, 40, 1769-1775 (2016).

[7] A. M. Wazwaz, Exact solutions to the double sinh-Gordon equation by the tanh method and a variable separated ODE. method, Comput. Math. Appl., 50 (2005) 1685-1696.

[8] A. M.Wazwaz, A sine-cosine method for handling nonlinear wave equations, Math. Comput.Modelling, 40 (2004) 499-508.

[9] C. Yan, A simple transformation for nonlinear waves, Phys. Lett. A 224 (1996) 77-84.

[10] E. Fan, H.Zhang, A note on the homogeneous balance method, Phys. Lett. A 246 (1998) 403-406.

[11] M. L. Wang, Exct solutions for a compound KdV-Burgers equation, Phys. Lett. A 213 (1996) 279-287.

[12] Emad H. M. Zahran and Mostafa M.A. Khater, The modified simple equation method and its applications for solving some nonlinear evolutions equations in mathematical physics, (Jokull journal- Vol. 64. Issue 5 - May 2014).

[13] Y. J. Ren, H. Q. Zhang, A generalized F-expansion method to find abundant families of Jacobi elliptic function solutions of the (2+1)-dimensional Nizhnik-Novikov-Veselov equation, Chaos Solitons Fractals, 27 (2006) 959-979.

[14] J. L. Zhang, M. L. Wang, Y. M. Wang, Z. D. Fang, The improved F-expansion method and its applications, Phys.Lett.A 350 (2006) 103-109.

[15] J. H. He, X. H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons Fractals 30 (2006) 700-708.

[16] H. Aminikhad, H. Moosaei, M. Hajipour, Exact solutions for nonlinear partial differential equations via Exp-function method, Numer. Methods Partial Differ. Equations, 26 (2009) 1427-1433.

[17] Z. Y. Zhang, New exact traveling wave solutions for the nonlinear Klein-Gordon equation, Turk. J. Phys., 32 (2008) 235-240.

[18] M. L. Wang, J. L. Zhang, X. Z. Li, The $\left(\frac{G'}{G}\right)$ –expansion method and travelling wave solutions of nonlinear evolutions equations in mathematical physics, Phys. Lett. A 372 (2008) 417-423.

[19] S. Zhang, J. L. Tong, W.Wang, A generalized $\left(\frac{G'}{G}\right)$ –expansion method for the mKdv equation with variable coe_cients, Phys. Lett. A 372 (2008) 2254-2257.

[20] Mostafa M. A. Khater, On the New Solitary Wave Solution of the Generalized Hirota-Satsuma Couple KdV System, GJSFR-A Volume 15 Issue 4 Version 1.0 (2015).

[21] E.H.M.Zahran and mostafa M.A. khater, Exact solutions to some nonlinear evolution equations

by the $\left(\frac{G'}{G}\right)$ −expansion method equations in mathematical physics, J□bkull Journal, Vol. 64, No. 5; May 2014.

[21] NizhumRahman, Md. NurAlam, Harun-Or-Roshid, SelinaAkter and M. Ali Akbar, Application of exp(-φ (ξ))-expansion method to find the exact solutions of Shorma-Tasso-Olver Equation, African Journal of Mathematics and Computer Science Research Vol. 7(1), pp. 1-6, February, 2014.

[23] Rafiqul Islam, Md. NurAlam, A.K.M. KaziSazzadHossain, Harun-Or-Roshid and M. Ali Akbar, Traveling Wave Solutions of Nonlinear Evolution Equations via Exp(-φ (ξ))- Expansion Method, Global Journal of Science Frontier Research Mathematics and Decision Sciences. Volume 13 Issue 11 Version 1.0 Year 2013.

[24] Mahmoud A.E. Abdelrahman, Emad H. M. ZahranMostafa M.A. Khater, Exact traveling wave solutions for power law and Kerr law non linearity using the exp(-φ (ξ))-expansion method . (GJSFR Volume 14-F Issue 4 Version 1.0).

[25] C. Q. Dai , J. F. Zhang, Jacobian elliptic function method for nonlinear differential difference equations, Chaos Solutions Fractals, 27 (2006) 1042-1049.

[26] E. Fan , J .Zhang, Applications of the Jacobi elliptic function method to special-type nonlinear equations, Phys. Lett. A 305 (2002) 383-392.

[27] S. Liu, Z. Fu, S. Liu, Q.Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69-74.

[28] Emad H. M. Zahran and Mostafa M.A. Khater, Exact Traveling Wave Solutions for the System of Shallow Water Wave Equations and Modified Liouville Equation Using Extended Jacobian Elliptic Function Expansion Method, American Journal of Computational Mathematics (AJCM) Vol.4 No.5 (2014).

[29] M. Alquran, A. Qawasmeh, Soliton solutions of shallow water wave equations by means of $\left(\frac{G'}{G}\right)$ −expansion method. Journal of Applied Analysis and Computation. Volume 4(3) (2014) 221-229.

[30] A. Qawasmeh, M. Alquran, Reliable study of some new fifth-order nonlinear equations by means of $\left(\frac{G'}{G}\right)$ −expansion method and rational sine-cosine method. Applied Mathematical Sciences. Volume 8(120) (2014) 5985-5994.

[31] A. Qawasmeh, M. Alquran, Soliton and periodic solutions for (2+1)-dimensional dispersive long water-wave system.Applied Mathematical Sciences. Volume 8(50) (2014) 2455-2463.

[32] Soliman, A. A., and H. A. Abdo. "New exact Solutions of nonlinear variants of the RLW, the PHI-four and Boussinesq equations based on modified extended direct algebraic method." *arXiv preprint arXiv:1207.5127* (2012).

[33]Deng, Xijun, Ming Zhao, and Xi Li. "Travelling wave solutions for a nonlinear variant of the PHI-four equation." *Mathematical and Computer Modelling* 49.3 (2009): 617-622.

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