



## Approximate generalized Jensen type mappings in proper Lie CQ\*-algebras

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### ABSTRACT

In this paper, we investigate the stability problems for proper Lie derivations associated to the generalized Jensen type functional equation

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) + \sum_{j=2}^n f\left(\frac{\sum_{i=1, i \neq j}^n x_i - (n-1)x_j}{n}\right) = f(x_1)$$

in a proper Lie CQ\*-algebra.

### KEYWORDS

proper Lie CQ\*-algebras, proper Lie CQ\*-derivations, proper Lie JCQ\*-derivations, generalized Hyers-Ulam stability, generalized Jensen type mappings

### SUBJECT CLASSIFICATION

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## INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations was originated from a question of Ulam [24] concerning the stability of group homomorphisms as follows:

*Let  $(G_1, *)$  be a group and  $(G_2, \diamond, d)$  be a metric group with the metric  $d(\cdot, \cdot)$ . Given  $\epsilon > 0$ , does there exist a  $\delta(\epsilon) > 0$  such that if a mapping  $h : G_1 \rightarrow G_2$  satisfies the inequality  $d(h(x * y), h(x) \diamond h(y)) < \delta(\epsilon)$  for all  $x, y \in G_1$ , then there is a homomorphism  $H : G_1 \rightarrow G_2$  with  $d(h(x), H(y)) < \epsilon$  for all  $x \in G_1$ ?*

If the answer is affirmative, we would say that the equation of a homomorphism  $H(x * y) = H(x) \diamond H(y)$  is stable. The famous Ulam stability problem was partially solved by Hyers [8] for linear functional equation of Banach spaces. Let  $f : X \rightarrow Y$  be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all  $x, y \in X$  and for some  $\epsilon > 0$ . Then, there exists a unique additive mapping  $T : X \rightarrow Y$  such that

$$\|f(x) - T(x)\| \leq \epsilon$$

for all  $x \in X$ . Hyers's theorem was generalized by Aoki [1] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference  $\|f(x + y) - f(x) - f(y)\|$  to be controlled by  $\epsilon(\|x\|^p + \|y\|^p)$ . Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias may be called the generalized Hyers-Ulam-Rassias stability. In 1994, a generalization of Rassias theorem was obtained by Găvruta [5], who replaced  $\epsilon(\|x\|^p + \|y\|^p)$  by a general control function  $\phi(x, y)$ . Cădariu and Radu [3] applied the fixed point method to investigation of the stability of a Jensen functional equation. Since then, the stability problems of many algebraic, differential, integral, operatorial equations have been extensively investigated [9, 10, 13, 14] and the references therein.

The theory of finite dimensional complex Lie algebras is an important part of Lie theory. It has several applications in physics and connections with other parts of mathematics. With an increasing amount of theory and applications concerning Lie algebras of various dimensions, it is becoming necessary to ascertain which tools are applicable for handling them. The miscellaneous characteristics of Lie algebras constitute such tools and have also found applications. Recently, many research papers have been published about the generalized Hyer-Ulam-Rassias stability of homomorphisms and derivations in  $C^*$ -algebras, Lie  $C^*$ -algebras,  $JC^*$ -algebras,  $CQ^*$ -algebras [4, 7, 11, 15, 16, 18, 19, 20, 22] and the references therein.

We recall some basic facts concerning  $CQ^*$ -algebras [2, 23].

**Definition 1.1.** Let  $A$  be a linear space and  $A_0$  is a  $*$ -algebra contained in  $A$  as a subspace.  $A$  is called a quasi  $*$ -algebra over  $A_0$  if the following three conditions holds:

- (i) the right and left multiplications of an element of  $A$  and an element of  $A_0$  are defined and linear;
- (ii)  $x_1(x_2a) = (x_1x_2)a$ ,  $(ax_1)x_2 = a(x_1x_2)$  and  $x_1(ax_2) = (x_1a)x_2$  for all  $x_1, x_2 \in A$  and all  $a \in A_0$ ;
- (iii) an involution  $*$ , which extends the involution of  $A_0$ , is defined in  $A$  with the property that  $(ab)^* = b^*a^*$  whenever the multiplication is defined.

Quasi  $*$ -algebras arise in natural way as completions of locally convex  $*$ -algebras whose multiplication is not jointly continuous; in this case, one has to deal with topological quasi  $*$ -algebras. A quasi  $*$ -algebra  $(A, A_0)$  is called topological if a locally convex topology  $\tau$  on  $A$  is given such that:

- (i) the involution is continuous and the multiplications are separately continuous;
- (ii)  $A_0$  is dense in  $A[\tau]$ .

Throughout this paper, we suppose that a locally convex quasi  $*$ -algebra  $(A, A_0)$  is complete. Many authors have considered a special class of quasi  $*$ -algebras, called proper  $CQ^*$ -algebras, which arise as completions of  $C^*$ -algebras.

**Definition 1.2.** Let  $A$  be a Banach module over the  $C^*$ -algebra  $A_0$  with involution  $*$  and  $C^*$ -norm  $\|\cdot\|_0$  such that  $A_0 \subset A$ . Then the pair  $(A, A_0)$  is called a proper  $CQ^*$ -algebra if

- (i)  $A_0$  is dense in  $A$  with respect to its norm  $\|\cdot\|_0$ ;
- (ii)  $(ab)^* = b^*a^*$  for all  $a, b \in A$ , whenever the multiplication is defined;
- (iii)  $\|y\|_0 = \sup_{a \in A, \|a\|_0 \leq 1} \|ay\|$  for all  $y \in A_0$ .

A proper  $CQ^*$ -algebra  $(A, A_0)$  is said to have a unit  $e$  if there exists an element  $e \in A_0$  such that  $ae = ea = a$  for all  $a \in A$ . A proper  $CQ^*$ -algebra with an identity is called a unital proper  $CQ^*$ -algebra. In this paper, we will always assume that  $(A, A_0)$  is an unital proper  $CQ^*$ -algebra.

**Definition 1.3.** A proper  $CQ^*$ -algebra  $(A, A_0)$ , endowed with a bilinear multiplication  $[\cdot, \cdot] : (A \times A_0) \cup (A_0 \times A) \rightarrow A$ , is called the Lie bracket, which satisfies the following properties:

- (i)  $[x_1, x_2] = -[x_2, x_1]$  for all  $(x_1, x_2) \in (A \times A_0) \cup (A_0 \times A)$ ;



(ii)  $[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_1, [x_2, x_3]]$  for all  $x_1, x_2, x_3 \in A_0$

is called a proper Lie CQ\*-algebra.

Now, we consider a mapping  $f : A_0 \rightarrow A$  satisfying the following functional equation:

$$f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) + \sum_{j=2}^n f\left(\frac{\sum_{i=1, i \neq j}^n x_i - (n-1)x_j}{n}\right) = f(x_1) \tag{1.1}$$

for all  $x_i, x_j \in A_0$ , where  $n$  is a fixed positive integer with  $n \geq 2$ . Gordji et al. [7] establish the stability of  $n$ -Lie homomorphisms and Jordan  $n$ -Lie homomorphisms on  $n$ -Lie algebras associated with the functional equation (1.1) by the fixed point method. Kim et al. [12] proved the stability and superstability problems of derivations for the functional equation (1.1) on Lie C\*-algebras by the direct method.

Motivated by the above works, we prove the generalized Hyers-Ulam stability problems for proper Lie derivations associated with the generalized Jensen type functional equation (1.1) on proper Lie CQ\*-algebras. In section 2, we give Lie CQ\*-derivations in proper Lie CQ\*-algebras associated with the functional equation (1.1). In section 3, we give Lie JCQ\*-derivations in proper Lie Jordan CQ\*-algebras associated with the functional equation (1.1).

Throughout this paper, we assume that  $(A, A_0)$  is a proper Lie CQ\*-algebra associated with the C\*-norm  $\|\cdot\|_0$  and norm  $\|\cdot\|$ . For convenience, we use the following abbreviation for any given mapping  $f : A_0 \rightarrow A$ ,

$$D_\mu f(x_1, \dots, x_n) = f\left(\frac{1}{n} \sum_{i=1}^n \mu x_i\right) - \mu \sum_{j=2}^n f\left(\frac{\sum_{i=1, i \neq j}^n x_i - (n-1)x_j}{n}\right) - f(\mu x_1)$$

for all  $\mu \in T^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$  and all  $x_1, \dots, x_n \in A_0 (n \geq 2)$ .

## DERIVATIONS IN PROPER LIE CQ\*-ALGEBRAS

In this section, we prove the generalized Hyers-Ulam stability results of proper CQ\*-derivations in proper CQ\*-triples for the functional equation  $D_\mu f(x_1, \dots, x_n) = 0$ .

Let  $(A, A_0)$  be a proper Lie CQ\*-algebra with respect to the Lie product

$$[z, x] = \frac{zx - xz}{2}$$

for all  $x \in A$  and all  $z \in A_0$ . A  $\mathbb{C}$ -linear mapping  $\delta : A_0 \rightarrow A$  is called a proper Lie CQ-derivation if

$$\delta([z, x]) = [z, \delta(x)] + [\delta(z), x]$$

for all  $z, x \in A_0$ . In addition, if  $\delta$  satisfies the additional condition

$$\delta(a^*) = \delta(a)^*$$

for all  $a \in A_0$ . Then, it is called a proper Lie CQ\*-derivation.

**Lemma 2.1.** [17] Let  $f : A_0 \rightarrow A$  is an additive mapping such that  $f(x)\mu = \mu f(x)$  for all  $x \in A_0$  and all  $\mu \in T^1$ . Then  $f$  is  $\mathbb{C}$ -linear.

**Theorem 2.2.** Suppose that  $f : A_0 \rightarrow A$  is a mapping with  $f(0) = 0$  for which there exist mappings  $\varphi : A_0^n \rightarrow [0, \infty)$ ,  $\psi : A_0^2 \rightarrow [0, \infty)$  and  $\eta : A_0 \rightarrow [0, \infty)$  be mappings such that

$$\|D_\mu f(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n) \tag{2.1}$$

$$\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\| \leq \psi(w_0, w_1) \tag{2.2}$$

$$\|f(w_2^*) - f(w_2)^*\| \leq \eta(w_2) \tag{2.3}$$

for all  $\mu \in T^1$  and all  $x_1, \dots, x_n, w_0, w_1, w_2 \in A_0 (n \geq 2)$ . Let  $L (< 1)$  be a constant such that

$$\varphi\left(\frac{x_1}{n}, \dots, \frac{x_n}{n}\right) \leq \frac{L}{n} \varphi(x_1, \dots, x_n) \tag{2.4}$$

$$\psi\left(\frac{w_0}{n}, \frac{w_1}{n}\right) \leq \frac{L}{n^2} \psi(w_0, w_1) \tag{2.5}$$

$$\eta\left(\frac{w_2}{n}\right) \leq \frac{L}{n} \eta(w_2) \tag{2.6}$$

for all  $x_1, \dots, x_n, w_0, w_1, w_2 \in A_0$ . Then there exists a unique Lie CQ\*-derivation  $\delta : A_0 \rightarrow A$  satisfying

$$\|f(x) - \delta(x)\| \leq \frac{L}{1-L} \varphi(x, 0, \dots, 0) \tag{2.7}$$

for all  $x \in A_0$ .



Proof. Letting  $\mu = 1$  and  $x_1 = x, x_2 = \dots = x_n = 0$  in (2.1), we get

$$\left\| n f\left(\frac{x}{n}\right) - f(x) \right\| \leq \varphi(x, 0, \dots, 0), \tag{2.8}$$

for all  $x \in A_0$ . Replacing  $x$  by  $\frac{x}{n^j}$  and multiplying  $n^j$  both the sides of (2.8),

$$\left\| n^{j+1} f\left(\frac{x}{n^{j+1}}\right) - n^j f\left(\frac{x}{n^j}\right) \right\| \leq n^j \varphi\left(\frac{x}{n^j}, 0, \dots, 0\right),$$

for all  $x \in A_0$  and all integer  $j \in \mathbb{Z}^+$  with  $j = 0, 1, 2, \dots$ . Hence, we obtain

$$\left\| n^m f\left(\frac{x}{n^m}\right) - n^k f\left(\frac{x}{n^k}\right) \right\| \leq \sum_{j=k}^{m-1} n^j \varphi\left(\frac{x}{n^j}, 0, \dots, 0\right) \leq \sum_{j=k}^{m-1} L^j \varphi(x, 0, \dots, 0) \tag{2.9}$$

for all  $x \in A_0$  and all nonnegative integers  $m, k$  with  $m \geq k$ . It follows from (2.9) that the sequence  $\{n^m f\left(\frac{x}{n^m}\right)\}$  is Cauchy in  $A$  for all  $x \in A_0$ . Since  $A$  is complete, it converges. So, we can define a mapping  $\delta : A_0 \rightarrow A$  defined by

$$\delta(x) = \lim_{m \rightarrow \infty} n^m f\left(\frac{x}{n^m}\right) \tag{2.10}$$

for all  $x \in A_0$ . Passing the limit  $m \rightarrow \infty$  with  $k = 0$  in (2.9), we have

$$\|f(x) - \delta(x)\| \leq \lim_{m \rightarrow \infty} \|n^m f\left(\frac{x}{n^m}\right) - f(x)\| \leq \sum_{j=0}^{\infty} n^j \varphi\left(\frac{x}{n^j}, 0, \dots, 0\right) \leq \sum_{j=0}^{\infty} L^j \varphi(x, 0, \dots, 0),$$

which implies the inequality (2.7) holds for all  $x \in A_0$ . On the other hand, it follows from (2.4), (2.5), (2.6) and  $L < 1$  that

$$\lim_{m \rightarrow \infty} n^m \varphi\left(\frac{x_1}{n^m}, \dots, \frac{x_n}{n^m}\right) = 0, \quad \lim_{m \rightarrow \infty} n^{2m} \psi\left(\frac{w_0}{n^m}, \frac{w_1}{n^m}\right) = 0, \quad \lim_{m \rightarrow \infty} n^m \eta\left(\frac{w_2}{n^m}\right) = 0.$$

Substituting  $\mu = 1$  and  $(x_1, \dots, x_n)$  by  $\left(\frac{x_1}{n^m}, \dots, \frac{x_n}{n^m}\right)$  in (2.1), we have

$$\begin{aligned} \|D_1 \delta(x_1, \dots, x_n)\| &= \lim_{m \rightarrow \infty} n^m \|D_1 f\left(\frac{x_1}{n^m}, \dots, \frac{x_n}{n^m}\right)\| \\ &\leq \lim_{m \rightarrow \infty} n^m \varphi\left(\frac{x_1}{n^m}, \dots, \frac{x_n}{n^m}\right) = 0 \end{aligned}$$

for all  $x_1, \dots, x_n \in A_0$ . Then  $D_1 \delta(x_1, \dots, x_n) = 0$ , and so the mapping  $\delta$  is additive [12]. Letting  $x_1 = x, x_2 = \dots = x_n = 0$  in (2.1), we get  $\|\mu f(x) - f(\mu x)\| \leq \varphi(x, 0, \dots, 0)$  for all  $x \in A_0$ . Thus,

$$\begin{aligned} \|\mu f(x) - f(\mu x)\| &= \lim_{m \rightarrow \infty} n^m \left\| \mu f\left(\frac{x}{n^m}\right) - f\left(\mu \frac{x}{n^m}\right) \right\| \\ &\leq \lim_{m \rightarrow \infty} n^m \varphi\left(\frac{x_1}{n^m}, \dots, \frac{x_n}{n^m}\right) = 0, \end{aligned}$$

which implies  $\mu f(x) = f(\mu x)$  for all  $\mu \in T^1$  and all  $x \in A_0$ . It follows from Lemma 2.1 that the mapping  $\delta$  is  $\mathbb{C}$ -linear.

Now, it follows from (2.2), (2.5) and (2.10) that

$$\begin{aligned} &\|\delta([w_0, w_1]) - [\delta(w_0), w_1] - [w_0, \delta(w_1)]\| \\ &= \lim_{m \rightarrow \infty} n^{2m} \left\| f\left(\frac{[w_0, w_1]}{n^{2m}}\right) - \left[ f\left(\frac{w_0}{n^m}, \frac{w_1}{n^m}\right) - \left[ \frac{w_0}{n^m}, f\left(\frac{w_1}{n^m}\right) \right] \right\| \\ &\leq \lim_{m \rightarrow \infty} n^{2m} \psi\left(\frac{w_0}{n^m}, \frac{w_1}{n^m}\right) = 0, \end{aligned}$$

which proves

$$\delta([w_0, w_1]) = [\delta(w_0), w_1] + [w_0, \delta(w_1)]$$

for all  $w_0, w_1 \in A_0$ . And it follows from (2.3), (2.6) and (2.10) that

$$\|\delta(w_2^*) - \delta(w_2)^*\| = \lim_{m \rightarrow \infty} n^m \left\| f\left(\frac{w_2^*}{n^m}\right) - f\left(\frac{w_2}{n^m}\right)^* \right\| \leq \lim_{m \rightarrow \infty} n^m \eta\left(\frac{w_2}{n^m}\right) = 0$$

for all  $w_2 \in A_0$ . Thus, we obtain that the mapping  $\delta : A_0 \rightarrow A$  satisfies

$$\delta(w_2^*) = \delta(w_2)^*$$

for all  $w_2 \in A_0$ .

Finally, let  $\delta' : A_0 \rightarrow A$  be another proper Lie CQ\*-derivation on  $A_0$  satisfying (2.7). Then, we have

$$\begin{aligned} \|\delta(x) - \delta'(x)\| &\leq \lim_{m \rightarrow \infty} n^m \left\| f\left(\frac{x}{n^m}\right) - \delta'\left(\frac{x}{n^m}\right) \right\| \\ &\leq \lim_{m \rightarrow \infty} \frac{n^m L}{1-L} \varphi\left(\frac{x}{n^m}, 0, \dots, 0\right) = 0 \end{aligned}$$





for all  $x \in A_0$ . So, we can conclude that  $\delta(x) = \delta'(x)$  for all  $x \in A_0$ . Therefore, the mapping  $\delta$  is a unique proper Lie CQ\*-derivation on  $A_0$  satisfying (2.7). This completes the proof.

**Corollary 2.3.** Let  $r (> 1)$  and  $\theta$  be positive real numbers. Suppose that a mapping  $f : A_0 \rightarrow A$  satisfies

$$\begin{aligned} \|D_1 f(x_1, \dots, x_n)\| &\leq \theta(\|x_1\|_0^r + \dots + \|x_n\|_0^r) \\ \|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\| &\leq \theta(\|w_0\|_0^{2r} + \|w_1\|_0^{2r}) \quad (2.11) \\ \|f(w_2^*) - f(w_2)^*\| &\leq \theta\|w_2\|_0^r \end{aligned}$$

for all  $x_1, \dots, x_n, w_0, w_1, w_2 \in A_0$ . Then there exists a unique proper Lie CQ\*-derivation  $\delta : A_0 \rightarrow A$  such that

$$\|f(x) - \delta(x)\| \leq \frac{n^r \theta}{n^r - n} \|x\|_0^r \quad (2.12)$$

for all  $x \in A_0$ .

**Proof.** Letting  $\varphi(x_1, \dots, x_n) = \theta(\|x_1\|_0^r + \dots + \|x_n\|_0^r)$  for all  $x_1, \dots, x_n \in A_0$  and  $L = n^{1-r}$ , we obtain the claimed stability result (2.12). This completes the proof.

**Corollary 2.4.** Let  $r, r_j (j = 1, 2, \dots, n)$  and  $\theta$  be positive real numbers such that  $0 < \sum_{j=1}^n r_j < 1$ . Suppose that a mapping  $f : A_0 \rightarrow A$  satisfies

$$\begin{aligned} \|D_\mu f(x_1, \dots, x_n)\| &\leq \theta(\|x_1\|_0^{r_1} \times \dots \times \|x_n\|_0^{r_n}) \quad (2.13) \\ \|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\| &\leq \theta(\|w_0\|_0^r \cdot \|w_1\|_0^r) \\ \|f(w_2^*) - f(w_2)^*\| &\leq \theta\|w_2\|_0^r \end{aligned}$$

for all  $x_1, \dots, x_n, w_0, w_1, w_2 \in A_0$  and all  $\mu \in T^1$ . Then  $f$  is a proper Lie CQ\*-derivation on  $A_0$ .

**Proof.** Letting  $\varphi(x_1, \dots, x_n) = \theta(\|x_1\|_0^{r_1} \times \dots \times \|x_n\|_0^{r_n})$ ,  $\psi(w_0, w_1) = \theta(\|w_0\|_0^r \cdot \|w_1\|_0^r)$ ,  $\eta(w_2) = \theta\|w_2\|_0^r$  in Theorem 2.2, we have the conditions (2.4), (2.5), (2.6). Putting  $x_1 = \dots = x_n = 0$  and  $\mu = 1$  in (2.13), we obtain  $f(0) = 0$ . Furthermore, if we put  $x_1 = x, x_2 = \dots = x_n = 0$  and  $\mu = 1$  in (2.13), then we have  $f(x) = n f\left(\frac{x}{n}\right)$  for all  $x \in A_0$ . It is easy to see that by induction, we get  $f(x) = n^m f\left(\frac{x}{n^m}\right)$  for all  $x \in A_0$  and all  $m \in \mathbb{Z}^+$ . Now, it follows from Theorem 2.2 that  $f$  is a proper Lie CQ\*-derivation on  $A_0$ . This completes the proof.

## DERIVATIONS IN PROPER LIE JCQ\*-ALGEBRAS

In this section, we give the generalized Hyers-Ulam stability of Lie JCQ\*-derivations in proper CQ\*-algebras associated with the functional equation  $D_\mu f(x_1, \dots, x_n) = 0$ .

A proper Lie CQ\*-algebra  $(A, A_0)$ , endowed with Jordan product

$$z \circ x = \frac{zx + xz}{2}$$

for all  $x \in A_0$  and all  $z \in A_0$  is called a proper Lie JCQ\*-algebra.

**Definition 3.1.** Let  $(A, A_0)$  be a proper Lie JCQ\*-algebra. A  $\mathbb{C}$ -linear mapping  $\delta : A_0 \rightarrow A$  is called a proper Lie JCQ\*-derivation if  $\delta$  satisfies a proper Lie CQ\*-derivation and

$$\delta(x \circ y) = x \circ \delta(y) + \delta(x) \circ y \quad (3.1)$$

for all  $x, y \in A_0$ .

**Theorem 3.2.** Let  $\varphi, \psi, \eta$  be as Theorem 2.2. Suppose that a mapping  $f : A_0 \rightarrow A$  with  $f(0) = 0$  satisfies (2.1), (2.2), (2.3) and

$$\|f(w_0 \circ w_1) - w_0 \circ f(w_1) - f(w_0) \circ w_1\| \leq \psi(w_0, w_1) \quad (3.2)$$

for all  $w_0, w_1 \in A_0$ . Then there exists a unique proper Lie JCQ\*-derivation  $\delta : A_0 \rightarrow A$  such that

$$\|f(x) - \delta(x)\| \leq \frac{L}{1-L} \varphi(x, 0, \dots, 0) \quad (3.3)$$

for all  $x \in A_0$ .

**Proof.** By the same reasoning as in the proof of Theorem 2.2, there exists a unique  $\mathbb{C}$ -linear mapping  $\delta$  satisfying (2.7). The mapping  $\delta : A_0 \rightarrow A$  is defined by

$$\delta(x) = \lim_{m \rightarrow \infty} n^m f\left(\frac{x}{n^m}\right)$$

for all  $x \in A_0$ . It is sufficient to show that  $f$  satisfies the condition (3.1) of Definition 3.1. It follows from (2.5) and (3.2) that



$$\begin{aligned} & \|\delta(w_0 \circ w_1) - \delta(w_0) \circ w_1 - w_0 \circ \delta(w_1)\| \\ &= \lim_{m \rightarrow \infty} n^{2m} \left\| f\left(\frac{w_0 \circ w_1}{n^{2m}}\right) - f\left(\frac{w_0}{n^m}\right) \circ \frac{w_1}{n^m} - \frac{w_0}{n^m} \circ f\left(\frac{w_1}{n^m}\right) \right\| \\ &\leq \lim_{m \rightarrow \infty} n^{2m} \psi\left(\frac{w_0}{n^m}, \frac{w_1}{n^m}\right) = 0, \end{aligned}$$

which gives

$$\delta(w_0 \circ w_1) = \delta(w_0) \circ w_1 + w_0 \circ \delta(w_1)$$

for all  $w_0, w_1 \in A_0$ . Therefore, we conclude the mapping  $\delta$  is a proper Lie JCQ\*-derivation on  $A_0$ . This completes the proof.

**Corollary 3.3.** Let  $r (> 1)$ ,  $\theta$  and  $\epsilon$  be positive real numbers. Suppose that a mapping  $f: A_0 \rightarrow A$  satisfies

$$\begin{aligned} & \|D_1 f(x_1, \dots, x_n)\| \leq \epsilon, \\ & \|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\| \leq \theta(\|w_0\|_0^{2r} + \|w_1\|_0^{2r}), \\ & \|f(w_2^*) - f(w_2)^*\| \leq \theta\|w_2\|_0^r, \\ & \|f(w_0 \circ w_1) - w_0 \circ f(w_1) - f(w_0) \circ w_1\| \leq \theta(\|w_0\|_0^{2r} + \|w_1\|_0^{2r}) \end{aligned}$$

for all  $x_1, \dots, x_n, w_0, w_1, w_2 \in A_0$ . Then there exists a unique proper Lie JCQ\*-derivation  $\delta: A_0 \rightarrow A$  such that

$$\|f(x) - \delta(x)\| \leq \frac{n\epsilon}{n^r - n} \quad (3.4)$$

for all  $x \in A_0$ .

**Corollary 3.4.** Let  $r_1, r, \theta$  be as Corollary 2.4. Suppose that a mapping  $f: A_0 \rightarrow A$  satisfies

$$\begin{aligned} & \|D_\mu f(x_1, \dots, x_n)\| \leq \theta(\|x_1\|_0^{r_1} \times \dots \times \|x_n\|_0^{r_n}) \\ & \|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\| \leq \theta(\|w_0\|_0^r \cdot \|w_1\|_0^r) \\ & \|f(w_2^*) - f(w_2)^*\| \leq \theta\|w_2\|_0^r \\ & \|f(w_0 \circ w_1) - w_0 \circ f(w_1) - f(w_0) \circ w_1\| \leq \theta(\|w_0\|_0^r \cdot \|w_1\|_0^r) \end{aligned}$$

for all  $x_1, \dots, x_n, w_0, w_1, w_2 \in A_0$ . Then  $f$  is a proper Lie JCQ\*-derivation on  $A_0$ .

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