A solution of Fractional Laplace's equation by Modified separation of variables

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ABSTRACT

This paper applies the Modified separation of variables method (MSV) suggested by Pishkoo and Darus towards obtaining a solution for fractional Laplace's equation. The closed form expression for potential function is formulated in terms of Meijer's G-functions (MGFs). Moreover, the relationship between fractional dimension $\mathcal{D}$ and parameters of Meijer's G-function for spherical Laplacian in R $\mathcal{D}$, for radial functions is derived.

Keywords: Fractional differential equation; Laplace's equation; Meijer's G-function; Modified separation of variables.

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INTRODUCTION

It is a known fact that shapes and forms occur in nature which cannot be described by Euclidean geometry. Examples of these include, branching of trees, the roughness of ocean floor, the geometry of clouds, the dust in planetary disks etc. All of these are instances where high spatial complexity occurs. Such structures are not easily describable because we cannot assign a characteristic length to them [1]. Mandelbrot [1, 2] first introduced the term, "Fractal" to differentiate pure geometries from other types which do not fit into a simple classification. Fractal is used to describe the degree of irregularity or fragmentation of a sample or structure which is identical at all scales. In the fractal media, some of the regions or domains are not filled with medium particles and these unfilled domains are called porous domains. Fractal models of media are becoming increasingly popular because just a small number of parameters are needed to define a medium of great complexity and rich structure. In order to use fractional derivatives and fractional integrals for media on fractal, we must use some continuous model [1, 3-4]. Most times, the real fractal structure can be disregarded and the medium on fractal can be described by some fractional continuous mathematical model. The order of the fractional integral is equal to the fractal dimension. Fractional integrals can be considered as approximations of integrals on fractals [5, 6]. Fractals are characterized by a fractional dimension, $\mathfrak{D}$. Hence complex structures can be modeled at microscopic and macroscopic levels [7, 8].

The concept of fractional dimensional space has widespread applications in several areas of physics, essentially to describe the effective parameters of physical systems [3, 9]. It is important to apply a generalization of electromagnetic theory in fractional space in order to extract full benefits of fractal models, which are becoming popular due to small number of parameters that define a medium of greater complexity and a rich structure. Fractional calculus is used by different authors to describe fractional solutions to many electromagnetic problems as well as fractional dimensional space [10-12]. It was proposed that smoothing of microscopic characteristics over the physically infinitesimal volume transforms the initial fractal distribution into fractional continuous model that uses fractional integrals [13]. Solutions to Poisson's and Laplace equation for scalar potential in fractional space have been discussed in [14, 15]. Stillinger [16] has developed the formalism for constructing a generalization of an integer dimensional Laplacian operator into a non-integer dimensional space. Laplace's equation in fractional space is derived by using modified vector differential operators:

$$\nabla^2 \Phi_D = 0. \tag{1.1}$$

Plamer and Stavrinou [9] generalized the Stillinger's results to n orthogonal coordinates and Laplacian operator in $\mathfrak{D}$ dimensional fractional space in three-spatial coordinates is given as:

$$\nabla^2{}_D = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z}, \tag{1.2}$$

where, parameters $(0 < \alpha_1 < 1, 0 < \alpha_2 < 1 \ and \ 0 < \alpha_3 < 1)$ are used to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $\mathfrak{D} = \alpha_1 + \alpha_2 + \alpha_3$.

On the other hand, forms of Laplacian in R^2, R^3, and R^n (n is integer) dimensional space in three coordinates systems are as follows [28]:

Polar Laplacian in R2 for Radial functions

$$\nabla^2 U(\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)U(\rho) \tag{1.3}$$

Spherical Laplacian in R3 and Rn for Radial Functions

$$\nabla^2 U(r) = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)U(r) \tag{1.4}$$

$$\nabla^2 U(r) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)U(r) \tag{1.5}$$

Cylindrical Laplacian in R3 for Radial functions

$$\nabla^2 U(\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)U(\rho) \tag{1.6}$$

Three analytical solution methods are used to solve differential equations namely; Green's function method, Variational method, and Separation of variables. Physically, the Green's function represents the field produced by a point source. By using the Green's function, the solution of Maxwell equations can be represented by an integral defined over the source region or on a closed surface enclosing the source. Mathematically, the solution of a partial differential equation $u = \hat{L}^{-1}f$, where $\hat{L}$ stands for the inverse of $\hat{L}u = \tilde{f}$ can be expressed as $\tilde{L}$ and is often represented by an integral operator whose kernel is the Green's function. The principle of least action leads to the development of the Lagrangian and Hamiltonian formulations of classical mechanics. Based on the principle of least action, the differential equations of a

given physical system are derivable by minimizing the action of the system. The original problem, governed by the differential equations, is thus replaced by an equivalent variational problem [28].

The method of separation of variables (SV) is usually used to seek a solution of the form of the product of functions, each of which depends on one variable only, so that the solution of original partial differential equations may reduce to the solution of ordinary differential equations [29]. In this paper the newly method called "Modified separation of variables" (MSV) is used to solve fractional Laplace's equation.

Our previous work had focused on introduction of the Modified separation of variables method (MSV), and applying it to solve partial differential equation related to the Reaction-Diffusion process, Laplace's equation, and Schrodinger equation which leads to represent its solution in terms of Meijer's G-functions [17-20]. In [21] we discuss the fractional transformations of Meijer's G-functions by illustrating how G-functions are more general in form than hypergeometric functions. This paper uses the MSV method to solve fractional Laplace's equation and finally it deduces the solution in terms of Meijer's G-functions.

The paper has been organized as follows. Section 2 gives notation and basic definition of Meijer's G-function. Section 3 introduces the Modified separation of variables (MSV) suggested by Pishkoo and Darus in [17]. Section 4 consists of main results of the paper, in which the Modified separation of variables method has been developed for fractional Laplace's equation.

MEIJER G-FUNCTIONS

In mathematics, the G-function was introduced by Cornelis Simon Meijer (1936) as a very general function intended to include all elementary functions and most of the known special functions as particular cases, for instance:

$$\sin z = \sqrt{\pi} G_{0,2}^{1,0}\left(\frac{z^2}{4}\Big|_{\frac{1}{2},0}^{-}\right), \quad -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$$

$$\sinh z = -\sqrt{\pi} i G_{0,2}^{1,0}\left(\frac{-z^2}{4}\Big|_{\frac{1}{2},0}^{-}\right), \quad \frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$$

$$\cos z = \sqrt{\pi} G_{0,2}^{1,0}\left(\frac{z^2}{4}\Big|_{0,\frac{1}{2}}^{-}\right), \quad \forall z$$

$$\cosh z = \sqrt{\pi} G_{0,2}^{1,0}\left(\frac{-z^2}{4}\Big|_{0,\frac{1}{2}}^{-}\right), \quad \forall z$$

$$\ln z = G_{2,2}^{1,2}\left(z-1\Big|_{1,0}^{1,1}\right), \quad \forall z$$

$$J_v(z) = G_{0,2}^{1,0}\left(\frac{z^2}{4}\Big|_{\frac{v}{2},\frac{-v}{2}}^{-}\right), \quad -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$$

A definition of the Meijer's *G*-function is given by the path integral in the complex plane, called Mellin-Barnes type integral see [22]-[26].

For the *G*-function

$$G_{p,q}^{m,n}\left(z\Big|_{b_1,...,b_q}^{a_1,...,a_p}\right) \tag{1.7}$$

The integers m; n; p; q are called orders of the G-function, ap and bq are called "parameters" and in general, they are complex numbers. The definition holds under the following assumptions:

$0 \leq m \leq q$ and $0 \leq n \leq p$, where $m, n, p,$ and q are integer numbers. For example the orders 1,0,0,2 are the same for functions $\sin z, \cos z, \sinh z, \cosh z$; but their parameters are different.

The Meijer's G-function $y(z) = G_{p,q}^{m,n}\left(z\Big|_{b_1,...,b_q}^{a_1,...,a_p}\right)$ satisfies the linear ordinary differential equation of the generalized hypergeometric type whose order is equal to max (p, q). (See [22], [27-29])

$$\left[(-1)^{p-m-n} z \prod_{j=1}^{p}\left(z\frac{d}{dz} - a_j + 1\right) - \prod_{k=1}^{q}\left(z\frac{d}{dz} - b_k\right)\right] y(z) = 0. \tag{1.8}$$

The fundamental question is that why do we suggest working with G-functions despite their complex forms? The first reason is that each of G-functions immediately has differential equation (8) on their side. For instance when we discuss about function $\ln z$ there is no image of its differential equation,

while for $\ln z = G_{2,2}^{1,2}(z-1|_{1,0}^{1,1})$, we can easily obtain its differential equation. The second reason will be shown when we use the following Modified Separation of Variables method to solve partial differential equations or linear ordinary differential equations.

MODIFIED SEPARATION OF VARIABLES (MSV) METHOD [17]

In solving Partial Differential Equations (PDE) initial and boundary conditions are necessary to obtain specific solutions, and depending on the geometry of the problem, these differential equations are separated into Ordinary Differential Equation (ODEs) in which each involves a single coordinate of a suitable coordinate system. With Cartesian, cylindrical and spherical coordinates, the boundary conditions are important in determining the nature of the ODE solutions obtained from the PDE.

The Separation of Variables is employed to obtain solutions to the many partial differential equations. The method consists of assuming the solution as a product of functions, each depending on one coordinate variable only. The method proposed in [17], i.e. the "Modified separation of variables", leads to obtain the solution as a product of the Meijer's G-functions, with each one depending on one variable by equating ordinary linear differential equation of the Meijer's G-functions and each one of the ODEs.

Four steps are followed in this method, namely:

1) Choosing a convenient coordinates system by considering the boundary conditions and boundary surfaces.

2) Obtaining ODEs from PDE by writing the solution as a product form of different variables and putting it into PDE.

3) Starting with (8), the orders m, n, p, and q, are chosen and the variable z is changed to $\acute{z}$ such that Eq.(8) converts into each of ODEs, and this is three times for time independent problem (e.g., x, y, z coordinates) and four times for time dependent problem (e.g., x, y, z, t). If this particular step is done, solving ODEs is not required because Equation (8) is an equation that does not need to be solved. It is solved, and the solutions are MGFs.

4) Using boundary conditions and obtaining the exact solution.

MAIN RESULTS- Form of Polar Laplacian in $R^{\mathbb{F}}(\mathbb{F} = \alpha_1 + \alpha_2)$

For each $r > 0$ $u = u(x(r,\theta), y(r,\theta))$

$$x(r,\theta) = r\cos\theta, \quad y(r,\theta) = r\sin\theta$$

$$u_r = u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta.$$

$$\frac{\alpha_1 - 1}{x}\frac{\partial u}{\partial x} = \frac{\alpha_1 - 1}{x}u_x = (\alpha_1 - 1)[\frac{1}{r}u_r - \frac{\tan\theta}{r^2}u_\theta]. \tag{1.9}$$

$$\frac{\alpha_2 - 1}{y}\frac{\partial u}{\partial y} = \frac{\alpha_2 - 1}{y}u_y = (\alpha_2 - 1)[\frac{1}{r}u_r + \frac{\cot\theta}{r^2}u_\theta]. \tag{1.10}$$

From

$$u_{rr} = u_{xx}\cos^2\theta 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta$$

and

$$u_{\theta\theta} = r^2(u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta) - r(u_x\cos\theta + u_y\sin\theta).$$

We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}. \tag{1.11}$$

Combining (9), (10), and (11), we obtain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial r^2} + \frac{(\alpha_1 + \alpha_2 - 1)}{r}\frac{\partial u}{\partial r} + (\alpha_1 - 1)(-\frac{\tan\theta}{r^2}\frac{\partial u}{\partial \theta}) + \quad (1.12)$$

$$(\alpha_2 - 1)(\frac{\cot\theta}{r^2}u_\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Finally Polar Laplacian in R$^{\mathcal{D}}$ for Radial function $(u = u(r), \frac{\partial u}{\partial \theta} = 0)$, where $F = \alpha_1 + \alpha_2$, is as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial r^2} + \frac{(\alpha_1 + \alpha_2 - 1)}{r}\frac{\partial u}{\partial r}. \quad (1.13)$$

MGF's solution for fractional Laplace's equation in R^F

We here consider two cases when (8) is reduced to first and second order ordinary differential equation, respectively:

CASE 1. Setting $m = 1, n = 0, p = 1, q = 1$ in (8) yields

$$[z\,(z\,\frac{d}{dz} - a_1 + 1)] - (z\,\frac{d}{dz} - b_1)]G_{1,1}^{1,0}(z\,|_{b_1}^{a_1}) = 0.$$

Simplifying gives

$$\frac{dG_{1,1}^{1,0}(z\,|_{b_1}^{a_1})}{dz} = \frac{z\,(a_1 - 1) - b_1}{z\,(z - 1)}dz,$$

Now putting $a_1 - 1 = b_1 = c$ leads to the solution z^c

$$G_{1,1}^{1,0}(z\,|_{c}^{c+1}) = z^c. \quad (1.14)$$

CASE 2. Setting $m = 1, n = 2, p = 2, q = 2$ in (8) yields

$$[-z\,(z\,\frac{d}{dz} - a_1 + 1)(z\,\frac{d}{dz} - a_2 + 1) - (z\,\frac{d}{dz} - b_1)(z\,\frac{d}{dz} - b_2)]G_{2,2}^{1,2}(z\,|_{b_1,b_2}^{a_1,a_2}) = 0.$$

By changing z to $z-1$, we have

$$[-(z - 1)[(z - 1)\frac{d}{dz} - a_1 + 1][(z - 1)\frac{d}{dz} - a_2 + 1] - [(z - 1)\frac{d}{dz} - b_1][(z - 1)\frac{d}{dz} - b_2]G_{2,2}^{1,2}(z - 1|_{b_1,b_2}^{a_1,a_2}) = 0.$$

Comparing above equation with the form of Polar Laplacian in R^2 or Cylindrical Laplacian in R^3 for Radial functions, namely

$$[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz}]u(z) = 0.$$

Equality condition for these two differential equations leads to $a_1 = a_2 = b_1 = 1$, $b_2 = 0$, and the solution in terms of Meijer's G-function is

$$u(z) = G_{2,2}^{1,2}(z - 1|_{1,0}^{1,1}) = \ln z. \quad (1.15)$$

Moreover, comparing again but this time with the form of Spherical Laplacian in R^3 for Radial Functions namely (4), we obtain parameters of G-function as follows:

$$a_1 - a_2 = 1, b_1 = 1, b_2 = 0.$$

And the solution is

$$u(z) = G_{2,2}^{1,2} \binom{a_1, a_1-1}{1,0} \tag{1.16}$$

Thus general solution gives us

$$u(z) = A + BG_{2,2}^{1,2} \binom{a_1, a_1-1}{1,0} + CG_{1,1}^{1,0}(z \mid_c^{c+1}) + DG_{1,1}^{1,0}(z \mid_{-c}^{-c+1}), MSV method, \tag{1.17}$$

Where *A, B, C, D* are constants, and they can be determined by boundary conditions.

Finally and most significantly we obtain the relation between fractional dimension $\mathcal{D}$ in general and parameters of G-functions for Spherical Laplacian in R^D for Radial Functions, namely (5)

$$\mathcal{D} = \alpha_1 + \alpha_2 = 2 + a_1 - a_2, \ b_1 = 1, \ b_2 = 0. \tag{1.18}$$

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