



Gravity Quantized in Dimension 4

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Abstract

This is a sequel to the article 'A More or Less Well-Behaved Quantum Gravity Lagrangean in Dimension 4?' in *Advanced Studies in Theoretical Physics*, Torbrand Dhrif[6]. We give a simple Quantum Gravity Lagrangean that behaves well, up to the standards of particle physics. Feynman calculus for cross-sections, and the diagrams involved, should behave good. The action is naively renormalizable, has critical dimension and is invariant under scalings in dimension 4. It implies standard Einstein gravity for a massless graviton.



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Note the identity

$$D^2 = \square + R = \square + R_{ab} \frac{[\Gamma^a, \Gamma^b]}{4}$$

where $R_{ab} = R_{ab}^\alpha T_\alpha$, the Riemann curvature tensor, is a Lie algebra valued 2-form, with T_α a basis of the defining representation the Lie algebra $SO(1,3)$. We use the Dirac or Clifford Algebra representation

$$\Gamma^a = \theta^a + \theta^{*a}.$$

Here θ^{*a} and θ^a generate a finite dimensional Fock-algebra via

$$\delta^{ab} = \eta_{wick-rotated}^{ab} = [\theta^{*a}, \theta^b]_+ = \{\theta^{*a}, \theta^b\}.$$

and

$$0 = [\theta^a, \theta^b]_+ = \{\theta^a, \theta^b\}.$$

Also

$$0 = [\theta^{*a}, \theta^{*b}]_+ = \{\theta^{*a}, \theta^{*b}\}.$$

The $[\theta^a]$ is a ON basis on the tangent space $T_p(X)$ over a point p and $[\theta^a]$ a raised ON basis on the cotangent space $T_p^*(X)$ over the same point p , just like in the notation of E. Cartan, who wrote the metric g in terms of the vielbeins θ_μ^a as $g_{\mu\nu} = \theta_\mu^a \delta_{ab} \theta_\nu^b$, $\delta_{ab} = \eta_{ab,wick-rotated}$, η_{ab} the Kronecker delta for Lorentzian signature, and

$$D = \Gamma^\mu (\partial_\mu + \omega_\mu + A_\mu) = \Gamma^a \nabla_a$$

$$\Gamma^a = \theta_\mu^a \Gamma^\mu.$$

We thus also conclude(see a side condition below),

$$|D \theta|^2 = \theta^* D^2 \theta = \theta^* (\square + R) \theta = \theta^* \square \theta + R = R$$

with R the Ricci scalar ¹. Here θ is the graviton or vierbein. Here we have suppressed a term, including a coupling constant, $16\pi G$. Notice that this is the Hilbert-Einstein action

$$S_{Einstein} = \int_X Ricci = \int_X R \sqrt{g} d^D x$$

with the side condition

$$\theta^* \square \theta = 0$$

imposed, which basically states that gravitons propagate with the speed of light. So with our side condition

$$L_{Gravity} = |D \theta|^2 = R = L_{Einstein}.$$

Counting dimensions

$$L_{Gravity} := |D \theta|^2$$

has physical dimension 4 (Compare to a YM-term if you wish, say in QED, it has the same dimension and looks very similar.), thus naively renormalizable in dimension 4. Let's prove this in a more obvious manner: Our volume measure is $\sqrt{g} d^D x$. Then our action functional becomes

¹ We sometimes suppress a minus-sign or a imaginary unit i in the following.



$$S = \int |\mathbb{D} \theta|^2 \sqrt{g} d^D x.$$

We set $e^a := \theta^a / L^a$, where L^a is the length in direction a . This e^a has dimension 0 and is θ^a after a normalization to make it give probabilities which are normalized to a total probability of the number 1. Thus e^a_μ has mass dimension 1. Thus finally

$$(\theta^a_\mu)^* (\square + \mathbb{R}) (\theta^a_\nu)_{normalized} = (e^a_\mu)^* (\square + \mathbb{R}) (e^a_\nu)$$

has mass dimension 4 or length dimension -4 . The space-time measure has length dimension D . So we have renormalizability by dimension counting in $D=4$, and the critical dimension is also $D=4$. Note the identity

$$\sqrt{g} d^D x = \theta^1 \wedge \theta^2 \cdots \wedge \theta^D = (T \times V) (\theta^1 / T) \wedge (\theta^2 \wedge \cdots \wedge \theta^D) / V.$$

Here V is the volume of space and T is the length of time, and we set space-time to be tubular or foliated with spatial leaves of volume V . This becomes

$$\sqrt{g} d^D x = e^1 \wedge \cdots \wedge e^D \times V \times T.$$

For information on why the vielbeins can be interpreted as generating probability currents via exterior algebra see my book, [5], it's basically a statement of compatibility, or that the covariant gradient and covariant divergence of monomials generated by vielbeins vanishes. See the expressions

$$J = *\theta^1 / V = e^2 \wedge e^3 \wedge e^4, DJ = 0, D^* J = 0.$$

so we equate the current J to the above form. Evaluate the current on a cycle on when expressed in the orthonormal frame (θ^a) . Actually if you set $\theta^0 = du = Du$, where u is some smooth enough function and d the exterior derivative, you end up very near to quaternionic geometry, see Hitchin et al. [11].

I now state the total action or Lagrangean for the standard model with gravity but not including symmetry braking terms or effective action terms that may arise;

$$L_{total} = |\mathbb{D} \theta|^2 + \bar{\psi} (i\mathbb{D} + m) \psi + \frac{1}{4} |R + F|^2.$$

The vertices for this theory, not including couplings to gauge and Yang-Mills theory are (two gravitons and one graviboson)

$$k_B^\mu \Gamma_\mu, \Gamma_\nu T^\alpha$$

where k_B is the Graviboson 4-momentum. T^α is a basis of the Lie algebra $\mathfrak{SO}(1,3)$. The other vertex

$$\Gamma_\mu, \Gamma_\nu f_\gamma^{\alpha\beta} T^\gamma$$

is for two gravitons and two gravibosons. The $f_\gamma^{\alpha\beta}$ are structure constants for $\mathfrak{SO}(1,3)$.

I should point out that we have to work with L^2 -spaces, that is well-defined Hilbert spaces, and also the mathematical theory of currents, which are generalizations of distributions to differential forms. In the most trivial applications and toy models one uses a compact smooth setting, but L^2 -spaces of distribution valued forms are probably the first candidates for appropriate spaces for us to use in mathematical physics.

I summarize by the following;

- This theory or Lagrangean is naively renormalizable by dimension counting and scale invariant in dimension 4. It has critical dimension 4. This is as good as the standard model in particle physics.
- Factorization of norm squared of matrix elements is evident. Feynman calculus for cross-sections goes through, which is a great simplification, and produces metric terms which are much easier to handle in this theory.
- The vielbein and graviboson are the fundamental fields. Actually the equation of motion will make the connection 'graviboson' the ON Levi-Cevita connection, that is the compatible torsion-less connection. So, from the point of view of elementary physics this looks right.
- This Lagrangean implies standard Einstein Gravity



$$Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

with obvious notation when the graviton or vielbein is on-shell light-like. Actually I think that the light-like criterion is obvious, gravity would not be a long range force otherwise.

• I conjecture that there is no need for ghosts in the gravity sector above, or, say, a Faddeev-Popov quantization. We alter the Lagrangean slightly

$$|\Omega + F|^2 \mapsto |D(A + \omega)|^2.$$

This implies ², after a bit of calculus, the equations $\nabla \cdot \omega = 0$ and $\nabla \cdot A = 0$ upon variation. These implied equations are the usual Lorentz gauge.

• I think we have relations to 8/10-dimensional string/super-gravity models with certain versions of the Maldacena conjecture or theorem, See Torbrand Dhrif[7], as long as the bulk has trivial topology such as the forward hyperboloid of some of the simplest dS/AdS-spaces,

$$\rho^2 = t^2 - x_1^2 - x_2^2 - \dots - x_n^2$$

all variables real, but with Wick rotations allowed in all coordinates, in the Green-Schwarz formalism after complexification of this four dimensional model. Particle physicists and quantum field theorists do work with a complexification by standard, e.g the Wick rotation procedure. Then the topology is namely the same, it's all in space-time, and this gives equivalent topological functors, like cohomological theories or physics expressed in cohomological theory such as BRST-theory, topological quantum field theories, anomaly theory or index theory. Remember, these theories could have pathologies or other peculiarities. In as far as string/sugra theory resemblences this is a good sign, because we think those models are at least somewhat nice. Yet our theory, notice well, lives in dimension 4.

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² This latter expression on the right side is well defined in view of our representation of the Dirac or Clifford algebra, as stated before.