On the Zitterbewegung Transient Regime in a Coarse-Grained Space-Time

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ABSTRACT

In the present contribution, by studying a fractional version of Dirac's equation for the electron, we show that the phenomenon of Zitterbewegung in a coarse-grained medium exhibits a transient oscillatory behavior, rather than a purely oscillatory regime, as it occurs in the integer case, $\alpha=1$. Our result suggests that, in such systems, the Zitterbewegung-type term related to a trembling motion of a quasiparticle is tamed by its complex interactions with other particles and the medium. This can justify the difficulties in the observation of this interesting phenomenon. The possibility that the Zitterbewegung be accompanied by a damping factor supports the viewpoint of particle substructures in Quantum Mechanics.

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1 Introduction

Dirac's equation unifies both Quantum Mechanics and Special Relativity by providing a relativistic description of the electron's spin; it predicts the existence of antimatter and is able to reproduce accurately the spectrum of the hydrogen atom. It also embodies the 'Zitterbewegung' (ZB) effect as an unexpected quivering motion of a free relativistic quantum particle, like the electron, for instance. This name was coined by Schrödinger, who first observed that, in describing relativistic electrons by the Dirac's equation, the components of the relativistic quadri-velocity do not commute with the free-electron Hamiltonian, with the consequence that the electron's velocity is not a constant of the motion even in the absence of external fields. Such an effect must be of a quantum nature, as it does not obey Newton's laws of classical dynamics. Schrödinger calculated the resulting time dependence of the electron's velocity and position, concluding that, in addition to its classical motion, the electron experiences very fast periodic oscillations [1].

One of the motivations for analyzing ZB-models is to describe the spin of the electron, S, and its magnetic moment, μ, as generated by a local circulation of mass and charge. Experiments indicate the possibility for an internal structure for the electron, considering it as an extended object (wavelength 10^{-3} m $< \delta < 10^{-16}$ m, where δ is the supposed dimension for the electron [2]). Another motivation is the possible existence of a non-vanishing electric dipole moment for the electron due to the separation between the center of mass, which is related to the Foldy-Wouthuysen position operator, and the center of charge, which corresponds to Dirac's position operator, x [3].

The trembling electron's motion should occur also in crystalline solids, in classical systems, in macroscopic sonic crystals, in materials like bi- and mono-layer graphene and nanotubes; it can also be predicted to occur in the presence of an external applied magnetic field. In spite of the great interest in the phenomenon of ZB, its physical origin has remained mysterious. It was recognized that the ZB in vacuum is due to an interference of states corresponding to positive and negative electron energies [1].

Experimental difficulties to observe the ZB in vacuum are considerable because the predicted frequency of the trembling is very high: $\hbar\omega_z = 2m_0 c^2 \simeq 1 MeV$, and its amplitude is very small: $\lambda_c = \hbar/m_0 c \simeq 3.86 \times 10^{-3}$Å [4], and they are not accessible to current experimental techniques.

Fractional Calculus (FC) is one of the generalizations of classical calculus. It provides a redefinition of mathematical tools and it seems very useful to deal with anomalous [5, 6, 7, 8, 9, 10] and frictional systems. Several applications of FC may be found in the literature [11, 12, 13]. Presently, areas such as field theory and gravitational models demand new conceptions and approaches which might allow us to understand new systems and could help in extending well-known results.

In this work, we investigate the fractional coarse-grained aspects of the electron's ZB in a coarse-grained medium. Here, we take the viewpoint that the fractionality associated to the complex interactions may be responsible for a damping of the ZB oscillations that, in turn, justify the difficulties in their experimental observations. This claim highlights that our motivation to adopt the FC is more physical than a simple mathematical extension. This becomes more explicit once we argue that the possible justification for the difficulties in the experimental measurements are originated from the complexity of the interactions of the electrons, considered as a pseudo-particle "dressed" by the interactions and the medium. Here, we look at the dynamical system as an open system that can interact with the environment and we argue that FC can be an important tool to study open classical and quantum systems. [14].

Our efforts to justify the use of FC in fractional systems with dissipative systems and the relationship with complex systems, coarse-grained medium, limit scale energy for the interactions and non-integer dimensions may be found in the papers of [15, 16, 17, 18, 19].

For problems related to the quantization of field theories, the reader could consult [20, 21, 22, 23]. Also, in connection with our work, a fractional Riemann–Liouville Zeeman effect and an attempt to implement gauge invariance in fractional field theories and an angular momentum algebra proposed with the Riemann-Liouville formalism are reported in the paper of [24]. Low-energy nuclear excitations have been studied in connection with a fractional symmetric rigid rotor in order to calculate barionic excitations [25].

Here, we claim that the use of an approach to FC, based on a sequential form of the modified Riemann-Liouville (MRL) fractional calculus [26], is more appropriate to describe the dynamics associated with field theory and particle physics in the space of nondifferentiable solution functions, or in the scenario of a coarse-grained space-time.

Some backgrounds on coarse-grained media, references on fractal space-time and the efforts to build up a solid foundation for the construction of a geometry and field theory in fractional spaces may be found in [15, 16, 17, 18, 19] and references therein.

Here, to achieve our goal, we pursue an investigation of the fractional Dirac's equation for the electron, looking at dynamical systems as open interacting structures. We show that the ZB phenomenon in a coarse-grained medium exhibits a transient oscillatory behavior, rather than a purely oscillatory regime, as it occurs in the integer case, α=1 . Our result suggests that, in such systems, the ZB-type term related to a trembling motion of the quasiparticle is tamed by the complex interactions of the quasiparticles with other particles and the medium. This may be the argument to justify the difficulties in the observation of this rather interesting phenomenon. The possibility that the ZB be accompanied by a damping supports the viewpoint of a particle substructure in Quantum Mechanics.

Our paper is outlined as follows: In Section 2, we consider some relevant remarks on the fractional derivative. In

section 3, we present the fractional approach to Zitterbewegung and finally, in Section 4, our Discussion and Conclusions are cast.

2 Some Remarks on the Fractional Derivative

In the sequel, we adopt an alternative approach by considering a fractional coarse-grained space-time rather than a fractional space of functions, meaning that neither the space nor the time are infinitely thin, but they rather exhibit "thickness". As the use of certain calculation rules is essential to our approach, we briefly comment on this point, before presenting these rules.

The Riemann-Liouville and Caputo approaches to FC are well-known and have their rules rigorously proved, as the reader may find in the standard textbooks [27, 28, 29, 30, 31]. These well-tested definitions for fractional derivatives, referred to as Riemann-Liouville's and Caputo's, have been frequently used for several applications. In spite of their efficacy, they have some dangerous pitfalls. For this reason, an interesting definition for fractional derivative [32, 33], the so-called modified Riemann-Liouville (MRL) fractional derivative, has been proposed which is less restrictive than other definitions. Its basic expression is as follows:

$$D^\alpha f(x) = \lim_{x \to 0} \hbar^{-\alpha} \sum_{k=0}^\infty \binom{\alpha}{k} f(x + (\alpha - k)h) =$$
$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} (f(t) - f(0)) dt; \quad (0 < \alpha < 1). \tag{1}$$

But, by strictly referring to the context of the modified Riemann-Liouville (MRL) formalism, to our mind it seems worthy to notice that the chain rule, as well as the Leibniz product rule, had their mathematical validity proven only recently, in view of the formal proof for the Taylor expansion [34]. To be more precise, in the MRL approach, the fractional Taylor expansion is the mathematical basis for the validation of the chain rule and also for the Leibniz product rule. We also emphasize that these rules can then be viewed as good approximations. Then, we point out that the Leibniz rule used here is a good approximation that comes from the first two terms of the fractional Taylor series development, which holds only for nondifferentiable functions [32, 33] and are as good and approximated as the classical integer one. The mentioned rules are quite similar to their counterparts in the integer-order calculus.

Following the MRL definition, we find that derivative of a constant is zero, and second, we can use it for differentiable and non-differentiable functions as well. For further details on MRL formalism, the reader can follow [17, 32, 33, 35], which contain all the basic for the formulation of a fractional differential geometry in coarse-grained space, and refers to an extensive use of coarse-grained phenomenon.

Here, another important comment is worthy and concerns certain definitions referred to as local fractional derivatives, like the ones introduced by [36, 37, 38], with several works related to this approach; we also quote the papers of refs. [39, 40, 41] and the approaches with the Hausdorff derivative and with the so-called fractal derivative [42, 43]. All of the mentioned treatments seem to be applicable to power-law phenomena. For some alternative definitions of fractional derivatives, see [44] and the interesting work [45]. Some relevant comments and remarks on the similarities between different approaches or even on fractional q-calculus may be found in [12, 17, 43, 46, 47, 48].

In a recent article [49], one of the authors show that the Leibniz rule for fractional derivatives in a coarse-grained medium treated as a Hölder-space yields a modified chain rule. The latter can be safely applied, in agreement with alternative versions of fractional calculus, in these classes of spaces or even in the local versions of FC.

We could revisit the ZB phenomena by applying the Balankin's [50, 51] approach based on mapping to a fractional continuum [52] and by adopting a local version of fractional called Hausdorff derivative. Doing so, an expansion of the fractional Newton binomial, that appears as a pre-factor in the derivative of [50, 51], can be shown to lead to a q-derivative [53] as a lower-order term [54]. The resulting equations would be non-linear due to a local factor, with integer-order derivatives acting on a continuum Euclidean space of fractional metric as a result of the mapping [50]. Alternatively, by applying the approaches found in [36, 37, 38, 45, 55], one could attain results close to the ones obtained here. Using another definition of the local fractional derivative [56], which seems to be actually based on the MRL approach, but appear in the form of a limit operation and can be seen by considering the first two terms of a truncated fractional Taylor's series, would manifest a global pre-factor, identical to that of the MRL formalism. This global factor, could be considered as a statistical average, that can be seen as a Mellin transform of exponential factors weighted by some statistical probability factor . We can speculate that this could be interpreted in such a way to attribute to each spatial point, , a probability to influence the system in a range The use of the global factor would yield similar equations to those here described.

The connections above indicate that the MRL approach is consistent with a local approach with a global pre-factor, when the fractional Leibniz rule or the chain rule are used.

Now that we have set up these fundamental considerations, we are ready to carry out the calculations of main interest in our paper.

3 Fractional Approach to the Zitterbewegung

In recent work [17], we have built up a fractional Dirac's equation in a coarse-grained scenario by taking into account a fractional Weyl's equation, a fractional angular momentum algebra, by introducing a mass parameter and imposing that the equations be compatible with the fractional energy-momentum relation.

By extending the concept of helicity to include it in a fractional scenario, we write down the left- and the right-handed Weyl's equations from first principles in this extended framework [17]. Next, by coupling the different fractional Weyl sectors by means of a mass parameter, we arrive at the fractional version of Dirac's equation as

$$(i\hbar\gamma^\mu\partial_\mu^\alpha - \mathbf{1}m^\alpha c^\alpha)\psi_\alpha. \tag{2}$$

Where

$$\partial_\mu^\alpha = \left(\frac{M_{t,\alpha}}{c^\alpha}\frac{\partial^\alpha}{\partial t^\alpha}; M_{x,\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\right). \tag{3}$$

With

$$\hat{p}^\alpha = -i\hbar^\alpha M_{x,\alpha}\frac{\partial^\alpha}{\partial x^\alpha}, \tag{4}$$

we have the fractional commutation relation $\left[\hat{x}_i^\alpha, \hat{p}_j^\alpha\right] = \Gamma(\alpha+1)\hbar^\alpha M_\alpha \delta_{i,j}\mathbf{1}$.

The fractional Dirac Hamiltonian is then [17]

$$H^\alpha = c^\alpha\vec{\alpha}.\vec{p}^\alpha + m^\alpha c^{2\alpha}\beta. \tag{5}$$

Applying the operator eq.(5) to a state-vector ψ_α and using that $i\hbar^\alpha{}_0^J D_t^\alpha\psi_\alpha = H^\alpha\psi_\alpha$, yields

$$i\hbar^\alpha{}_0^J D_t^\alpha\psi_\alpha = (c^\alpha\vec{\alpha}.\vec{p}^\alpha + m^\alpha c^{2\alpha}\beta)\psi_\alpha = H^\alpha\psi_\alpha. \tag{6}$$

We now write that the state-vector $\psi_\alpha(\vec{r},t)$ as below:

$$\psi_\alpha(\vec{r},t) = U_\alpha(t,t_0)\psi_\alpha(\vec{r},t_0), \tag{7}$$

where $U_\alpha(t,t_0)$ is the fractional temporal evolution operator to be determined.

This determination can be set as follows:

$$i\hbar^\alpha{}_0^J D_t^\alpha U_\alpha(t,t_0)\psi_\alpha(\vec{r},t_0) = H^\alpha U_\alpha(t,t_0)\psi_\alpha(\vec{r},t_0). \tag{8}$$

So, the evolution equation may be written as

$$i\hbar^\alpha{}_0^J D_t^\alpha U_\alpha(t,t_0) = H^\alpha U_\alpha(t,t_0). \tag{9}$$

Noticing that $_0^J D_t^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha)$ we can see that the solution to the eq. (9) is the fractional evolution operator in a coarse-grained scenario, that is given by $U_\alpha(t,t_0) = E_\alpha(-\frac{i}{\hbar^\alpha}H^\alpha t^\alpha)$. Here, we have supposed that $D_t^\alpha H^\alpha = 0$, and $E_\alpha(x)$ is the one-parameter Mittag-Leffler function.

Going now into the Heisenberg representation, an operator A_α^H can be written as

$$A_\alpha^H = U_\alpha^+(t,0)A_\alpha^S U_\alpha(t,0), \tag{10}$$

where A_α^S is the operator A_α in the Schrödinger representation.

In order to set up a fractional version of Heisenberg evolution equation, we can use the fractional Leibniz rule in the MRL approach. The result is

$$_0^J D_t^\alpha A_\alpha^H(t) = -\frac{i}{\hbar^\alpha}[A_\alpha^H, H^\alpha]. \tag{11}$$

We can now calculate the fractional evolution of the position as

$$_0^J D_t^\alpha x_\alpha^H(t) = -\frac{i}{\hbar^\alpha}[x_\alpha^H, H^\alpha]. \tag{12}$$

The commutation relations yield:

$$[H^\alpha, x_\alpha^H] = [c^\alpha\vec{\alpha}.\vec{p}^\alpha, x_\alpha^H] = -c^\alpha\vec{\alpha}i\Gamma(\alpha+1)\hbar^\alpha, \tag{13}$$

so that the fractional evolution equation turns out to the be the fractional Breit equation:

$$_0^J D_t^\alpha x_\alpha^H(t) = \Gamma(\alpha+1)c^\alpha\vec{\alpha}. \tag{14}$$

As a next step, we proceed to calculate $_0^J D_t^\alpha\vec{\alpha}$.

Using that the Poisson bracket,

$$\{H^\alpha, \vec{\alpha}\} = 2c^\alpha\vec{p}^\alpha, \tag{15}$$

and the fractional Heisenberg equation, we get that

$$_0^J D_t^\alpha\vec{\alpha} = \frac{i}{\hbar^\alpha}(-2c^\alpha\vec{p}^\alpha + 2H^\alpha\vec{\alpha}) \tag{16}$$

or

$$i\hbar^\alpha {}_0^J D_t^\alpha \vec{\alpha} = 2H^\alpha \vec{\eta}_\alpha, \tag{17}$$

where $\hat{\eta}_\alpha$ is given by

$$\vec{\eta}_\alpha \equiv \vec{\alpha} - c^\alpha (H^\alpha)^{-1} \vec{p}^\alpha. \tag{18}$$

Since ${}_0^J D_t^\alpha (H^\alpha)^{-1} \vec{p}^\alpha = 0$ we can write that

$$_0^J D_t^\alpha \vec{\alpha} = {}_0^J D_t^\alpha \vec{\eta}_\alpha. \tag{19}$$

Now, assuming that the conservation relations hold

$$_0^J D_t^\alpha H^\alpha = {}_0^J D_t^\alpha \vec{p}^\alpha = 0, \tag{20}$$

and by verifying that $\{H^\alpha, \hat{\eta}_\alpha\} = 0$, then, we can write the fractional differential equation for $\hat{\eta}_\alpha$ as

$$_0^J D_t^\alpha \vec{\eta}_\alpha = -\frac{i}{\hbar^\alpha} 2H^\alpha \vec{\eta}_\alpha. \tag{21}$$

The solution to the eq.(21) above is

$$\vec{\eta}_\alpha = \vec{\eta}_\alpha(0) E_\alpha \left(-\frac{i}{\hbar^\alpha} 2H^\alpha t^\alpha\right); \tag{22}$$

which can be rewritten as

$$\vec{\alpha} = c^\alpha (H^\alpha)^{-1} \vec{p}^\alpha + \vec{\eta}_\alpha(0) E_\alpha \left(-\frac{i}{\hbar^\alpha} 2H^\alpha t^\alpha\right). \tag{23}$$

With the help of eq. (14), we have

$$_0^J D_t^\alpha x_\alpha^H(t) = \Gamma(\alpha+1) c^{2\alpha} (H^\alpha)^{-1} \vec{p}^\alpha + \Gamma(\alpha+1) c^{2\alpha} \hbar^\alpha \vec{\eta}_\alpha(0) E_\alpha \left(-\frac{i}{\hbar^\alpha} 2H^\alpha t^\alpha\right). \tag{24}$$

The fractional integration of the equation above results as below:

$$x_\alpha^H(t) = \Gamma(\alpha+1) c^{2\alpha} (H^\alpha)^{-1} \vec{p}^\alpha t^\alpha + \frac{i}{2} \Gamma(\alpha+1) c^{2\alpha} (H^\alpha)^{-1} \hbar^\alpha \vec{\eta}_\alpha(0) E_\alpha \left(-\frac{i}{\hbar^\alpha} 2H^\alpha t^\alpha\right) +$$
$$- \frac{i}{2} \Gamma(\alpha+1) c^{2\alpha} (H^\alpha)^{-1} \hbar^\alpha \vec{\eta}_\alpha(0). \tag{25}$$

It can readily be checked that, for α=1, the result is the same as in the integer case [57].

Now, for the Mittag-Leffler function the relation below holds:

$$E_\alpha(-ix) = E_{2\alpha}(-x^2) - ix E_{2\alpha,1+\alpha}(-x^2), \tag{26}$$

where $E_{\gamma,\delta}(x)$ is the two-parameter Mittag-Leffler function.

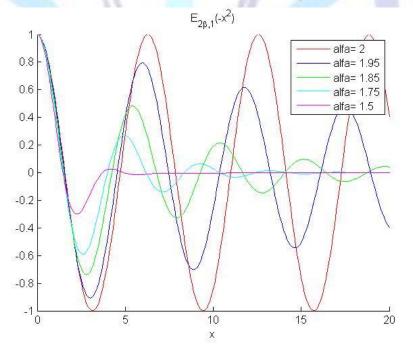So, the real part of eq.(25) can be represented as depicted in the plot of Fig.1.

Figure 1: ($x_0 E_{2\beta,1}(-\omega^2 t^2)$. $In\ figure, \alpha = 2\beta.$)

4 Discussion and Conclusions

With the result cast in the previous Section, the evolution of the particle's position can be determined, at every instant of time, by a numerical simulation. It exhibits a transient oscillatory behavior rather than a pure oscillatory regime, as in the integer case ($\alpha=1$). This can justify the difficulties in the experimental observation of the ZB phenomenon.

It is worthy to mention that quantum simulations of one-dimensional Dirac's equations for free relativistic electrons carried out by Gerritsma et al.[58] show that the ZB of an electron prepared in the form of a Gaussian wave packet decays in time. This reproduces the same behavior observed in the crystalline solids in Ref. [4], where the authors claim that the ZB represents the basic way of electron propagation in a periodic potential and observe that the ZB of an electron prepared in the form of a Gaussian wave packet decays in time.

We can argue that the damping in the oscillations can be understood by considering electrons as pseudo-particles, "dressed" by the interactions and the medium and by looking at the dynamical system as an open system that can interact with the environment. The complexity of the interactions affect the value of the α-fractional parameter.

Other important point to remark here is the possibility of establishing connection between different formalisms, as we have pointed out in Section 2. Indeed, by applying Balankin's approach with a version of local fractional derivative, namely, the Hausdorff derivative, and by considering an expansion of the fractional Newton binomial, we are led to a q-derivative as a lower-order term. This may indicate that the use of q-calculus in the context of non-additive entropy is really justifiable, at least to the first order, in the realm of complexity, fractals and multifractals, where power-law phenomena take place. Efforts to better investigate this particular point are in progress and we shall be reporting on that soon [54]. Also, some higher-order terms in the fractional binomial expansion could be included in order to yield better consistent theories. The perspective for the construction of non-linear models, with integer derivatives, in the context of complex systems is also a viable possibility, by the mapping with the fractional continuum and the local fractional Hausdorff derivative. Actually, non-linear fractional theories may arise from the direct use of some versions of local fractional derivatives.

On the other hand, in the framework of Particle Physics, all massive leptons, including their corresponding neutrinos [59] should manifest the ZB effect. The confirmation of this claim in intimately related to the experimental limitations and also to the quantum-mechanical uncertain principle.

In fact, chirality corresponds to a property of fundamental importance in the study of neutrino physics [60, 61]. The effects of chiral oscillations can be explained as a consequence of the ZB phenomenon which emerges whenever solutions to Dirac's equation are used to describe the space-time evolution of a wave packet of massive particles like neutrinos [3]. In the work of Ref. [62], there are suggestions that a *spin-1/2* particle produced in a localized condition is subject to chiral oscillations remnant from Zitterbewegung.

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