



## Relativistic analysis of application of Helmholtz theorem to classical electrodynamics

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### ABSTRACT

In this work we discuss the relationship between the instantaneous-action-at-a-distance solutions of Maxwell's equations obtained using Helmholtz theorem and the Lorentz's invariant solutions of the same equations obtained using Special Relativity postulates. We show that Special Relativity postulates are not consistent with Helmholtz's theorem in the presence of charges and currents, but in the vacuum, without charges and currents, Helmholtz's theorem and Special Relativity agree because the instantaneous-action-at-a-distance solutions can be eliminated using a gauge transformation.

### Keywords

Helmholtz theorem; Lorentz gauge; space-time theories.

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## Introduction

A very common procedure in the treatment of wave propagation in an elastic solid dictates that the wave field  $\mathbf{u}$  can be decomposed into transversal and longitudinal components:

$$\mathbf{u} = \mathbf{u}_s + \mathbf{u}_i, \quad (1)$$

where  $\mathbf{u}_s = \nabla \times \mathbf{a}$ ,  $\mathbf{u}_i = \nabla b$  with  $\mathbf{a}$  as any continuous vector function and  $b$  as any continuous scalar function. The irrotational component  $\mathbf{u}_i$  generates primary waves (p-waves), while the solenoidal component  $\mathbf{u}_s$  is responsible for the secondary waves (s-waves). It is a well-known result that the propagation velocity of s-waves is less than the finite propagation velocity of p-waves in the solid.

In general, decomposition like (1) is known as a Helmholtz decomposition, a result obtained using Helmholtz theorem. When such decomposition is directly applied to Maxwell equations of electrodynamics a quite surprising result is obtained, at least when it is compared to the results obtained for an elastic solid: electric field solenoidal component ( $\nabla \cdot \mathbf{E}_s = 0$ ) satisfies a D'Alembert equation, with well-known retarded solutions, while the electric field irrotational component ( $\nabla \times \mathbf{E}_i = 0$ ) satisfies Poisson equation, so it is an *instantaneous* field.

An obvious objection against this result is that the irrotational component is not a physically independent field, it must be a mathematically introduced delusion, or it must somehow depend on the physically relevant component (see, e.g. [6] and [7]).

However, as we shall show below (Section 3), the instantaneous component is not only part of the solution of the field equations in the Coulomb gauge, the gauge where it naturally appears, but in the Lorentz one too using the Helmholtz theorem only. But this is not the puzzling matter, indeed, instantaneous action at a distance (IAAAD) can be used if it is retarded, just like in the Wheeler-Feynman theory. It is interesting to note that using the Helmholtz decomposition we get a solution of Maxwell's equations for the irrotational component of the electric field that is not Lorentz invariant but a Galileo one, so, the entire electric field is not Lorentz invariant or Galileo invariant but an invariant for the semi-direct product  $SO(3) \oplus_s T(4)$  of space rotations and space-time translations.

So, the objective of this paper is an analysis of the explicit symmetry breaking of the Lorentz group introduced by the Helmholtz decomposition with a very different perspective than that used in the previous work [5]. In that work a 3-dimensional formalism was used to tackle the question of energy transmission in *classical electrodynamics*, but this is not quite correct if we want to offer a depth analysis of the IAAAD solutions of the Maxwell's equations in flat, 4-dimensional spaces. As we shall see the symmetry properties of the Maxwell's equations that are essential to us does not depend on the signature of the space-time metric, but only on the flatness condition and the general structure of the space involved, because with both these conditions we determine the symmetry groups.

Our results are, roughly, as follows:

(1) Without the presence of charges and currents, in the vacuum, the Helmholtz decomposition is fully compatible with the Lorentz group, that is: there is no symmetry breaking, because with the help of a gauge transformation we achieve that  $\mathbf{E} = \mathbf{E}_s$ .

(2) In the presence of charges and convection currents in the vacuum, the Helmholtz theorem introduces an irrotational and isolated electric field that is not Lorentz invariant, hence there is symmetry breaking. And we cannot make this component zero without paradoxical consequences, as we shall see.

Hence a conclusion is in order: the symmetry group of the full electric field decomposed according to Helmholtz theorem is not the Lorentz or the Galileo group, but a group common to both as it is, for example,  $SO(3) \oplus_s T(4)$ .

But obviously this symmetry group tells us that the space-time is Euclidean, a quite incorrect conclusion.

The organization of the paper is as follows: in the next section we present a brief and rough exposition of the Minkowski, the Galilean and Newtonian space-time, as general frameworks for electrodynamics with a given symmetry group, in the next one we show how in the Lorentz gauge the instantaneous non-retarded solutions appear as a consequence of Helmholtz theorem, which in the same section is explained as a purely mathematical result. In the following sections we explain and demonstrate our main results. We close the paper with an assessment of the results achieved in the conclusions of the last section.

## The space-time theories

To settle the framework of the next section, we proceed to discuss three important space-time theories. The first one is the Special Theory of Relativity (STR) with its Minkowski space-time, the second one is the 4-dimensional Galilean space-time (GT) closely related to the third one, the Newtonian space-time (NT). Following Friedman [1], a Space-Time theory (STT) is defined once we give a 4-dimensional manifold  $M$ , the geometric objects  $\Phi_1, \dots, \Phi_M$  that satisfy the field equations of the theory, and the vector field  $X$  that define the geodesics and it is usually called "congruence". We follow this pattern here.



The first geometric object of any of our space-time theories is an affine connection  $D$  and the first field equation which we need is the condition that the Riemann curvature tensor<sup>1</sup>  $K$  is zero everywhere on the manifold  $M$ . This is a condition common for STR, NT and GT, they must be flat. The next geometric object which we need is a metric tensor, and here is where our STTs differ: for the Minkowski case we just need a 4-dimensional symmetric non-singular tensor  $\eta$  such that  $D_X\eta = 0$  for all tangent vector fields  $X$ , but for the NT and GT cases we need more geometric structures. For the NT case we need, in fact, two singular metric tensors because the time measures and the space measures are independent, so, there are more geometric objects than in the STR case<sup>2</sup>. Let us call these two tensors  $h$  and  $t$ . They<sup>3</sup> must satisfy  $D_Xh = 0, D_Xt = 0$  and these are other field equations for the NT and GT case. These conditions are enough for the GT but not for the NT. For the NT case we need the introduction of a time-like vector field  $V$  such that  $D_XV = 0$ , and this last condition on  $V$  define the NT. The integral curves defined by  $V$  are used in the Newtonian space-time to relate different 3-spaces.

Following Friedman, the field equations that give a structure to the space-time define not only the inertial frames as the frames where the metrics get a canonical form, but its specific symmetry groups as well.

We get these symmetry groups as isometries of the metric that leave invariant the flat connection.

Let us start a discussion of STR.

It was shown by Friedman [1] that sufficient conditions to obtain the Lorentz group on STR are the invariance of the covariant derivative and the invariance of the metric under the transformations. In this way linearity of the transformations is obtained from the covariant derivative invariance, and the coefficients of the linear transformations from the Killing's equations for the flat metric.

The usual postulate:

(A) The linear transformations that relate with two inertial frames  $\Sigma, \Sigma_0$  leave invariant the velocity of light.

This is not enough to obtain the proper Lorentz group without the use of extra assumptions. There is, however, a second postulate in STR, which is necessary in any space-time theory:

(B) The equations describing physical events are the same in every frame.

With (B) we obtain a group representation of the space-time group that acts on the fields and relates them on different inertial frames leaving invariant the form of the field equations.

To be more specific, we choose a particular connected component of the Lorentz group which we denote with  $SO^+(3,1)$  the proper (orientation preserving with  $\det = +1$ ) orthochronous (preservation of time direction) Lorentz group.  $SO^+(3,1)$  is normal sub-group of the Lorentz group, sometimes known as the restricted Lorentz group. We shall not discuss the discrete symmetries of space inversion and time reversal.

If we have at our disposal the restricted Lorentz group we get precision using the postulate (B), because now we know that the equations which describe physical events in any inertial frame must be invariant in front of the group  $SO^+(3,1)$ . Hence the invariance condition comes from the structure of the space-time and is previous, logically, to the field equations for the physical fields. Hence, historically, Maxwell equations were written for a Galilean or Newtonian space-time. The ether reference frame was such a Galilean frame with the property that the field equations were not invariant.

So, using the transformation properties of the differential operators involved in Maxwell equations we erect a representation of  $SO^+(3,1)$  which transforms the physical fields in such a way that the equations remain invariant. If we make the semi-direct product of  $SO^+(3,1)$  with space-time translations we get the Poincare group, but space-time translations leave invariant the co-vectors (1-forms), so we may skip them.

Now we go on to discuss the GT and NT.

There are well-known 4-dimensional formulations of the Newtonian space-time ([1], [2]), and a rough description of it. It consists of 3-spaces attached to the temporal line, so, each 3-space consists of all the simultaneous points, while the

<sup>1</sup> The curvature tensor  $R$  is defined for all pairs of vectors  $X, Y$  tangent to  $M$  as  $R(X, Y) = D_X D_Y - D_Y D_X - D_{[X, Y]}$  where  $[X, Y]$  is the Lie bracket. The Riemann curvature tensor is defined then as  $K(\omega, X, Y, Z) = \omega(R(X, Y), Z)$  where  $\omega$  is a 1-form, so,  $K$  is a (1,3)-tensor which we can build using relation  $D$ .

<sup>2</sup> In fact, we can see from the mathematical point of view, how these two metric tensor arise. Suppose that we have two bundles given by  $\langle M, \tau, T \rangle, \langle M, \varepsilon, N \rangle$  with  $\dim T = 1, \dim N = 3$ , with  $\tau: M \rightarrow T, \varepsilon: M \rightarrow N$ . So, the fiber  $\tau^{-1}(t)$  over  $t \in T$  is an instantaneous space isomorphic at a 3-space, while the fiber  $\varepsilon^{-1}(p)$  over  $p \in N$  is a relation of duration through time of point  $p$ . We build a singular metric tensor on each of these bundles, giving separate measures of time and space.

<sup>3</sup> Friedman uses  $h$  for a 2-contratensor, here we use the  $h$  for a 2-cotensor, that is, the metric tensor on each instantaneous 3-space.



time lines are the integral lines of time-like vector field  $V$  introduced before. These 3-spaces define a foliation of the 4-dimensional space, and each one is a space of simultaneity because all its points, and metric relations, are defined at one time instant. The congruence of  $V$  is used to define the notion of duration or time interval among the 3-spaces.

This space-time is radically different from STR. The group acting on NT leaves invariant the time component, defining the absolute time, and each 3-space, defining the absolute rest. Each of these elements of the NT is independent of each other. NT, like STR, is flat, so there coordinate systems exist where the singular tensors  $t$  and  $h$  have components

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{NM}$$

The transformation group in the Newtonian case is obtained using the same pattern described for the STR, giving as the result the group  $SO(3) \oplus_s T(4)$  ([1], chapt. 3). For the GT case we shall give the following brief description of its symmetry groups.

One of its symmetry groups is  $SO(3) \subset SO(4)$ .  $SO(3)$  is a normal subgroup of  $SO(4)$ , with  $\det = +1$ , in such a way that we have a fiber bundle  $\pi: SO(4) \rightarrow S^3 \simeq SO(4)/SO(3)$  with fiber isomorphic to  $SO(3)$ . This is enough to define absolute rest (NT case), but we need a relative space<sup>4</sup> plus an absolute time, so we introduce the Galileo transformations on our  $\mathfrak{R}^4$  as follows: we choose an element  $\langle x, t \rangle \in \mathfrak{R}^4$  and define an action  $g: SO(3) \times \mathfrak{R}^4 \times \mathfrak{R} \times \mathfrak{R}^4 \rightarrow \mathfrak{R}^4$  with  $g_{(0,v,s)}(x, t) = \langle Ox + vt, t + s \rangle$ . The Galileo group  $Gal(4)$  is a semi-direct product of  $SO(3)$ ,  $\mathfrak{R}^4$  and  $\mathfrak{R}$ . We may complete  $SO(3)$  to the Euclidian group introducing the space translations to get the full Galilean group. But when we use a frame of covectors the space-time translations are clearly trivial.

We shall consider in detail two space-times.

▮ The flat Minkowski space-time:  $M = \langle \mathfrak{R}^4, \eta, SO^+(3,1) \rangle$  with the Lorentz transformations that we take in the form

$$x_\mu = \sum_\nu \Lambda_{\mu\nu} \bar{x}_\nu. \tag{LT}$$

▮ The flat Galilean space-time:  $N = \langle \mathfrak{R}^4, h, t, Gal(4) \rangle$  with the Galileo transformations

$$x_\mu = \sum_\nu (v_g^\mu t_\nu \bar{x}_\nu + o_{\mu\nu} \bar{x}_\nu) + \xi_\mu, \tag{GT}$$

here  $v_g^\mu$  has components  $\langle 1, v_1, v_2, v_3 \rangle$  and  $t_\nu$  is  $\langle 1, 0, 0, 0 \rangle$  numerically the same on all inertial frames (absolute time such that  $h_{\mu\nu} = t_\mu t_\nu$ ). The matrix  $o_{\mu\nu}$  is such that  $o_{\mu\nu} = 0$ , for  $\mu, \nu = 0$  the matrix with components  $o_{ij}$ , with latin indexes running from 1 to 3, is a member of  $SO(3)$ . In this way the transformation (GT) includes the pure Galileo transformation, the rotation and the translation. Greek indices run from 0 to 3, Latin indices run from 1 to 3.

Each one of these space-times can be considered as the framework for electrodynamics if the equations are invariant with respect to its symmetry group. However, the use of Helmholtz theorem tells us that the right space is the NT space  $Ne = \langle \mathfrak{R}^4, h, t, V, SO(3) \oplus_s T(4) \rangle$  that is a completely incorrect result.

In the next section we discuss Helmholtz theorem and we show how the *instantaneous non-retarded solutions* arise from Coulomb and Lorentz gauges.

## Helmholtz decomposition

Any continuous and differentiable vector function  $E$  can be decomposed in an irrotational component  $E_i$  and a solenoidal one  $E_s$ . The idea is quite simple: at every point of space we construct a local 3-dimensional coordinate system, we choose one of the axes as a propagation axis, and this is the axis of the irrotational component, while the other two axes are the polarization axes, where the solenoidal component is projected. So, it is not difficult to suppose that

$$E = E_i + E_s. \tag{2}$$

Fourier transforms show us that  $\mathbf{k} \times E_i(\mathbf{k}, t) = 0$ ,  $\mathbf{k} \cdot E_s(\mathbf{k}, t) = 0$ , so if  $\mathbf{k}$  is the vector in the propagation direction, the irrotational component is parallel to it and the solenoidal one is orthogonal.

The decomposition (2) is known as Helmholtz theorem, and in any Banach space  $\mathfrak{S}$  we can always get a decomposition of the form  $\mathfrak{S} = N \oplus N^\perp$  with  $N^\perp$  the orthogonal space to  $N$ . But this decomposition is not the Helmholtz decomposition yet. We shall see this in detail at the end of the section.

If we apply (2) to the Maxwell equations we get for the solenoidal fields (see [5]):

<sup>4</sup> In this relative space there is a law of transformation of velocities, or, to be more specific, the usual Galilean addition of velocities is valid.



$$\nabla \times \mathbf{E}_s = -\frac{1}{c} \frac{\partial \mathbf{B}_s}{\partial t}, \tag{S1}$$

$$\nabla \times \mathbf{B}_s = \frac{1}{c} \frac{\partial \mathbf{E}_s}{\partial t} + \frac{4\pi}{c} \mathbf{J}_s, \tag{S2}$$

$$\nabla \cdot \mathbf{E}_s = 0, \tag{S3}$$

$$\nabla \cdot \mathbf{B}_s = 0. \tag{S4}$$

And for the irrotational field we obtain:

$$\nabla \times \mathbf{E}_i = 0, \tag{I1}$$

$$\nabla \cdot \mathbf{E}_i = 4\pi\rho, \tag{I2}$$

$$\frac{\partial \mathbf{E}_i}{\partial t} = -4\pi\mathbf{J}_i. \tag{I3}$$

We applied the Helmholtz theorem to the convection current  $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_s$ . The equations (S1-4) can be reduced to the following two equations:

$$\square \mathbf{E}_s = \frac{4\pi}{c^2} \frac{\partial \mathbf{J}_s}{\partial t}, \tag{s1}$$

$$\square \mathbf{B}_s = -\frac{4\pi}{c} \nabla \times \mathbf{J}_s. \tag{s2}$$

Meanwhile the equations (I1-3) are reduced to

$$\Delta \mathbf{E}_i = 4\pi\nabla\rho, \tag{i1}$$

$$\frac{\partial \mathbf{E}_i}{\partial t} = -4\pi\mathbf{J}_i, \tag{i2}$$

Where  $\Delta$  is the Laplace operator,  $\square$  is the D'Alembert one. Obviously, solutions of (s1-2) are retarded, while solutions of (i1-2) are instantaneous. This one and other simple considerations lead us to propose two mechanisms of energy transfer in electromagnetic theory ([5]-[7]).

Helmholtz decomposition appears quite naturally for the electric field in the Coulomb gauge. It is very easy to write down, in general, the following expression for the electric field:

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \tag{3}$$

If we use the Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$  for the vector potential  $\mathbf{A}$  we can look at (3) as a Helmholtz decomposition with  $\mathbf{A} = \mathbf{A}_s$ . For the Lorentz gauge an expression like (3) is valid, but now using Helmholtz theorem we obtain  $\mathbf{A} = \mathbf{A}_s - \nabla\aleph$ , where  $\mathbf{A}_i = -\nabla\aleph$ . In this way the equation (3) in the Lorentz gauge becomes:

$$\mathbf{E} = -\nabla\left(\varphi - \frac{1}{c} \frac{\partial \aleph}{\partial t}\right) - \frac{1}{c} \frac{\partial \mathbf{A}_s}{\partial t}. \tag{L1}$$

This is a Helmholtz-like decomposition. Let us show how instantaneous non-retarded solutions arise in this gauge. The field equations for (L1) are:

$$-\square \nabla \aleph = -\frac{4\pi}{c} \mathbf{J}_i, \tag{L2}$$

$$\square \mathbf{A}_s = -\frac{4\pi}{c} \mathbf{J}_s, \tag{L3}$$

$$\square \varphi = -4\pi\rho, \tag{L4}$$

$$-\Delta \aleph + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0. \tag{L5}$$

Equation (L5) is the Lorentz gauge. We must find a way to solve the coupled equations (L2), (L4), (L5) for the scalar functions  $\varphi$  and  $\aleph$ . We claim that when we achieve this goal, instantaneous non-retarded solutions arise.

Let us give a proof of this claim. We start with the first differential consequence of (L5):

$$-\Delta \frac{1}{c} \frac{\partial \aleph}{\partial t} = -\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2}, \tag{L6}$$

Which we can use in (L4) to get



$$\Delta\varphi = -4\pi\rho + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot(\nabla\mathfrak{N}). \tag{L7}$$

From equation (L7) we can write down

$$\Delta\left(\varphi - \frac{1}{c}\frac{\partial\mathfrak{N}}{\partial t}\right) = -4\pi\rho \tag{L8}$$

With the solution

$$\varphi - \frac{1}{c}\frac{\partial\mathfrak{N}}{\partial t} = -4\pi\int\frac{\rho}{r}d^3x, \tag{L9}$$

Showing that the electric field in the Lorentz gauge is the addition of instantaneous and retarded solution. Hence the conclusion is the following: *Helmholtz theorem predicts an instantaneous field in the Coulomb and the Lorentz gauges.*

However, the equations (S1-S4), (I1-I3) do not depend on the gauge: they can be deduced from using only Helmholtz decomposition, because we are using the electric and magnetic fields directly, not the potentials. So, there are grounds to believe that no mathematical trick will avoid the instantaneous field, but with the help of STR we can take it away, as we shall see in the next section. In other words: only in the flat, relativistic, Minkowski space-time there are no instantaneous solutions of Maxwell equations.

Now we shall discuss the Helmholtz theorem in the context of differential forms.

To get a decomposition of the space of 2-forms  $\Lambda^2(\mathfrak{R}^4)$  over  $\mathfrak{R}^4$  we use a projection operator  $P_V$  defined according to:  $P_V F = dx_0 \wedge (\iota_V F)$  where  $\iota_V$  is the contraction operator with the vector field  $V$ . So we easily get:<sup>5</sup>  $(1 - P_V)F = \iota_V(dx_0 \wedge F)$ . Hence it is clear that for the electromagnetic field tensor  $F$  we have the decomposition:

$$F = (1 - P_V)F + P_V F = \iota_V(dx_0 \wedge F) + dx_0 \wedge (\iota_V F). \tag{3a}$$

So:

$$\Lambda^2(\mathfrak{R}^4) = \Lambda_1 \oplus \Lambda_2$$

Decomposes our tensor space  $\Lambda^2(\mathfrak{R}^4)$  on a spatial  $\Lambda_1$  part and a temporal  $\Lambda_2$  one.

Let us choose for our vector field  $V$  a time-like vector field  $\iota_V dx_0 = 1$ , such that  $\iota_V dx_i = 0, i = 1,2,3$  and an inertial reference system such that we can get:

$$(1 - P_V)F = \frac{1}{2}\sum_{\mu\nu} F_{\mu\nu} dx_\mu \wedge dx_\nu - dx_0 \wedge \left(\sum_{i=1}^3 F_{0i} dx_i\right), \quad P_V F = dx_0 \wedge \left(\sum_{i=1}^3 F_{0i} dx_i\right). \tag{3b}$$

The form we choose for  $V$  is quite convenient to our objectives, and can be achieved using a rectification theorem. Besides, the field  $V$  now introduced can be identified with the congruence field of NT space-time previously discussed. This is not Helmholtz decomposition yet, but it is very instructive to achieve the Helmholtz decomposition in our 4-dimensional setting using identity (3a). To do that, we have to remember that we must find an adequate characterization of a curl and a divergence on a 4-dimensional manifold. This characterization in terms of 2-covectors involves its specific form, because a divergence and a curl must be obtained from the closure condition on the 2-covector; which is a natural condition on the electromagnetic field tensor  $F$ . We can see that the decomposition obtained using (3a) is such that each of its pieces behaves differently when the exterior derivative is used:

$$d(\iota_V(dx_0 \wedge F)) \text{ is a divergence-like,}$$

$$d(dx_0 \wedge (\iota_V F)) \text{ is a curl-like.}$$

Therefore our Helmholtz theorem in 4-dimensional space for any 2-covector  $F$  is given by the following two local conditions:

$$d(\iota_V(dx_0 \wedge F)) = 0, \tag{1H}$$

$$d(\iota_V F) = 0. \tag{2H}$$

Let us state the previous remarks as a:

### Helmholtz theorem

Necessary and sufficient conditions for the decomposition (3a) of a 2-cotensor  $F$  in a 4-dimensional manifold for the equivalence with Helmholtz decomposition are ( $L_V$  is the *Lie derivative* with respect to the vector field  $V$ ):

1. The congruence field  $V$  is a time-like rectifiable vector field;

<sup>5</sup> The decomposition comes from the following identity:  $\iota_V(dx_0 \wedge F) = \iota_V(dx_0)F - dx_0 \wedge \iota_V F$ . If we use a time-like vector field  $V$  such that  $\iota_V(dx_0) = 1$ , we obtain the decomposition.



2.  $d(t_V(dx_0 \wedge F)) = 0$ ;
3.  $d(t_V F) = 0 = L_V F = 0$ .

## Proof

Let us suppose the conditions are fulfilled, then every congruence can be written as  $V = \frac{\partial}{\partial x_0}$  using the right coordinates, and the decomposition of  $F$  is (3a). If  $t_V F$  is a closed 1-form, then  $(1 - P_V)F$  contains the electric field with its non-integrated part. The condition  $d(t_V F) = 0$  is, in 3-dimensional Gibbs' notation, equivalent to  $\nabla \times \mathbf{E}_i = 0$ .

So, the condition  $d(t_V(dx_0 \wedge F)) = 0$  is equivalent to a zero divergence, as it is required.

On the other hand, let us suppose that (3a) is a Helmholtz decomposition, that is, when it is written in 3-dimensional Euclidean space, we obtain a typical Helmholtz decomposition. Hence, we should have:

$$\begin{aligned} t_V dx_0 &= 1, \\ t_V dx_i &= 0, \quad i = 1, 2, 3, \\ d(t_V(dx_0 \wedge F)) &= 0, \\ d(dx_0 \wedge (t_V F)) &= 0. \end{aligned}$$

The first two conditions are obtained when the congruence vector field is time-like rectifiable vector field, and the other two are the ones stated in the theorem; so, the conditions are necessary and sufficient. **QED**

In this way we can reproduce the Helmholtz theorem of 3-dimensional space in the very simple expression, which is almost a tautology:

$$F = F^s + F^i$$

with  $F^s = t_V(dx_0 \wedge F)$  and  $F^i = dx_0 \wedge (t_V F)$ .

There is an interesting literature on the subject of the Helmholtz theorem and its applicability to time-dependent or time-independent vector fields showing that, ultimately, it is widely applicable [8-14]. Our **Helmholtz theorem** shows that a Helmholtz-like decomposition can be achieved for closed 2-cotensors in a 4-dimensional space when a quite trivial identity and some additional conditions are used. Obviously in the cited literature the Helmholtz theorem is obtained using a known identity of 3-dimensional vector analysis plus some manipulations of weak solutions (Green functions) of the Poisson equation (a nice explanation is contained in ref. [14]. Following this way the Helmholtz decomposition is obtained. However, conditions on the infinity are necessary to rule out the boundary terms.

In our case the decomposition of the 2-cotensor we make is obtained using an identity for any  $p$ -tensor, identifying the pieces that must have a zero divergence and a zero curl because in our 4-dimensional setting they are not obvious from the notation, as in the 3-dimensional Gibbs notation, but we need not boundary conditions; or, at least, we believe that we have not used them implicitly or explicitly. So, our Helmholtz theorem is not a big achievement, but perhaps a useful reformulation for our objective.

## The STR and Helmholtz theorem

In this section we shall enunciate the results in the form of theorems, to facilitate the discussion. When we say that STR and the Helmholtz theorem are compatible or incompatible we mean that some premises of one or another are contradictory. In this section we shall write down Maxwell equations with the help of differential forms, that is, we shall use an explicit invariant formulation.

So, the equations (i1-i3) for the irrotational component  $\mathbf{E}_i$  are written with the help of the field tensor:

$$F^i = dx_0 \wedge \left( \sum_{\nu=0}^4 F_{0\nu}^i dx_\nu \right) = dx_0 \wedge \left( \sum_{k=0}^3 F_{0k}^i dx_k \right) = dx_0 \wedge \omega \quad (4)$$

with  $\omega = \sum_k F_{0k}^i dx_k + f dx_0$  in the form:

$$dF^i = 0, \quad (5a)$$

$$d\omega = -4\pi J^i, \quad (5b)$$

$$d * F^i = * \theta, \quad (5c)$$

$$J^i = dx_0 \wedge \left( \sum_{k=1}^3 F_k^i dx_k \right), \quad (5d)$$

$$\theta = 4\pi \rho dx_0, \quad (5e)$$



The Hodge operator  $*$  is taken with respect to the Newtonian metric, so the inner product of forms is:  $(\sigma|\theta) = \sigma_{i_1, \dots, i_p} \theta_{j_1, \dots, j_p} g_{i_1 j_1} \dots g_{i_p j_p}$  where  $\sigma, \theta$  are  $p$ -co-tensors with components  $\sigma_{i_1, \dots, i_p}, \theta_{j_1, \dots, j_p}$  and  $g_{ij}$  the components of the metric. We use coordinates  $(x_0, x_1, x_2, x_3)$  for our flat space times according to the canonical metrics.

The solenoidal component  $F^s$  is:

$$F^s = \frac{1}{2} \sum_{\mu\nu} F_{\mu\nu}^s dx_\mu \wedge dx_\nu, \tag{6}$$

while the equations (s1), (s2) are:

$$dF^s = 0, \tag{7a}$$

$$d * F^s = * \lambda, \tag{7b}$$

$$\lambda = \sum_{k=1}^3 J_k^s dx_k. \tag{7c}$$

Now we must introduce the group transformations. We shall do this by means of its action upon differential generators  $dx_0, \dots, dx_3$ . So we have the Lorentz transformations:

$$\phi_L^* dx_\mu = \sum_\nu \Lambda_{\mu\nu} d\bar{x}_\nu \tag{8}$$

and the Galileo transformations:

$$\phi_G^* dx_\mu = \sum_\nu v_{\mu g} t_\nu d\bar{x}_\nu + \sum_\nu o_{\mu\nu} d\bar{x}_\nu. \tag{9}$$

Our calculations will show how the Helmholtz decomposition behaves from one inertial frame to another. Before proceed we must comment a little on our use of the concepts of "invariance" and "covariance". As we shall see from the lemmas, sometimes the tensor  $F$  is invariant under a transformation  $\varphi$ , which means that  $\varphi^* F = F$ , so the form of  $F$  is preserved, but its components are not, because under the transformation these components are "covariant", that is, they follow a tensor transformation law as must be, so its form is no preserved but linearly changed. We know that transformations are natural operations in relation to the operator  $d$  in the DeRham complex, hence we have that for any transformation  $\varphi$  we can write down  $\varphi^* d = d\varphi^*$ , so, to check invariance of the Maxwell equations we just need to check the invariance of  $F$ . When we follow this path we obtain the tensor transformation law of the components of  $F$ . As we shall see, sometimes this transformation law can be absorbed in the form of  $F$ , but sometimes not. When we obtain the last situation we say that the Maxwell equations are not invariant under the transformation. Clearly this procedure is the non-infinitesimal form of the infinitesimal condition of invariance given by:  $L_X F = 0$  where  $L$  is the Lie derivative,  $X$  is the infinitesimal generator of the transformation and  $F$  a tensor.

**Lemma 1.** The irrotational component of the electromagnetic field tensor is Galileo invariant.

**Proof**

The simplest way to show this is to write down (9) in temporal and spatial pieces:

$$\phi_G^* dx_0 = d\bar{x}_0, \tag{10}$$

$$\phi_G^* dx_i = v_{ig} d\bar{x}_0 + \sum_j o_{ij} d\bar{x}_j, \tag{11}$$

So, when we applied this to the irrotational part  $F^i$  of the electromagnetic field tensor we get:

$$\begin{aligned} \phi_G^* F^i &= \phi_G^* dx_0 \wedge \left( \sum_k F_{0k}^i \circ \phi_g \phi_g^* dx_k \right) = \\ &= d\bar{x}_0 \wedge \left( \sum_k \phi_g^* F_{0k}^i \left[ v_{kg} d\bar{x}_0 + \sum_j o_{kj} d\bar{x}_j \right] \right) = d\bar{x}_0 \wedge \left( \sum_j \left( \sum_k \phi_g^* F_{0k}^i o_{kj} \right) d\bar{x}_j \right). \end{aligned} \tag{12}$$

If we define  $\bar{F}_{0j}^i = \sum_k \phi_g^* F_{0k}^i o_{kj}$  we finally obtain:

$$\phi_G^* F^i = dx_0 \wedge \left( \sum_j \bar{F}_{0j}^i d\bar{x}_j \right). \tag{13}$$

**QED .**





This is an expected result, because the irrotational component of the electromagnetic field tensor is like a Newtonian gravitational theory. But it is very interesting, probably quite surprising, that Maxwell equations play no role on the proof. The proof relies on the form of the tensor field only.

**Lemma 2.** The irrotational component of the electromagnetic field tensor is not Lorentz invariant.

**Proof**

we write down:

$$\phi_L^* dx_\mu = \sum_\nu \Lambda_{\mu\nu} d\bar{x}_\nu \tag{14}$$

for the Lorentz transformation. Then

$$\begin{aligned} \phi_L^* F^i &= \left( \sum_\nu \Lambda_{0\nu} d\bar{x}_\nu \right) \wedge \left( \sum_j \phi_L^* F_{0j}^i \left( \sum_\nu \Lambda_{j\nu} d\bar{x}_\nu \right) \right) \\ &= \left( \sum_\nu \Lambda_{0\nu} d\bar{x}_\nu \right) \wedge \left( \sum_{\nu,j} \phi_L^* F_{0j}^i \Lambda_{j\nu} d\bar{x}_\nu \right). \end{aligned} \tag{15}$$

We define  $\bar{F}_{0\nu}^i = \sum_j \phi_L^* F_{0j}^i \Lambda_{j\nu}$  to get

$$\phi_L^* F^i = \left( \sum_\nu \Lambda_{0\nu} d\bar{x}_\nu \right) \wedge \left( \sum_\nu \bar{F}_{0\nu}^i d\bar{x}_\nu \right). \tag{16}$$

Finally:

$$\phi_L^* F^i = \sum_{k,j} (\Lambda_{0k} \bar{F}_{0j}^i - \Lambda_{0j} \bar{F}_{0k}^i) d\bar{x}_k \wedge d\bar{x}_j + \Lambda_{00} d\bar{x}_0 \wedge \left( \sum_k \bar{F}_{0k}^i d\bar{x}_k \right). \tag{17}$$

**QED .**

Another expected result. But, again, we can see that the non-invariance comes from a tensor field's form.

**Lemma 3.** The transversal component of the electromagnetic field tensor is not Galileo invariant.

**Proof**

We take as transversal component:

$$F^s = \frac{1}{2} \sum_{\mu,\nu} F_{\mu\nu}^s dx_\mu \wedge dx_\nu, \tag{18}$$

hence under a Galileo transformation we have

$$\begin{aligned} \phi_G^* F^s &= \sum_{\mu,\nu} \phi_G^* F_{\mu\nu}^s \left[ d\bar{x}_0 \wedge \left( \sum_\lambda (o_{\nu\lambda} v_{\mu g} - v_{\nu g} o_{\mu\lambda}) d\bar{x}_\lambda \right) + \right. \\ &\quad \left. \sum_{\lambda,\eta} (o_{\mu\lambda} o_{\nu\eta} - o_{\mu\eta} o_{\nu\lambda}) d\bar{x}_\lambda \wedge d\bar{x}_\eta \right] = \\ &= \frac{1}{2} d\bar{x}_0 \wedge \sum_\lambda \bar{F}_\lambda^s d\bar{x}_\lambda + \frac{1}{2} \sum_{\lambda,\eta} \bar{F}_{\lambda\eta}^s d\bar{x}_\lambda \wedge d\bar{x}_\eta \end{aligned}$$

with  $\bar{F}_\lambda^s = \sum_{\mu,\nu} (o_{\nu\lambda} v_{\mu g} - v_{\nu g} o_{\mu\lambda}) \phi_G^* F_{\mu\nu}^s$ ,  $\bar{F}_{\lambda\eta}^s = \sum_{\mu,\nu} (o_{\mu\lambda} o_{\nu\eta} - o_{\mu\eta} o_{\nu\lambda}) \phi_G^* F_{\mu\nu}^s$ . **QED.**

**Lemma 4.** The transversal component of the electromagnetic field tensor is Lorentz invariant.

**Proof:**

$$\phi_L^* F^s = \frac{1}{2} \sum_{\mu,\nu} \phi_L^* F_{\mu\nu}^s \left( \sum_{\lambda,\eta} (\Lambda_{\mu\lambda} \Lambda_{\nu\eta} - \Lambda_{\mu\eta} \Lambda_{\nu\lambda}) d\bar{x}_\lambda \wedge d\bar{x}_\eta \right) = \frac{1}{2} \sum_{\lambda,\eta} \bar{F}_{\lambda\eta}^s d\bar{x}_\lambda \wedge d\bar{x}_\eta,$$

where  $\bar{F}_{\lambda\eta}^s = \sum_{\mu,\nu} (\Lambda_{\mu\lambda} \Lambda_{\nu\eta} - \Lambda_{\mu\eta} \Lambda_{\nu\lambda}) \phi_L^* F_{\mu\nu}^s$ . **QED.**



These quite trivial lemmas allow us to say that the electromagnetic field tensor  $F$ , when it is decomposed using Helmholtz theorem, is *not Lorentz invariant*, so this theorem cannot be used within the framework of the STR. The trouble with the Helmholtz theorem and the STR comes from the fact that the Helmholtz theorem allows us to isolate an electric field  $E_i$  without a magnetic field counterpart in a given inertial frame. This electric field is compatible with a Galilean space-time  $N$  or a Newtonian one,  $Ne$ , because of its Galileo and Euclidean covariance but not with a Minkowski space-time  $M$  without absolute time. It is instantaneous because there is not any limit on velocity on  $N$ , while this is not possible in  $M$ .

This should have been clear from the outset, just remembering the following cite from J.L. Synge ([4], p. 321):

*"... a purely electric field (or a purely magnetic field) is not a relativistic concept, since this pure character is not preserved under a Lorentz transformation"* .

Obviously this is quite in line with the general idea of STR: the space and the time are not two different physical concepts, there is just one space-time. So, there are not electric and magnetic fields, there is just one electromagnetic field. Therefore we get the following:

**Theorem 1.** Helmholtz theorem is not compatible with a relativistic space-time electrodynamics.

**Proof:**

Helmholtz theorem predicts an isolated instantaneous irrotational electric field, which satisfy non Lorentz invariant equations. **QED.**

There is, however, a certain instance where Helmholtz theorem is quite in line with a relativistic space-time electrodynamics (STR): the case when it is possible to eliminate the irrotational component. This can be done with the help of a gauge transformation when there is not charged matter present.

**Theorem 2.** Helmholtz theorem and STR are compatible, if and only if, there is no charged matter in the vacuum.

**Proof:**

Let us suppose we have no charged matter. Hence, the field equations for the irrotational component are:

$$dF^i = 0, \quad d * F^i = 0, \quad d\omega = 0, \quad (19)$$

while for the solenoidal one are:

$$dF^s = 0, \quad d * F^s = 0. \quad (20)$$

If we remember that the Helmholtz theorem allows us to write down  $F = F^i + F^s$  while we can relate two solutions to the Maxwell field equations using  $F = F_+ + dA$  we obtain

$$F_+ = F^i - dA + F^s. \quad (21)$$

So, if we choose  $F^i = dA = dx_0 \wedge \omega$  we obtain

$$F_+ = F^s \quad (22)$$

And the equations for the gauge are:

$$d * dA = d * (dx_0 \wedge \omega) = 0. \quad (23)$$

With no difficulty we can show that (23) is a Laplace equation for the components of  $\omega$ , that is, for the electric field components (a harmonic function, as this must be according to the Hodge theorem).

In this way we get the necessary transformation properties, obtaining with a gauge transformation the compatibility of Helmholtz theorem and STR when there are not charges present.

Now suppose that Helmholtz theorem and STR are quite compatible, so, the field tensor is Lorentz invariant. But this can be the case only if  $F^i$  is not present. But if this is the case we have two possibilities:

1. We used a gauge transformation whose gauge is determined by equations (5).
2.  $F^i$  is always identically zero.

If we used a gauge transformation determined by equation (5), the field equations for  $F^s$  are not the Maxwell equations but equations (7). Hence, Gauss law for the electric field is not correct except in the absence of charged matter.

If  $F^i$  is identically zero the equations describing the electromagnetic field are equations (7), which are the Maxwell equations only when there is no charged matter. **QED.**

In the proof of these theorems there is a dominant role for Maxwell equations, unlike in the previous lemmas.

Obviously we can use another gauge transformation to eliminate the solenoidal component, but in that case we obtain a non-Lorentz invariant theory in a relative Galilean frame or in an absolute Newtonian one, that is: the electromagnetic analogue of gravitational theory with relative space and absolute time or with absolute space and



absolute time. This means that the structure of the space-time is not determined by the mathematical structure of the electromagnetic field equations for the vacuum, and must be decided by an experiment .

The theorems give us an insight into the nature of the Helmholtz theorem: when we use it we introduce a description of the electromagnetic field where the density of charges and the density of currents produce, if we accept that charges and currents are the sources of the fields, *quite different fields*. On the other hand, if the fields are the producers of charges and currents, hence the Helmholtz theorem tell us that the production of a charge comes from an irrotational electric field alone, while the production of a current comes from electric and magnetic solenoidal fields jointly with an irrotational time-dependent field.

If we eliminate one of the components of  $F$  in the Helmholtz decomposition we cannot fully describe the electromagnetic field *except when a charged matter is absent*. A static electric field is irrotational and a static magnetic field produces a solenoidal current, so, from the start, a description of the electromagnetic field with irrotational and solenoidal field is motivated.

When there is a *time variation* of the electric field, we know that some magnetic field appears, but at the same time, a solenoidal component of the electric field is added to the irrotational one. It is not *clear why the irrotational component must be rejected, or why the electric field cannot be decomposed according to the Helmholtz theorem*. The STR gives us the following answer to this question: we reject the description of the electromagnetic field in terms of irrotational and solenoidal components because *it is not a Lorentz invariant description*. Hence, we can say that *the presence of charges is the origin of this incompatibility*. But we know that charges change the space-time connectivity, because at their positions they represent singularities of the field equations solutions. So we can establish our results in a more suggestive language: "*Helmholtz theorem is compatible with STR on any simple connected space-time*". Obviously this result is grounded in the Maxwell equations, unlike the symmetry properties of the irrotational and solenoidal components of the electromagnetic field tensor.

In a previous publication [5] we introduced the irrotational component because this component can be considered the cause of the transference of the momentum and the energy when the solenoidal field is zero along some axis (the x-axis in that case). As we have seen, that example is not really within the realm of STR, because the irrotational electric field does not satisfy a Lorentz invariant equation, but it is within the realm of classical electrodynamics written on each of the 3-dimensional folia of  $Ne$ . And this is possible because, for that specific configuration of charges and currents, the solenoidal component contributes nothing.

## Conclusions

We have been discussing the following propositions:

- (p) The Maxwell equations ( $dF = 0, d * F = *J$ );
- (q) The Helmholtz theorem ( $F = F^i + F^s$ );
- (r) The special theory of relativity ( $M$  the Minkowski space-time).

We have considered that the special theory of relativity has as an outcome a specific space-time structure given by the Minkowski space-time. So there is a quite interesting question to ask: *is the structure of a space-time determined by the electromagnetic field equations?* The answer is well-known: no, only a theory like the General Theory of Relativity can achieve that goal.

But, when we combine the Maxwell equations with the Helmholtz theorem a surprising prediction is obtained: the symmetry group of the space-time must be the Euclidean group, because this group is at the intersection of the Galileo and Lorentz groups. So instantaneous action-at-a-distance solutions are possible and the whole critique of simultaneity is left aside. Therefore somewhere a mistake is present, because the special theory of relativity is a well-confirmed theory. Clearly, if we are correct, (p) and (r) are well-confirmed theories, while (q) is just a mathematical result, therefore, we can use all of them. The conclusion obtained using (p), (r) and (q) is not relevant from the mathematical point of view, because as a mathematical result is not really important if the space-time is Euclidean or Minkowskian. That is: Helmholtz theorem and Maxwell equations determine a form for the space-time. However from the physical side this result is quite shocking.

We believe that the reason lies in the fact that Helmholtz theorem is a mathematical theorem with no physical grounds, so, there is no a priori reason to avoid its use, except that if we use it we contradict a well-confirmed theory. Hence, if we propose to avoid the use of the Helmholtz theorem, this is not because it is incorrect on mathematical grounds, but on physical grounds when combined with Maxwell equations.

So, the only mistake lies in the fact that when we use a 3-dimensional notation we are not considering the structure of space-time, a structure which cannot be determined from Maxwell equations but defined independently. The special theory of relativity is such independent determination, but even this theory cannot avoid the *IAAAD solutions* of the Maxwell equations, but only gives a mathematical framework: the Minkowski space-time  $M$  where IAAAD solutions, which are the manifestation of the 3-dimensional folia of the  $Ne$  space, are impossible.

Then the recommendation *is no to avoid the use of the Helmholtz theorem*, but to establish *from the beginning* the space-time structure and its symmetry groups, or, more clearly, *to leave aside the widely used formulation of electrodynamics that suppose an indeterminate space-time structure*. This recommendation is in line with the teachings of Einstein in his 1905 paper on the electrodynamics of moving bodies, where a program for the building of physical theory is



outlined: first we must to know the form of the space and the time, second, we must find the laws that in that framework are invariant.

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