On the mathematical necessity of the co-existence of electromagnetic fields of the non-standard polarity with the standard electromagnetic fields in the case of the existence of Dirac monopoles

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Abstract

In this brief note we adduce the mathematical rationale of the necessity of the existence of the electromagnetic field of the non-standard polarity if the magnetic charges (Dirac monopoles) exist in reality.

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The present paper was inspired by our careful perusal of newly published brilliant work [1] about observation of Dirac monopoles which were identified, in both experiments and matching numerical simulations. Nevertheless the authors of [1] note that Maxwell´s equations refer neither to magnetic monopoles nor to the magnetic currents that arise from their motion. Indeed, the *presence* of magnetic charges implies the existence of a *vectorial* magnetic field (polar vectors) rather than a *pseudo-vectorial* one. But it is well known that from the system of complete Maxwell equations (CME) (i.e., equations with non-zero density of an electric charge ϱ and non-zero density of a conduction current $\mathbf{j}$) it follows that any solution $\mathbf{E}$ and $\mathbf{B}$ of this system

$$\left.\begin{aligned} \nabla\cdot\mathbf{E} &= 4\pi\varrho \\ \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{B} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j} \end{aligned}\right\} \qquad (1)$$

must be $\mathbf{E}$ as a *polar* vector and $\mathbf{B}$ as an *axial* vector in the microscopic consideration. One can satisfy oneself that this fact directly follows from the equation $\nabla\cdot\mathbf{E} = 4\pi\varrho$ and from other equations of CME. Actually, seeing that ϱ is a scalar rather than pseudo-scalar with respect to the spatial inversion transformation (K-system $\rightarrow K'$-system) for coordinates and the operator ∇ changes its sign, the vector $\mathbf{E}$ is a *polar* vector because it behaves as $\mathbf{E} \rightarrow \mathbf{E}' = -\mathbf{E}$, and thereagainst the vector $\mathbf{B}$ is an *axial* vector because it behaves as $\mathbf{B} \rightarrow \mathbf{B}' = \mathbf{B}$. Let us call this kind of the polarity of the electromagnetic fields as a *standard polarity*.

If one follows this claim, then it is obvious that so called *free* Maxwell equations (FME) (i.e., equations with *zero* density of an electric charge ϱ and *zero* density of a conduction current $\mathbf{j}$)

$$\left.\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned}\right\} \tag{2}$$

must, correspondingly, have only solutions of such polarity.

However, if we consider FME from a purely *formal mathematical point of view*, then this condition ($\mathbf{E}$ is a *polar* vector and $\mathbf{B}$ is an *axial* vector) is not obligatory. But have we a right to do such consideration? The point is that one of the authors of the present work argued [2] that a rigorous application of Gauss' law to constructing CME leads to realizing the fact that FME (2) and, correspondingly, so-called *free* electric and magnetic fields are not *a consequence* of CME: one just may *postulate* them. Jefimenko ([3] p. 17), nay, claims that sourceless (free) fields are impossible.

Thus in the present work we show that the system of equations (2) admits nonstandard solutions regarding the polarity of the vectors $\mathbf{E}$ and $\mathbf{B}$. In other words we show that such *nonstandard* solutions of FME exist, when the magnetic field $\mathbf{B}$ can be a *polar* vector as well as an axial vector, and the electric field $\mathbf{E}$ in turn can be an axial vector as well as f polar vector. Of course, we omit the consideration of the trivial case when one can make the following interchange in FME (2): $\mathbf{E} \rightarrow \mathbf{B}'$, $\mathbf{B} \rightarrow \mathbf{E}'$.

So let us approach a solution of the problem of the estimation $\mathbf{E}$ and $\mathbf{B}$ in the system (2) purely mathematically:

One can represent some solutions of Eq. (2) as follows:

$$\mathbf{E} = [C_1\mathbf{p}(\mathbf{r}) + C_2\mathbf{a}(\mathbf{r})]\sin(\Omega t) \quad \text{and} \quad \mathbf{B} = [C_2\mathbf{p}(\mathbf{r}) + C_1\mathbf{a}(\mathbf{r})]\cos(\Omega t), \tag{3}$$

where C_1, C_2 and Ω are constant and where the vectors $\mathbf{p}$ and $\mathbf{a}$ are the solution of the system

$$\operatorname{rot}\mathbf{p} = \frac{\Omega}{c}\,\mathbf{a}; \qquad \operatorname{rot}\mathbf{a} = \frac{\Omega}{c}\,\mathbf{p}, \tag{4}$$

such that $\mathbf{p}$ is a polar vector and $\mathbf{a}$ is an axial vector (c is the velocity of light in vacuum). This solution expressed by components (Cartesian and spherical ones) is:

$$\mathbf{p} = D\left\{\frac{-\alpha\Omega y}{cr^3}, \quad \frac{\alpha\Omega x}{cr^3}, \quad 0\right\} = \frac{\Omega\alpha\sin\theta}{cr^2}D\mathbf{e}_\varphi \tag{5}$$

and

$$\mathbf{a} = D\left\{\frac{\beta xz}{r^5}, \qquad \frac{\beta yz}{r^5}, \qquad \frac{2\alpha}{r^3} - \frac{\beta(x^2+y^2)}{r^5}\right\}$$

$$= \frac{2\alpha\cos\theta}{r^3} D\mathbf{e}_r + \frac{(\beta - 2\alpha)\sin\theta}{r^3} D\mathbf{e}_\theta \,, \qquad (6)$$

where D is a dimension constant $[D] = M^{1/2}L^{5/2}T^{-1}$;

$$\beta = 3\alpha - \frac{\Omega^2}{c^2} r^2 \sin\left(\frac{\Omega}{c} r\right),$$

$$\alpha = -\frac{\Omega}{c} r \cos\left(\frac{\Omega}{c} r\right) + \sin\left(\frac{\Omega}{c} r\right), \;\; r = (x^2 + y^2 + z^2)^{1/2} \,,$$

and $\mathbf{e}_\varphi$, $\mathbf{e}_r$, $\mathbf{e}_\theta$ are the *orts* of the spherical coordinate system. From the whole class of solutions of the system (4) we have chosen the simplest non-trivial one. The detailed way of obtaining this solution one can find in [4] or [5].

Let us consider the solution (3) for some different constants C_1 and C_2.

Let, for example, $C_1 = 1$ and $C_2 = 0$. In this case from Eqs. (3) we have the fields with the well-known *standard polarity* ($\mathbf{E}_s$ and $\mathbf{B}_s$):

$$\mathbf{E}_S = \mathbf{p}(\mathbf{r})\sin(\Omega t) \qquad \text{and} \qquad \mathbf{B}_S = \mathbf{a}(\mathbf{r})\cos(\Omega t). \qquad (7)$$

Now let us consider the case when in Eqs. (3) $C_1 = 0$ and $C_2 = 1$, in this case we have the electromagnetic fields (let us call this kind of fields the electromagnetic fields of a *non-standard polarity*) $\mathbf{E}_N$ and $\mathbf{B}_N$:

$$\mathbf{E}_N = \mathbf{a}(\mathbf{r})\sin(\Omega t) \qquad \text{and} \qquad \mathbf{B}_N = \mathbf{p}(\mathbf{r})\cos(\Omega t). \qquad (8)$$

We see now that this solution is already unusual in the sense of the *polarity*: $\mathbf{E}_N$ is an *axial* vector and $\mathbf{B}_N$ is a *polar* one! It is easy to satisfy oneself that the electromagnetic wave formed by this field spreads in the opposite direction with respect to the direction of the wave spreading by *S*-fields (7). Actually

$$\mathbf{E}_N \cdot \mathbf{B}_N = 0 \quad \text{but} \quad \mathbf{S}_N = \frac{c}{4\pi}(\mathbf{E}_N \times \mathbf{B}_N) = -\frac{c}{8\pi}(\mathbf{p} \times \mathbf{a})\sin(2\Omega t) \qquad (8a)$$

and the energy density

$$w_N = \frac{E_N^2 + B_N^2}{8\pi} = \frac{1}{8\pi}[|\mathbf{a}|^2\sin^2(\Omega t) + |\mathbf{p}|^2\cos^2(\Omega t)] \,. \qquad (8b)$$

Here, however, a certain question can arise: how are these "mathematical" fields rightful *physically*? Can one apply, for example, the definitions of the energy density and the energy-flux vector to fields of the non-standard polarity? It is easy to show that one can apply the mentioned definitions to these non-standard fields if the discussed fields are solutions of FME. Indeed, let us multiply both sides of $\nabla \times \mathbf{E} = -\left({}^{1}/_{c}\right) {}^{\partial \mathbf{B}}/_{\partial t}$ by $\mathbf{B}_N$ and both sides of $\nabla \times \mathbf{B} = \left({}^{1}/_{c}\right) {}^{\partial \mathbf{E}}/_{\partial t}$ by $\mathbf{E}_N$ (taking into account that $\mathbf{B}_N$ and $\mathbf{E}_N$ are solutions of these equations) and combine the resultant equations. Then we have

$$\frac{1}{c}\mathbf{E}_N \cdot \frac{\partial \mathbf{E}_N}{\partial t} + \frac{1}{c}\mathbf{B}_N \cdot \frac{\partial \mathbf{B}_N}{\partial t} = -(\mathbf{B}_N \cdot \nabla \times \mathbf{E}_N - \mathbf{E}_N \cdot \nabla \times \mathbf{B}_N). \tag{9}$$

Using the well-known formula of the vector analysis, we rewrite this relation in the form

$$\frac{1}{2c}\frac{\partial}{\partial t}(E_N^2 + B_N^2) = -\nabla \cdot (\mathbf{E}_N \times \mathbf{B}_N)$$

or after multiplying this relation by $\frac{c}{4\pi}$

$$\frac{\partial}{\partial t}\left(\frac{E_N^2 + B_N^2}{8\pi}\right) = -\mathrm{div}\,\mathbf{S}_N. \tag{10}$$

Now we show that the vector

$$\mathbf{S}_N = \frac{c}{4\pi}(\mathbf{E}_N \times \mathbf{B}_N) \tag{11}$$

and the relation $(E_N^2 + B_N^2)/(8\pi)$ are the energy-flux vector and the energy density of the field (8). First we integrate (10) over a volume and apply Gauss' theorem to the term on the right. Then we obtain[1]

$$\frac{d}{dt}\int \frac{E_N^2 + B_N^2}{8\pi}\,dV = -\oint \mathbf{S}_N \cdot d\mathbf{f}. \tag{12}$$

If the integral extends over all space, then the surface integral vanishes because the field is zero at infinity.[2] Then (12) becomes

$$\frac{d}{dt}\left\{\int \frac{E_N^2 + B_N^2}{8\pi}\,dV\right\} = 0. \tag{13}$$

Thus, for the closed system consisting of the electromagnetic field of a nonstandard polarity, the quantity with dimensions of energy in brackets in this equation is conserved. Therefore we can call the quantity $(E_N^2 + B_N^2)/8\pi$ the energy density of the electromagnetic field (8).

Now if we integrate over any *finite* volume, then the surface integral in (12) generally does not vanish, so that we can write the equation in the form

$$\frac{d}{dt}\left\{\int \frac{E_N^2 + B_N^2}{8\pi}\,dV\right\} = -\oint \mathbf{S}_N \cdot d\mathbf{f}. \tag{14}$$

On the left the change in the total energy of field per unit time stands. Therefore the integral $\oint \mathbf{S}_N \cdot d\mathbf{f}$ must be interpreted as the flux of field energy across the surface

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[1] Recall that in this case ${}^{d}/_{dt} = {}^{\partial}/_{\partial t}$.

[2] one can see this from Eqs. (5), (6)

bounding the given volume, so that the vector $\mathbf{S}_N$ is a density of this flux (the Poynting vector) that is the amount of the field energy passing through unit area of the surface in unit time.

Thus we can see that expressions for the energy density and the Poynting vector were obtained without subjecting the field to any *polarity* conditions. We are not going to consider in detail the *N*-solution of FME, instead, we emphasize that the waves formed by this solution spread in the opposite direction with respect to the waves formed by the *S*-solution (7). Absolute magnitudes of the Poynting vectors for *S* and *N* coincide, while the distribution of the energy density is different.

Now, referring to the findings of [2] namely that *free* electric field is not a consequence of *complete* Maxwell equations (CME), and remembering the phrase by Morris Kline from [6], namely "*What is especially remarkable about electromagnetic waves – and reminiscent of gravitation – is that we have not the slightest physical knowledge of what electromagnetic waves are. Only mathematics vouches for their existence, and only mathematics enabled engineers to invent the marvels of radio and television... Mathematicians and theoretical physicists* speak *of fields – the gravitational field, the electromagnetic field ..., and others – as though they were material waves which spread out into space and exert their effects somewhat as water waves pound against ships and shores. But these fields are fictions. We know nothing of their physical nature. They are only distantly related to observables such as sensations of light, sound, motions of objects, and the now perhaps too familiar radio and television ... But by formulating mathematically the laws of these fictional fields, which have no apparent counterparts in reality, and by deducing consequences of these laws, we obtain conclusions, which when suitably interpreted in physical terms can be checked against sense perceptions.* ", we can conclude the following.

The generally accepted so called *free* Maxwell equations (2) can be constructed by a *fusion* of two free Maxwell equations and accordingly can be separated into two distinct systems of the dissimilar polarity (the *standard* and the *non-standard* one):

$$\left.\begin{aligned} \nabla\cdot\mathbf{E}_s &= 0 \\ \nabla\times\mathbf{E}_s &= -\frac{1}{c}\frac{\partial\mathbf{B}_s}{\partial t} \\ \nabla\cdot\mathbf{B}_s &= 0 \\ \nabla\times\mathbf{B}_s &= \frac{1}{c}\frac{\partial\mathbf{E}_s}{\partial t} \end{aligned}\right\} \qquad \text{(FMES)}$$

and

$$\left.\begin{aligned} \nabla\cdot\mathbf{E}_N &= 0 \\ \nabla\times\mathbf{E}_N &= -\frac{1}{c}\frac{\partial\mathbf{B}_N}{\partial t} \\ \nabla\cdot\mathbf{B}_N &= 0 \\ \nabla\times\mathbf{B}_N &= \frac{1}{c}\frac{\partial\mathbf{E}_N}{\partial t} \end{aligned}\right\}. \qquad \text{(FMEN)}$$

Now taking into account the real (or possible) existence of magnetic charges [1] and the system (FMEN) we can write the *complete* Maxwell equations for electromagnetic fields of the *non-standard* polarity (CMEN):

$$\left.\begin{aligned} \nabla\cdot\mathbf{E}_N &= 0 \\ \nabla\times\mathbf{E}_N &= -\frac{1}{c}\frac{\partial\mathbf{B}_N}{\partial t}+\frac{4\pi}{c}\mathbf{j}_m \\ \nabla\cdot\mathbf{B}_N &= 4\pi\mu \\ \nabla\times\mathbf{B}_N &= \frac{1}{c}\frac{\partial\mathbf{E}_N}{\partial t} \end{aligned}\right\}, \qquad \text{(CMEN)}$$

where $\mathbf{j}_m$ is the magnetic current density, μ is the magnetic charges density and $\frac{1}{4\pi}\frac{\partial\mathbf{B}_N}{\partial t}$ may be some magnetic *displacement* current density.

We would like to mention here the system of equations describing the electromagnetic fields granting the possible existence of magnetic charges which was proposed by Griffiths in his well-known textbook [7][3]:

$$\left.\begin{aligned} &\text{(i)} & \nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\varrho_e \\ &\text{(ii)} & \nabla\cdot\mathbf{B} &= \mu_0\varrho_m \\ &\text{(iii)} & \nabla\times\mathbf{E} &= -\mu_0\mathbf{j}_m-\frac{\partial\mathbf{B}}{\partial t} \\ &\text{(iv)} & \nabla\times\mathbf{B} &= \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\mu_0\mathbf{j}_e \end{aligned}\right\}, \qquad \text{(Grif)}$$

where ϱ_m represents the magnetic charges density, and ϱ_e the electric charges density; $\mathbf{j}_m$ would be the magnetic current density, and $\mathbf{j}_e$ the electric current density. One can see that if we believe that ϱ_m has to be a true scalar quantity rather than a pseudo-scalar one and taking into account that ϱ_e is the true scalar quantity by definition one can conclude that in this case Eqs. (Grif) *in a certain sense* are inconsistent equations from the viewpoint of the vector analysis. Let us explain this statement: the system (Grif) must have solutions without the definite polarity when $\mathbf{E}=\mathbf{E}_a+\mathbf{E}_p$ and $\mathbf{B}=\mathbf{B}_a+\mathbf{B}_p$, where $\mathbf{E}_a$ and $\mathbf{B}_a$ are axial vectors and $\mathbf{E}_p$ and $\mathbf{B}_p$ are polar ones. But this system cannot have the simultaneous solutions of this type $\mathbf{E}=\mathbf{E}_p$ and $\mathbf{B}=\mathbf{B}_p$ for example.

So we have to conclude that in the case of the existence of magnetic charges which are the true scalars the *total* electromagnetic field must be described by *three* independent systems of equations describing electric and magnetic fields of *different nature and origin*. One of them corresponds to the *free* field of the *standard polarity* as well as to the *free* field of the *non-standard polarity*:

[3] Eq. (7.43) in [7]

$$\left.\begin{aligned}\nabla\cdot\mathbf{E}&=0\\ \nabla\times\mathbf{E}&=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\cdot\mathbf{B}&=0\\ \nabla\times\mathbf{B}&=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\end{aligned}\right\}. \qquad \text{(FME)}$$

The other two correspond to the fields produced by *magnetic* currents and charges and *electric* currents and charges, respectively:

$$\left.\begin{aligned}\nabla\cdot\mathbf{E}_N&=0\\ \nabla\times\mathbf{E}_N&=-\frac{1}{c}\frac{\partial\mathbf{B}_N}{\partial t}+\frac{4\pi}{c}\mathbf{j}_m\\ \nabla\cdot\mathbf{B}_N&=4\pi\mu\\ \nabla\times\mathbf{B}_N&=\frac{1}{c}\frac{\partial\mathbf{E}_N}{\partial t}\end{aligned}\right\}, \qquad \text{(CMEN)}$$

and

$$\left.\begin{aligned}\nabla\cdot\mathbf{E}_S&=4\pi\varrho\\ \nabla\times\mathbf{E}_S&=-\frac{1}{c}\frac{\partial\mathbf{B}_S}{\partial t}\\ \nabla\cdot\mathbf{B}_S&=0\\ \nabla\times\mathbf{B}_S&=\frac{1}{c}\frac{\partial\mathbf{E}_S}{\partial t}+\frac{4\pi}{c}\mathbf{j}\end{aligned}\right\}. \qquad \text{(CMES)}$$

Note that the set of the systems of equations (FME), (CMEN) and (CMES) admits more solutions than the system of equations (Grif). Furthermore, one can add that the simultaneous solution of these systems[4] begs for the electromagnetic field of the non-standard polarity to co-exist with the standard electromagnetic fields in the case of the existence of Dirac monopoles.

In conclusion, we would like to quote the words of Louis de Broglie: "*Only the motions of elements localized in space and time are really physical*" [8] he said.

References

[1] Ray M V, Ruokokoski E, Kandel S, Möttönen M and Hall D S 2014 Observation of Dirac monopoles in a synthetic magnetic field *Nature* **505** 657-665

[4] (FME), (CMEN) and (CMES)

[2] Chubykalo A, Munera H and Smirnov-Rueda R 1998 Is free electromagnetic field a consequence of Maxwell's equations or a postulate? *Found. Phys. Lett.* **11**(6) 573-584

[3] Jefimenko O D 1992 *Causality, Electromagnetic Induction and Gravitation* (USA Star City: Electret Scientific Company)

[4] Chubykalo A and Espinoza A 2002 Unusual formations of the electromagnetic field in vacuum *J. Phys. A: Math. Gen.* **35** 8043-8053

[5] Chubykalo A and Espinoza A 2002 Steady-state sphere-like and ring-like formations of the free electromagnetic field in vacuum *Il Nuovo C*imiento **117 B** (12) 1303-1315

[6] Kline M 1980 *Mathematics: The Loss of Certainty* (New York: Oxford Yuniversity Press)

[7] Griffiths D J 1999 *Introduction to Electrodynamics* (3rd ed. USA New Jersey: Prentice Hall)

[8] de Broglie L 1976 13 remarques sur divers sujets de physique théorique *Ann. Fond. L. de Broglie* **1**(3) 116-128