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HOMOTOPY ANALYSIS METHOD TO SOLVE BOUSSINESQ EQUATIONS

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ABSTRACT

In this paper, Homotopy analysis method is applied to the nonlinear coupled differential equations of classical Boussinesq system. We have applied Homotopy analysis method (HAM) for the application problems in [1, 2, 3, 4]. We have also plotted Domb-Sykes plot for the region of convergence. We have applied Pade for the HAM series to identify the singularity and reflect it in the graph. The HAM is a analytical technique which is used to solve non-linear problems to generate a convergent series. HAM gives complete freedom to choose the initial approximation of the solution, it is the auxiliary parameter h which gives us a convenient way to guarantee the convergence of homotopy series solution. It seems that more artificial degrees of freedom implies larger possibility to gain better approximations by HAM.

Indexing terms/Keywords

Homotopy Analysis Method; Coupled Boussinesq Equations; Pade approximations.

Academic Discipline And Sub-Disciplines

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1. Introduction

Boussinesq type equations can be used to model the nonlinear transformation of surface waves in shallow water due to the effects of shoaling, refraction, diffraction and reflection. Different linear dispersion relations can be obtained by expressing the equations in different velocity variables. In fluid dynamics, the Boussinesq approximation for water waves is an approximation valid for weakly non-linear and fairly long waves. The approximation is named after Joseph Boussinesq, who first derived them in response to the observation by John Scott Russell of the wave of translation (also known as solitary wave or soliton). The 1872 paper of Boussinesq introduces the equations now known as the Boussinesq equations.

The Boussinesq equation

$$u_{tt} - u_{xx} - 3u_{xx}^2 - u_{xxxx} = 0, \qquad (1)$$

was solved by Cao [4] using the inverse scattering technique and by Hirota [6] using direct method. Krishnan [2, 3] has found periodic wave solutions for equation (1). Rajaraman [12] studied coupled nonlinear differential equations of quantum field theory which are given by

$$\sigma_{xx} = -\sigma + \sigma^3 + d\rho^2\sigma,$$

$$\rho_{xx} = (f-d)\rho + \lambda\rho^3 + d\rho\sigma^2, \qquad (2)$$

where σ and ρ are real scalar fields and d, f, λ are parameters. Sachs [13] has constructed an infinite family of rational solutions of the completely integrable variant of the Boussinesq system given by

$$\begin{aligned} u_t + \rho_x + uu_x &= 0, \\ \rho_t + u_x + u_{xxx} + (\rho u)_x &= 0. \end{aligned} \qquad (3)$$

Using the variable $H = 1 + \rho$ in equations (3), we get

$$\begin{aligned} u_t + H_x + uu_x &= 0, \\ H_t + u_{xxx} + (uH)_x &= 0, \end{aligned} \qquad (4)$$

where u is the velocity and H is the total depth. Cao [4] has obtained more general soliton solutions of equation (4) by trigonometric functions transformation method using homogeneous balance method. Differential transform method [1] has also been applied and shown that it is a very fast convergent, precise and powerful tool for solving Boussinesq equations. Applying the similarity tranformation to equation (4) we get equation (14).

2. Basic Idea of HAM

Basic Idea of HAM is to construct a homotopy as follows

$$H(\phi; p, h, H) = (1-q)L[\phi(\eta; p, h, H) - u_0(\eta)] - phH(\eta)N[\phi(\eta; q, h, H)], \qquad (5)$$

Where

$$N[u(\eta)] = 0, \qquad (6)$$

is given equation.

where q $\in$ [0,1] is an embedding parameter, h an auxiliary parameter, $u_0(\eta)$ is an initial guess of u(η) and H(η) is a non zero auxiliary function,choose a linear operator L of order same as that of N.

When the embedding parameter q = 0 and q = 1 then

$$\phi(\eta;0) = u_0(\eta),\ \ \phi(\eta;1) = u(\eta), \qquad (7)$$

respectively. Thus as q increases from 0 to 1, the solution $\phi(\eta;q)$ varies from the initial guess $u_0(\eta)$ to the required solution u(η). By writing $\phi(\eta;q)$ in series as follows

$$\phi(\eta;q) = u_0(\eta) + \sum_{m=1}^{\infty} u_m(\eta) q^m, \qquad (8)$$

Where

$$u_m(\eta) = \frac{1}{m!}\frac{\partial^m \phi(\eta;q)}{\partial q^m} \quad at \quad q = 0. \tag{9}$$

Differentiating the equation (5) m times with respect to q, dividing them by m! and finally setting q = 0, we get the following m[th] order deformation equation.

$$L[u_m(\eta) - \chi_m u_{m-1}(\eta)] = hR_m(u_{m-1}), \tag{10}$$

Where

$$R_m(u_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\eta;q)]}{\partial q^{m-1}} \quad at \quad q = 0, \tag{11}$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{12}$$

The convergence of the series depends upon the auxiliary parameter h. The value of h is obtained by h curve [14]. If it is convergent at q = 1

$$u(\eta) = u_0(\eta) + \sum_{m=1}^{\infty} u_m(\eta). \tag{13}$$

3. Numerical Applications

Problem 1

Consider the following Boussinesq equation given in Cao [4]

$$2u_{\eta\eta} - u^3 + 3u^2 - 2u = 0, \quad u(0) = 3, \quad u'(0) = 0. \tag{14}$$

We consider the Linear operator as

$$L = \frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \eta}, \tag{15}$$

with the property

$$L(c_1 + c_2\eta) = 0. \tag{16}$$

Applying HAM explained in previous section and solving equation (15) we get the initial guess.

$$\phi 0 = 3, \tag{17}$$

The Nonlinear equation is written as

$$N[\phi] = 2\frac{\partial^2 \phi}{\partial \eta^2} - \phi^3 + 3\phi^2 - 2\phi, \tag{18}$$

Thus,

$$R_m = 2\phi''_{m-1}(\eta) - \sum_{k=0}^{m-1}\sum_{r=0}^{k} \phi_{m-1-k}\phi_{k-r}\phi_r + 3\sum_{k=0}^{m-1}\phi_{m-1-k}\phi_k - 2\phi_{m-1}. \tag{19}$$

Using equations (17) and (19) in (10) along with the initial condition $\phi_m(0) = 3, \quad \phi'_m(0) = 0$ where m = 1, 2, 3……., we get

$$\phi 0 = 3,$$
$$\phi 1 = -6h\eta - e^{-\eta} 6h + 3(1+2h),$$
$$\phi 2 = -3h(35+22h)\eta + 33h^2\eta^2 + e^{-\eta}(-54h - 54h\eta - 3h(35+22h)) + 3(1+53h+22h^2),$$
$$\phi 3 = -108e^{-2\eta}h - \frac{1}{2}h\eta(1248 + h(7236-1869\eta) + 2h^2(2478-942\eta+193\eta^2)) + e^{-\eta}(-702h(1+\eta) - 594h^3(1+\eta) - 27h^2(63+63\eta+\eta^2) - 6(68h+603h^2+413h^3)) + 3(1+406h+1773h^2+1024h^3).$$

and so on.

The HAM series solution for (14) is given by

$$u = \phi 0 + \phi 1 + \phi 2 + \phi 3 + \ldots\ldots\ldots\ldots \qquad (20)$$

This solution is graphically depicted in graphs and compared with Achala [1].

Problem 2

Consider the following Boussinesq equation given in Krishnan [2, 3]

$$2v_{\eta\eta} = 3v^3 - 9v^2 + 6v, \quad v(0) = 3, \; v(1) = 0. \qquad (21)$$

We consider the Linear operator as

$$L = \frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \eta}, \qquad (22)$$

with the property

$$L(c_1 + c_2\eta) = 0. \qquad (23)$$

Applying HAM explained in previous section and solving equation (22) we get the initial guess

$$\phi 0 = \left(\frac{3}{1-exp[1]}\right) - \left(\frac{3exp[1]exp[-\eta]}{1-exp[1]}\right), \qquad (24)$$

The Nonlinear equation is written as

$$N[\phi] = 2\frac{\partial^2 \phi}{\partial \eta^2} - 3\phi^3 + 9\phi^2 - 6\phi, \qquad (25)$$

Thus,

$$R_m = 2\phi''_{m-1}(\eta) - 3\sum_{k=0}^{m-1}\sum_{r=0}^{k}\phi_{m-1-k}\phi_{k-r}\phi_r + 9\sum_{k=0}^{m-1}\phi_{m-1-k}\phi_k - 6\phi_{m-1}. \qquad (26)$$

Using equations (24) and (26) in (10) along with the initial condition $\phi_m(0) = 3, \quad \phi_m(1) = 0$ where m = 1, 2, 3……., we get

$$\phi 0 = \left(\frac{3}{1-exp[1]}\right) - \left(\frac{3exp[1]exp[-\eta]}{1-exp[1]}\right),$$

$$\phi 1 = \frac{1}{2(-1+e)^3}e^{-3\eta}(-27e^3h + 162e^{2+\eta}h + 81e^{3+\eta}h + 36e^{3\eta}h\eta + 90e^{1+3\eta}h\eta + 36e^{2+3\eta}h\eta + e^{3+2\eta}(24h(1+\eta) - 2(\frac{3(2e-6e^2+6e^3-2e^4-181eh-179e^2h+118e^3h+26e^4h)}{2(-1+e)^4}) + \ldots\ldots\ldots$$

and so on.

The HAM series solution for (21) is given by

$$v = \phi 0 + \phi 1 + \phi 2 + \phi 3 + \ldots\ldots\ldots\ldots \quad (27)$$

This solution is graphically depicted in graphs and compared with Achala [1].

4. Results and Discussions

In figure 1 and 5 we draw u" versus h and the value of h is estimated by observing the horizontal region of this h curve. In figure 2 and 6 velocity curve is drawn for h = 0.1 and this curve exactly matches with the one obtained in Achala [1]. We have also estimated the radius of convergence of HAM solutions by Domb-Sykes plot presented in figure 3 and 7 along with the Radius of convergence. We have applied Pade approximation for the series solution of (20) and (27) by taking the degree as (1, 5) with h = 0.1 and is given by,

$$\frac{-55605.59859000003 + 5991.772517717633\eta}{1.0 + 0.7496654790041055\eta + 0.21304134428943589\eta^2 + 0.0036621694401157145\eta^3 - 0.016711420326742027\eta^4 - 0.004969834554931681\eta^5}, \quad (28)$$

$$\frac{48752.04898643494 - 1.489090292481979 \times 10^{10}\eta}{1.0 + 0.3564206345726206\eta - 0.18500880044423818\eta^2 - 0.25047758909171436\eta^3 - 0.06760984733896122\eta^4 + 0.18660825988100885\eta^5}. \quad (29)$$

We observe that the graphs of (28) and (29) exhibit singularities for large values of η. It is also observed in (28) that there is a singularity at η = 3.75168 presented in figure 4 and in (29) it exhibits a singularity at η = -1.25809 presented in figure 8.

Thus, we conclude that HAM solution is more accurate and can be applied to almost all nonlinear problems arising in real world problems.

5. Graphs

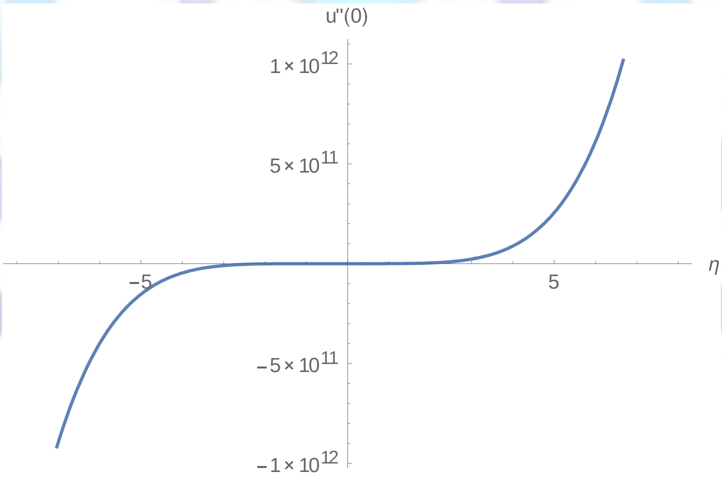

Figure 1 : h-curve

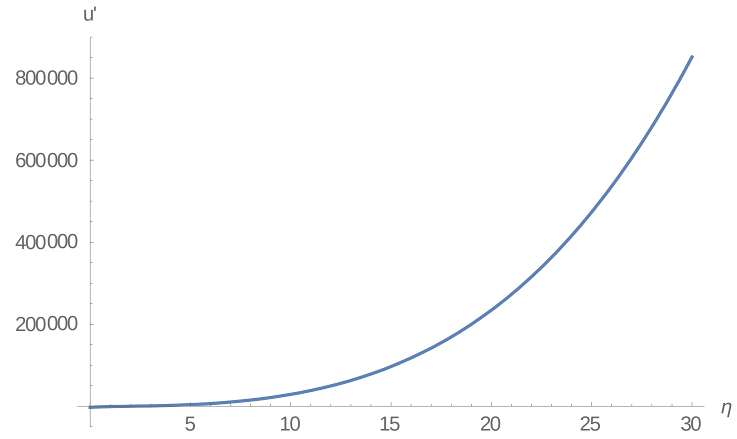

Figure 2 : Velocity Curve for h = 0.1 for 0 < η < 30

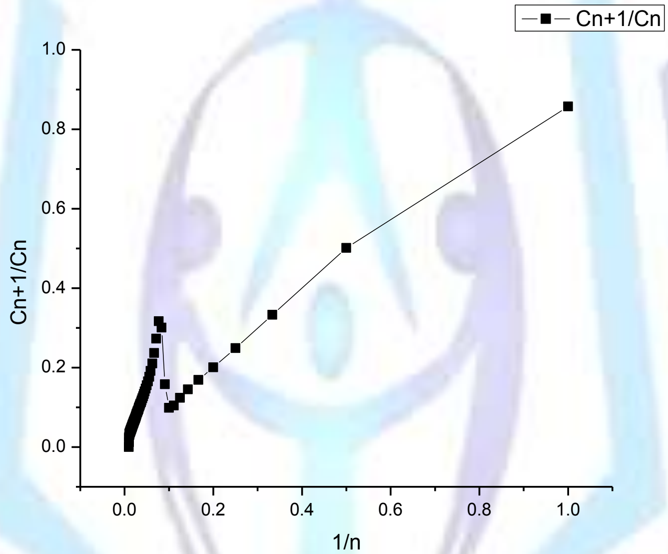

Figure 3 : Domb-Sykes plot for h = 0.1 and R = 1/0.02603

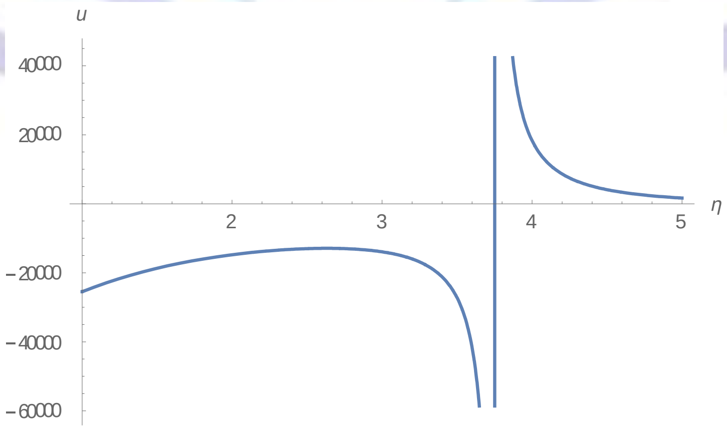

Figure 4 : We observe that there is a singularity in the range at η = 3:75168.

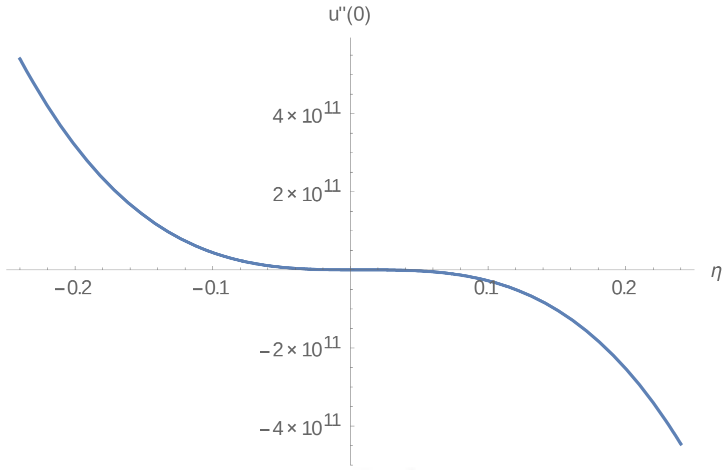

Figure 5 : h-curve

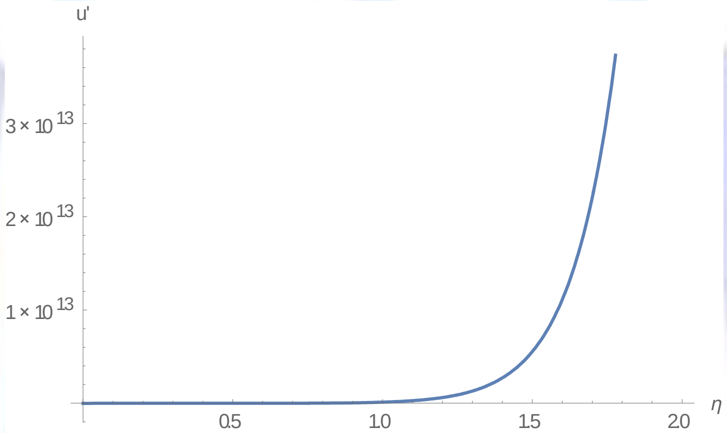

Figure 6 : Velocity Curve for h = 0.1

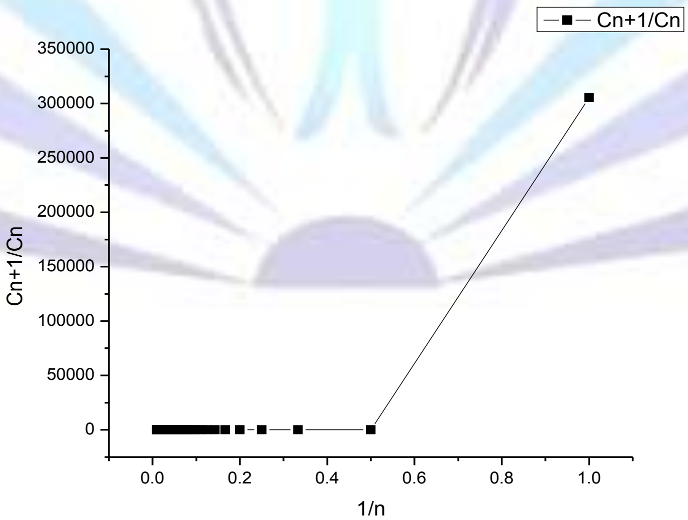

Figure 7 : Domb-Sykes plot for h = 0.1 and R = 1/0.17299

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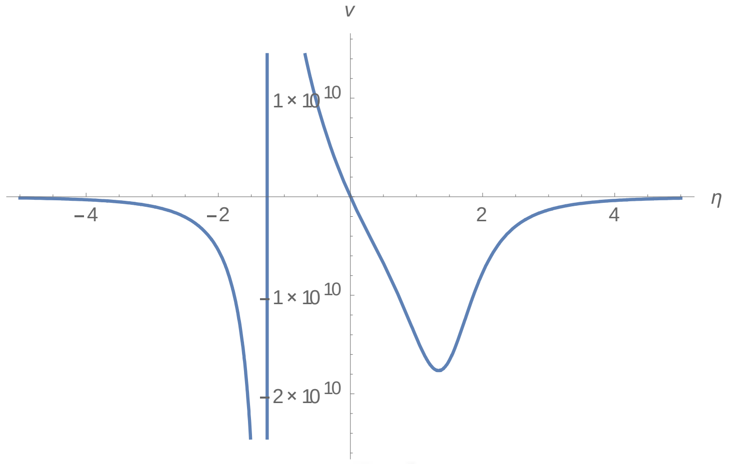

Figure 8 : We observe that there is a singularity in the range at η = -1.25809

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