ISSN 2347-3487

Generalization of Faraday's Law of Induction: Some Examples

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ABSTRACT

The induced electromotive force and Faraday's law of induction, due to a time-dependant magnetic field, are more conveniently written on the covering space. In this paper, we consider the induced electromotive force in the n^{th} loop on a covering space which is generated by the $(n+1)^{th}$ time derivative the external magnetic field enclosed that loop. The total induced electromotive force is derived by summing over all the contributions coming from the infinite winding numbers on the covering space. Illustrative examples of different time-dependent magnetic field are examined and analytical closed form expressions for the total induced electromotive force are derived. Our results, for all these examples, show the explicit dependence of the electromotive force on the ratio between the self-inductance and the resistance of the loop and they reduce to the well-known result when the limit of this ratio gpoes to zero.

Indexing terms/Keywords

Faraday's law of induction; Maxwell's equations; Covering space.

Academic Discipline And Sub-Disciplines

Theoretical physics, Electrodynamics.

SUBJECT CLASSIFICATION

41.20. Gz

TYPE (METHOD/APPROACH)

Line integration, Infinite sums, Mathematical functions.

INTRODUCTION

Faraday's of induction has been an important topic in electrodynamics and discussed in introductory physics textbooks [1-3]. Over the last decade, it has been discussed for better understanding for undergraduate students [4-8] and examined in different experimental measurements: Electrical conductivity in metallic tubes [9], quantitative measurement that involves damping [10], measurements for high frequency pulses [11], in measurements systems' design [12-14]. Some workers investigated the connection of Faraday's law of induction with energy conservation [15] and magnetic loss in dielectric material [16]. Other workers discussed the formulation of Faraday's law of induction via the magnetic vector potential [17, 18] and its manifestation in the Aharonov-Bohm ring [19]. Recently, one of us [20] had shown that higher order time – derivatives of the magnetic field contribute to the induced electromotive force generated in a loop that encloses a time-dependent magnetic field. The contribution of the n^{th} loop (so called winding number) is related to the $(n+1)^{th}$ time derivative of the magnetic field and therefore the total induced electromotive force is the infinite sum over all these contributions. In the present paper, we present illustrative examples for different time-varying magnetic fields and derive analytic close form expressions for the total induced electromotive force. In all of these examples, it is demonstrated that the induced electromotive force depends on the ratio between the self-inductance and the electrical resistance of the loop. Furthermore, our result reduces to the well-known electromotive force when that ratio goes to zero.

MATHEMATICAL FORMULATION

We consider an external time-varying magnetic field B_0that passes through a loop of resistance *R*. Due to Faraday's law of induction, each time the loop is traversed the magnetic field is updated. Using successive method and following the derivation given in ref. [20], the total induced electromotive force is the sum over all contributions coming from all the infinite loops (the so called winding numbers). It must be emphasized that this method utilizes the use of a covering space that was proposed long ago by Schulman [21] and was used by several authors [22-24]. The general formula for the total induced electromotive force is found to be

$$\varepsilon = -\sum_{n=0}\left(\frac{1}{R}\int \vec{c}\,.\overrightarrow{dA}\right)^{n}\int \vec{B}_0^{(n+1)}.\overrightarrow{dA} \quad , \qquad (1)$$

which, in terms of the self-inductance, is written as

$$\varepsilon = -\sum_{n=0}\left(\frac{L}{R}\right)^{n}\int \vec{B}_0^{(n+1)}.\overrightarrow{dA}. \qquad (2)$$

Eq. (2) shows that the induced electromotive force is a power series of the product of the n^{th} power of ratio between the self-inductance and resistance of the loop and the surface integral of the $(n+1)^{th}$ time-derivative of the external magnetic field. It is clear to note that the first term $(n=0)$ gives the well-known induced electromotive force, namely

$$\varepsilon_0 = -\int \frac{\partial \vec{B}}{\partial t}.d\vec{A} \qquad (3)$$

Furthermore, the result in Eq. (2) immediately gives the closed line integral of Faraday's law on the covering space;

$$\oint \vec{E}_n\,.\overrightarrow{dS} = -\left(\frac{L}{R}\right)^{n}\int \vec{B}_0^{(n+1)}.\overrightarrow{dA}, \qquad (4)$$

which by Stoke's theorem, yields the differential form of Faraday's law on the covering space, namely

$$\vec{\nabla}x\vec{E}_n = -\left(\frac{L}{R}\right)^{n}\vec{B}_0^{(n+1)}. \qquad (5)$$

The sum of all contributions from all winding numbers on the covering space, as shown in [20], gives the integral and differential forms of Faraday's law on the physical space;

$$\oint \vec{E}\,.\overrightarrow{dS} = \sum_{n=0}\oint \vec{E}_n\,.\overrightarrow{dS} = -\sum_{n=0}\left(\frac{L}{R}\right)^{n}\int \vec{B}_0^{(n+1)}.\overrightarrow{dA} \qquad (6)$$

$$\vec{\nabla}x\vec{E} = -\sum_{n}\left(\frac{L}{R}\right)^{n}\vec{B}_0^{(n+1)}. \qquad (7)$$

For later purposes, it is instructive to define a dimensionless quantity

$$s = \frac{R}{L}t \;\rightarrow\; \frac{d}{ds} = \frac{L}{R}\frac{d}{dt} \qquad (8)$$

Physically, s gives number of time constants. This immediately gives the induced electromotive force ε_n from the n^{th} winding number on the covering space and the total induced electromotive force in the physical space;

$$\varepsilon_n = \left(\frac{d}{ds}\right)^{n}\varepsilon_0 \qquad (9)$$

$$\varepsilon = \sum_{n=0}\left(\frac{d}{ds}\right)^{n}\varepsilon_0 \qquad (10)$$

DEMONSTRATIVE EXAMPLES

In this section, we provide some demonstrative examples, each with a specified time dependent function of the external magnetic field.

Example 1

We consider here an external time-varying magnetic field given by $B = B_0 \sin(\omega t)$, which in terms of *s*, is written as

$$B = B_0 \sin\left(\omega \frac{Ls}{R}\right) \quad (11)$$

This gives the first term for the induced electromotive force, namely

$$\varepsilon_0 = -A\omega B_0 cos\left(\omega \frac{Ls}{R}\right) \quad (12)$$

Defining $r = \frac{\omega L}{R}$ and using Eq. (9), one gets $\varepsilon_n = -\frac{d^n}{ds^n}[AB_0 \omega cos(rs)]$, which immediately gives

$$\varepsilon_{2n} = -(-1)^n AB_0 \omega r^{2n} \cos(rs), \quad n = 0, 1, 2, \ldots.. \quad (13)$$

$$\varepsilon_{2n-1} = -(-1)^n AB_0 \omega r^{2n-1} \sin(rs), \quad n = 1, 2, \ldots. \quad (14)$$

Thus, the total induced electromotive force is

$$\varepsilon = \sum_{n=0} \varepsilon_{2n} + \sum_{n=1} \varepsilon_{2n-1}$$

which, after some algebra, yields the result

$$\varepsilon = -\frac{AB_0\omega}{1 + r^2}[\cos(rs) - r sin(rs)] \quad (15)$$

The expression in Eq. (15) shows that when $\frac{l}{R} \to 0$ $means\, r \to 0$, it reduces to $-AB_0 \omega \cos(rs) = \varepsilon_0$.

Example 2

In this example we consider a time-varying magnetic field given by $B = B_0 e^{-\omega t}$, which can be written as $B = B_0 e^{-rs}$. This yield $\varepsilon_0 = AB_0 \omega e^{-rs}$ and by using Eq.(9), one gets the n^{th} term of the induced electromotive force, namely

$$\varepsilon_n = AB_0 \omega (-1)^n r^n e^{-rs} \quad (16)$$

Therefore the total induced electromotive force is given by

$$\varepsilon = \sum_n \varepsilon_n = AB_0 \omega e^{-rs} \sum_n (-1)^n r^n \quad (17)$$

which we write as

$$\varepsilon = AB_0 \omega e^{-rs} \left(\sum_{n=0}^{\infty} r^{2n} - \sum_{n=0}^{\infty} r^{2n+1} \right) \quad (18)$$

After some algebra, the above expression yields

$$\varepsilon = AB_0 \omega \frac{1}{1+r^2} e^{-rs} = \frac{1}{1+r^2} \varepsilon_0 \quad (19)$$

which, in the limit $r \to 0$, it reduces to ε_0.

Example 3

In our last example, we consider a magnetic field that is step function in time: $B = B_0 u(t)$, where $u(t)$ is the step function. This gives $\varepsilon_0 = -B_0 A\delta(t)$ which, in terms of *s*, is written as $\varepsilon_0 = -B_0 A\delta(\frac{L}{R}s)$. It is constructive to model the step function as

$$u(t) = \frac{1}{1+e^{-2kt}} \tag{20}$$

where *k* is a karge quantity. Definning $x = e^{-2kt}$, (thus $x \ll 1$), enables us to write the step function as

$$u(x) = \frac{1}{1+x} = \sum_{m=0}(-1)^m x^m \tag{21}$$

Using $\frac{d}{dt} = -2kx\frac{d}{dx}$, gives $\frac{du}{dt} = -2k\sum_{m=0}(-1)^m m x^m$ and therefore

$$\varepsilon_0 = 2kAB_0 \sum_{m=0} m(-1)^m x^m \tag{22}$$

Using Eq. (9), the n^{th} term of the induced electromotive force is written as

$$\varepsilon_n = \left(\frac{L}{R}\right)^n \left(-2kx\frac{d}{dx}\right)^n \varepsilon_0 \tag{23}$$

which, by using Eq. (22), can be written in the form

$$\varepsilon_n = (-1)^n B_0 A\left(\frac{L}{R}\right)^n (2k)^{n+1} \sum_{m=0} (-1)^m m^{n+1} x^m \tag{24}$$

Therefore, the total induced electromotive force is expressed as

$$\varepsilon = \sum_{n=0} \varepsilon_n = 2kB_0 A \sum_{m=0} m(-1)^m x^m \sum_{n=0}\left(-\frac{2Lkm}{R}\right)^n \tag{25}$$

Using $\sum_{n=0}\left(-\frac{2Lkm}{R}\right)^n = \frac{1}{1+\frac{2Lkm}{R}}$, one immediately gets

$$\varepsilon = 2kAB_0 \sum_{m=0} (-1)^m \frac{m}{1+\frac{2Lkm}{R}} x^m \tag{26}$$

In order to investigate the behavior of the indiced electromotive force in Eq. (26), we conside the following three cases:

Case 1. $t > 0$: In this case $x \to 0$ and each term in the sum of Eq. (26) vanishes and therefore $\varepsilon = 0$. This result is expected since there is no change in the magnetic flux for $t > 0$ and thus there is no induced electromotive force.

Csae 2. $t < 0$: In this case $x \to \frac{1}{x}$ which makes the step function written as $u = \sum_{m=0}(-1)^m x^{m+1}$. Therefore;

$$\varepsilon = AB_0 \sum_{m=0}(-1)^m \frac{m+1}{\frac{1}{2k}+\frac{L(m+1)}{R}} x^{(m+1)} \tag{27}$$

It is clear to see that when $x \to 0$, the induced electromotive force vanishes since each term in Eq.(27) vanishes.

Case 3. $t = 0$: in this case $x \to 1$ and neglecting the factor $1/2k$ in the denominator of Eq. (27), we get

$$\varepsilon = AB_0\left[\frac{1}{L/R} + \sum_{m=1}\frac{(-1)^m}{L/R}\right] \tag{28}$$

Obviously, the sum inside the bracket vanishes and thus $\varepsilon = AB_0\frac{1}{L/R}$, which goes to ∞ as $L/R \to 0$.

From the results of the previous three cases, we conclude that $\varepsilon = AB_0\delta(t)$.

RESULTS AND DISCUSSION

In the present work we have showed the advantage of using a covering space for the induced electromotive force due to a time varying magnetic field. This reflects the contribution of different winding numbers to the total induced electromotive force. The sum of all these contributions yields the induced electromotive force on the physical space. To illustrate the usefulness of the covering space, we have presented three examples with different time varying magnetic fields. In all these examples, the contributions from different winding numbers depend on different powers of the ratio between the inductance and resistance and on higher order time derivatives of the magnetic field. Furthermore, we derived closed form expressions for the induced electromotive force for the given examples. In addition, our results reduce to the well-known values of the induced eleoctromotive force in the limit when the ratio between the inductance and resistance goes to zero.

CONCLUSION

Three illustrative examples were presented to demonstrate the usefulness of a covering space in writing the induced electromotive force due to a time-varying magnetic field. Our results show the dependence of the induced electromotive force on higher order time derivatives of the magnetic field and on the ratio between the inductance and the resistance of the loop. Our general formulas for the induced electromotive force yield the well-known results when the ratio between the inductance and resistance goes to zero.

ACKNOWLEDGEMENT

I would like to thank Erasmus Mundus project team for providing financial support through staff mobility program during his visit to Technische Universitaet Berlin (TU-B), Germany. I would also like to thank Prof. R. Seiler and TU-B for the very kind hospitality they provide during my stay in Berlin.

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