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Point Spread Function For Elliptical Aperture Inclined at π/4 angle with x-axis

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ABSTRACT

Point spread function (PSF) for inclined elliptical aperture with an angle equal to π/4 with x-axis has been studied, by taking new coordinates m and n which are rotated by an angle π/4 to x and y- axes.
The properties of the obtained image have been found in different cases, for diffraction limited system and different types of aberrations, like focus error, spherical aberration and coma aberration.

Indexing terms/Keywords

Elliptical aperture, enclined elliptical aperture, Point Spread Function, Spherical aberration, Coma

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1. Introduction

Circular pupils are the most popular pupils in optical systems, because they are dealing with circular boundaries or circular wavefronts. However, in optics, the occurrence of the diffraction patterns of a circular aperture inclined at an angle to the beam axis is easily understood if one integrates over the elliptical aperture using the appropriate coordinate system[1]. The elliptical aperture can be in the case of off axis imaging and wide field optical systems, like the human eye[2]. It also found in other instruments shearing interferometry [3], or in imaging and testing fold mirrors [4].

Many studies on the elliptical aperture have been adopted, where in 1967, J. V. Cornacchio derived the normalized transfer function of an elliptical annular aperture for incoherent illumination[5]. Then in 1985, Y. P. Kathuria have been investigated in theoretical and experimental Fresnel and far-field diffraction of a coherent light beam with an elliptical aperture of aspect ratio between 0 and 1[6].

In 2003, A.V. Gitin and I. B. Movchan obtained Relationships to set up a program for calculating the profile of the leaf of an iris with an elliptical aperture of uniformly variable size[7]. In the next year, in 2004, A.R. Zakharian et.al. have analyzed the transmission of light through small elliptical apertures in a thin silver film at λ=1.0μm [8]. In 2012, Jian Liu et.al. studied the PSF for elliptical apertures of orientation greater than π/2[9]. While in 2014, Jose A. Díaz et.al. generalized an analytical form of orthonormal elliptical polynomials for any arbitrary aspect ratio to arbitrary orientation and gave expression for them up to the 4th order[10].

In this work, the PSF for inclined elliptical aperture with the major axis inclined at an angle equal to π/4 to the x-axis were studied. In next two sections, brief definitions of PSF and aberration function were stated. While in the other two sections a derivation of the PSF in the new coordinates were done, and finally the results and discussion were performed in section 6.

2. Point Spread Function (PSF)

The point spread function is the irradiance in the image of a point source in an optical system. This function may be obtained by the Fourier transform of the pupil function, where the amplitude on the image was given by[11].

$$F(u,v)=\frac{1}{A}\int_x\int_y f(x,y)e^{2\pi i(ux+vy)}dxdy \qquad (1)$$

Where f(x,y) represents the pupil function and equal

$$f(x,y)=\tau(x,y)e^{ikw(x,y)} \qquad (2)$$

and

$\tau(x,y)$ represents the real amplitude distribution in exit pupil coordinates (x,y), and it is called "pupil transparency". It equals one unit if the illumination is uniform.

$e^{ikw(x,y)}$ is the wave front of aberration function and $w(x,y)$ is the aberration factor, while A represent the exit pupil area.

Then the point spread function is then given by the complex square of the amplitude in the image [11]

$$G(u,v)=\left|F(u,v)\right|^2 \qquad (3)$$

3. Aberration Function

The wavefront aberration function is the distance from the reference sphere to the wavefront as a function of pupil coordinates .

The general wavefront aberration function can be expressed as a series of terms [13]

$$W(h;\rho,\psi)=\sum_{A,B,C} W''_{ABC}\, h^A \rho^B \cos^C\psi \qquad (4)$$

For orders above the first, the W''_{ABC} are the wavefront aberration coefficients.

There are five third-order terms, or the Seidel aberrations, which are, spherical, coma, astigmatism, field curvature, and distortion. There is also a piston-error term.

It is useful to include defocus as a term in aberration expansions. Its wavefront aberration is

$$W=W_{020}\,\rho^2 \qquad (5)$$

In spherical aberration the wavefront error is a figure of revolution in the pupil. The individual terms of the expansion have the form ρ^{2N}. The form that appears on axis, and which is independent of field position is

$$W = W_{020}\,\rho^2 + W_{040}\,\rho^4 + W_{060}\,\rho^6 + \ldots \qquad (6)$$

Where defocus has been included. The W_{040} term is the third-order term, the W_{060} is the fifth-order term.

In coma, the wavefront aberration varies linearly with field height, so the general form is hr2M+1cosy. Coma is an odd aberration. The wavefront expansion is

$$W = (W_{131}\,\rho^3 + W_{151}\,\rho^5 + \ldots)h\cos\psi \qquad (7)$$

And if the coordinates rotated by an angle f i. e. (in Cartesian coordinates XY)

$$x = X\cos\phi - Y\sin\phi$$

$$y = X\sin\phi + Y\cos\phi$$

then eq. (7) becomes, in Cartesian coordinates and for the third order term only

$$W = W131(x^2+y^2)(x\sin\phi + y\cos\phi) \qquad (8)$$

4. Deriving the Equation of Point Spread Function for inclined ellipse Aperture.

In this work an inclined ellipse at an angle $\pi/4$ to x- axis has been studied. This ellipse is of major axis equal to 1 unit (see fig. (1). And to make it easier an inclined coordinates m and n inclined to x and y axes by $\pi/4$ respectively will be taken. So, to do that, the equation of the principal axis, which is at an angle of $\pi/4$ to the x-axis, is y=x. i.e. y-x=0. While the equation of the minor axis, which is at an angle p/4 to the y-axis, is y=-x. i.e. y+x=0.

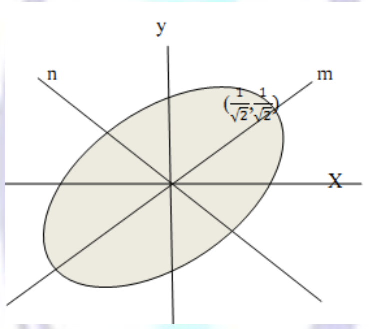

Fig.1: inclined ellipse by $\pi/4$ to x- axis

Let the new coordinates be m and n, where,

$m = y + x$ and $n = y - x$

i.e. $y = \frac{m+n}{2}$ and $x = \frac{m-n}{2}$

and to change dxdy to dmdn, Jacobian method is used as follows[14]

$$\frac{dmdn}{dxdy} = \begin{vmatrix} \frac{dm}{dx} & \frac{dm}{dy} \\ \frac{dn}{dx} & \frac{dn}{dy} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

So, dxdy=1/2 dmdn

equation (3) becomes

$$G(u,v) = n.f\left|\frac{1}{2}\int_m\int_n f(x,y)e^{2\pi i\left[u\frac{(m-n)}{2}+v\frac{(m+n)}{2}\right]}dndm\right|^2 \qquad (9)$$

Where $n.f$: normalizing factor.

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The boundaries of equation (9) can be get from the inclined ellipse equation $\frac{m^2}{a^2}+\frac{n^2}{b^2}=1$.

The limits of integration for m is from x+y=$\frac{-1}{\sqrt{2}}+\frac{-1}{\sqrt{2}}=-\sqrt{2}$ to $-\sqrt{2}$

And for n $\frac{m^2}{(\sqrt{2})^2}+\frac{n^2}{b^2}=1 \rightarrow$ n=$\pm b\sqrt{1-m^2/2}$

Assume that $z=2\pi u$ and $z_1=2\pi v$ then the normalized PSF is:

$$G(u,v)=n.f\left|\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-b\sqrt{1-m^2/2}}^{b\sqrt{1-m^2/2}} f(m,n)e^{i\left[z\frac{(m-n)}{2}+z1\frac{(m+n)}{2}\right]}dndm\right|^2 \quad .(10)$$

At contrast to the normal ellipse aperture (inclination angle=0) which have asymmetry shape around x and y-axes, it can be noticed, from fig(1), that the x and y lengths are equal in this inclined aperture, then, for simplicity, only the x part (x=(m-n)/2) will be taken, i.e. z1=0

$$PSF=G(u)=n.f\left|\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-b\sqrt{1-m^2/2}}^{b\sqrt{1-m^2/2}} f(m,n)e^{iz\left[\frac{(m-n)}{2}\right]}dndm\right|^2 \quad (11)$$

Equation (11) represents the point spread function for inclined ellipse aperture at angle $\pi/4$ to x- axis.

5 Diffraction-Limited System

An imaging system is said to be diffraction-limited if a diverging spherical wave, emanating from a point-source object, is converted by the system into a new wave, again perfectly spherical, that converges towards an ideal point in the image plane, where the location of that ideal image point is related to the location of the original object point through a simple scaling factor (the magnification), a factor that must be the same for all points in the image field of interest if the system is to be ideal. Therefore, the pupil function becomes $f(m,n)=1$ [15].

To find the normalizing factor which makes $G(u)=1$, when $u=0$, equation(7) becomes

$$PSF=1=n.f\left|\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-b\sqrt{1-m^2/2}}^{b\sqrt{1-m^2/2}} dndm\right|^2 \quad (12)$$

$$1=n.f\left|\int_{-\sqrt{2}}^{\sqrt{2}} b\sqrt{1-m^2/2}\,2dm\right|^2$$

Let $m/\sqrt{2}=\sin\theta \Rightarrow dm=\sqrt{2}\cos\theta d\theta$

$$1=n.f\left|\sqrt{2}\int_{-\sqrt{2}}^{\sqrt{2}} b\sqrt{1-\sin^2\theta}\cos\theta d\theta\right|^2 \quad 1=n.f\left(b^2\right).2\left|\int_{-\sqrt{2}}^{\sqrt{2}}\cos^2\theta d\theta\right|^2 \quad =n.f\left(b^2\right)2\left|\int_{\pi}^{0}\cos^2\theta d\theta\right|^2$$

$$1 = n.f\left(b^2\right)2\left|1/2\int_{5\pi/4}^{\pi/4}(1+\cos 2\theta)d\theta\right|^2 = n.f\left(b^2\right)\frac{1}{2}\left|[(\theta+\sin 2\theta)]_{5\pi/4}^{\pi/4}\right|^2$$

$$= n.f\left(b^2\right)\frac{\pi^2}{2} \quad \rightarrow \quad n.f. = \left(\frac{2}{\sqrt{2}b\pi}\right)^2$$

Which, of caurse, is equal the area of the ellipse with major axis on x-axis of unit radius $\left(\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2\right)^{1/2} = 1$ and minor axis $\left(\left(\frac{b}{2}\right)^2+\left(\frac{b}{2}\right)^2\right)^{1/2} = \frac{b}{\sqrt{2}}$

so equation (11) becomes:

$$PSF = \frac{1}{2b^2\pi^2}\left|\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-b\sqrt{1-m^2/2}}^{b\sqrt{1-m^2/2}} f(m,n)e^{iz\left[\frac{(m-n)}{2}\right]}dndm\right|^2 \qquad (13)$$

6. Results and discussion

The equations derived in the last section for PSF were programmed in MATCAD to get the results, as follows:

6.1 PSF for diffraction limited system

Equation (13) were programmed, and the results at the x-axis were shown in fig. (2.a). It has a maximum value =1 because of normalization. A symmetry is obvious at the two sides of z- axis (z=2πu) in spite of the asymmetry of the aperture at the two sides of x-axis and y-axis separately, but, it must not forget that the PSF is the square value of the amplitude PSF, so, there were no negative values.

When calculating the values of PSF at the y-axis (i.e. v=0) (see fig.2.b), the same values of that on x-axis were got. This happens because the length of the ellipse on x and y-axes were the same and equal $2\sqrt{\frac{2}{3}}$ (in the case when the ellipse is inclined at π/4 to x-axis and the length of the major axis is 1 and minor axis is 0.707).

When comparing with the normal ellipse, of same area, (the major axis coincide on the x-axis) which have the same dimensions, it found that the shape of PSF at the x-axis is the same as that of the circular aperture of area equal to □ , but of course, not the same values, were the normalization factor is different. And the normalized PSF does not depend on b (the minor axis). While at the y-axis, the width of PSF (the distance between the first two zeros at the two sides of the peak) is increased as the minor axis decreased, and the till the shape of PSF (in 2-D) be nearly that of the slit.

The above explanation of normal ellipse is true for the inclined ellipse if x and y-axes replaced by m and n axes (fig.2.c).

6.2 PSF with the presence of aberrations

Of course the effect of aberrations is a decreasing of the value of the peak of PSF. i.e the quality of the image is decreased. And this effect is different for different types of aberration as follows:

6.2.1 PSF with the presence of focus aberrations

Different values of focus aberration (0.25,0.5,0.75, and 1) were taken to calculate values of PSF, and they were shown in fig (4). And the peak is decreased from 1 for free of aberration to 0.867, 0.558, 0.257, and 0.083 respectively.

6.2.2 PSF with the presence of spherical aberrations

The effect of spherical aberration is less than focus error (the values of w40 is taken as that of focus error) and the peak changed from 1 for free of aberration to 0.895, 0.657, 0.435, and 0.31. (Fig.5).

Spherical aberration can be balanced with focus error by choosing appropriate values of the two errors. To make this clear let w20=- w40, (figure.(6)). It is obvious that the effect of the error would be less than the effect of focus or the spherical errors separately. The peak is changed from 1 for free of aberration to 0.987, 0.951, 0.892, and 0.815. In other words, the Strehl ratio (the ratio of the irradiance at the center of the aberrated diffraction image to that of a perfect image) would exceed.

6.2.2 PSF with the presence of coma aberration

When interring different values of coma aberration, (fig.7) and different values of orientation of axis (ψ=0,π/2,π,3π/2), it found that there were a shift in the peaks proportional to the amount of the coma aberration.

Now return to the normal ellipse, it found that at ψ=0 or π, there were no shift in the peaks while there were a shift increasing with the value of aberration when $\psi=\pi/2$ or 3π/2, and the shift when $\psi=\pi/2$ is the same amount of 3π/2 but in the reverse direction.

This can be explained as follows:

As stated before the equation of coma is

$W=w_{31}(x^2+y^2)(x\sin\psi+y\cos\psi)$

When ψ=0 and π, → $W=\pm w_{31}(x^2+y^2)\,y$

The transverse aberration can be written as [14]

$\varepsilon \propto -\frac{\partial W}{\partial x}$ = =-w31 2xy=$-w_{31}\,r^2\sin(2\theta)$

$\psi=\pi/2$ → $W=w_{31}(x^2+y^2)\,x$

$\varepsilon \propto -w_{31}(3x^2+y^2)=-w_{31}\,r^2(2+\cos(2\theta))$

$\psi=3\pi/2$, $\varepsilon \propto w_{31}(3x^2+y^2)=w_{31}\,r^2(2+\cos(2\theta))$

So, it is obvious from the above three equation that there is no shift in the first case while there is a shift in the latter two cases with a negative shift in the second case and a posetive shift in the third case.

This interpretation became true for inclined aperture but when taking the axis m (u=v) and the angles equal to (π/4, 3π/4, 5π/4, and 7π/4), that's mean there is no shift in the second and fourth cases, while there is a reverse and equal shifts in the first and third cases. (fig. (8))

This can be making clearer by the following:

$W=w_{31}(x^2+y^2)(x\sin\psi+y\cos\psi)$

When ψ=3π/4 or 7π/4, → $W=w_{31}(x2+y2)\,\frac{1}{\sqrt{2}}(x-y)=w_{31}(m^2+n^2)\,\frac{1}{\sqrt{2}}(n)$

The transverse aberration can be written as [13,16]

$\varepsilon \propto -\frac{\partial W}{\partial m} = -w_{31}\frac{1}{\sqrt{2}}\,2mn = -w_{31}\frac{1}{\sqrt{2}}\,\rho^2\sin(2\phi)$

For 7π/4 → $\varepsilon \propto w_{31}\frac{1}{\sqrt{2}}\,\rho^2\sin(2\phi)$

$\psi=\pi/4$ → $W=w_{31}(x^2+y^2)\,\frac{1}{\sqrt{2}}(x+y)=w_{31}(m^2+n^2)\,\frac{1}{\sqrt{2}}(m)$

$\varepsilon \propto -\frac{\partial W}{\partial m} = -w_{31}\frac{1}{\sqrt{2}}(3m^2+n^2) = -w_{31}\frac{1}{\sqrt{2}}\,\rho^2(2+\cos(2\phi))$

For $\psi=5\pi/4$ $= w_{31}\frac{1}{\sqrt{2}}(3m^2+n^2)=w_{31}\frac{1}{\sqrt{2}}\,\rho^2(2+\cos(2\phi))$

As in the case of spherical and focus aberrations, the coma can be balanced with shift error. As in figure (9). In fact, in spite of improving the shifting, the pattern does not really improve. This is explained by James C. Yant [16] that is simply a selecting of a point other than the Gaussian image point to represent the best centre of light concentration in the point image.

As stated before coma on axis must be zero, but the appeared on axis aberration does not depend on field position, so it is not coma and it is an additive term, James c. Yant refer it to tilt or decentred optical components in the system due to misalignment.

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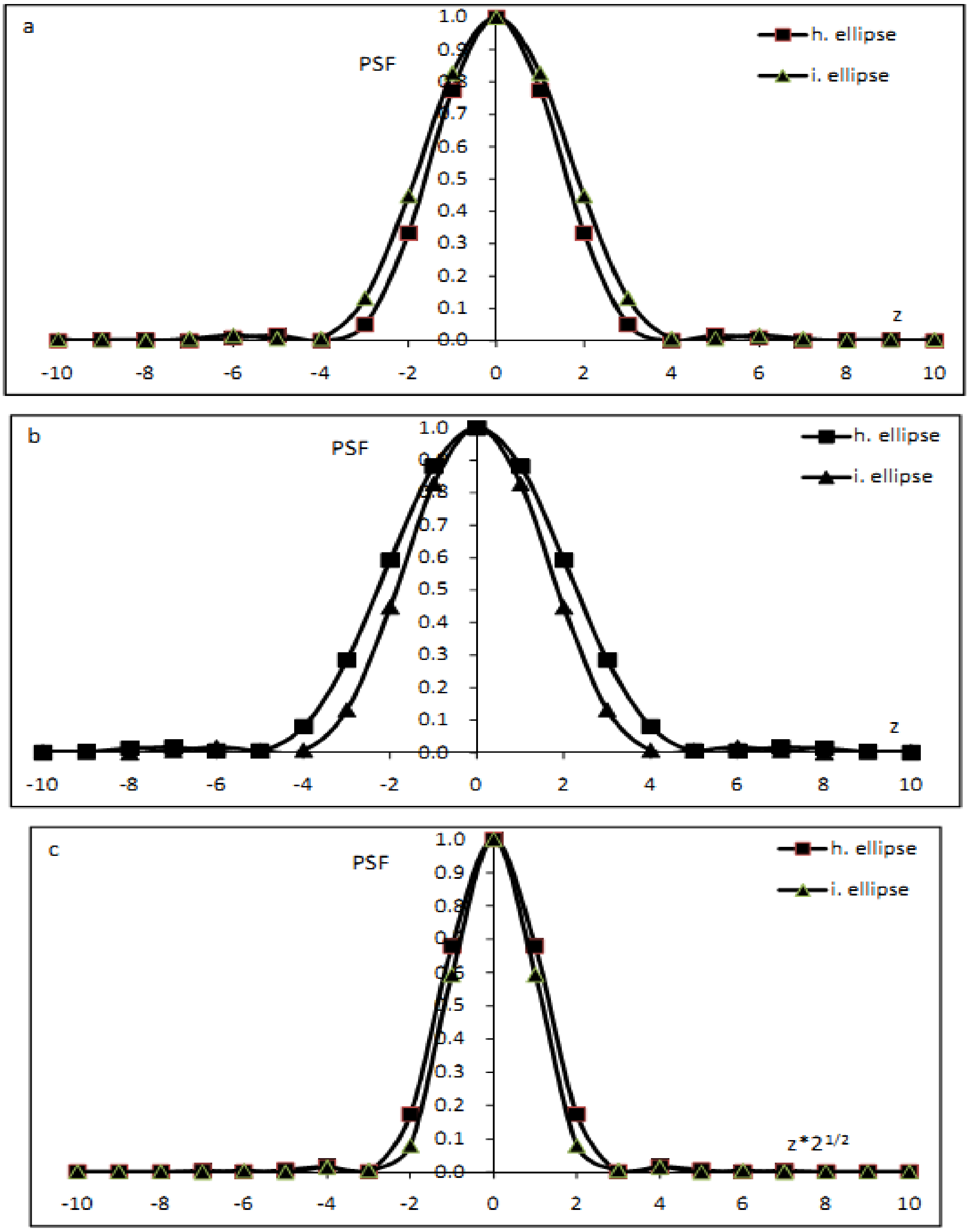

Fig.1: Diffraction limited PSF for horizontal ellipse and inclined ellipse

a) in x- axis (v=0).

b) in y- axis(u=0).

c) in the m-axis (u=v).

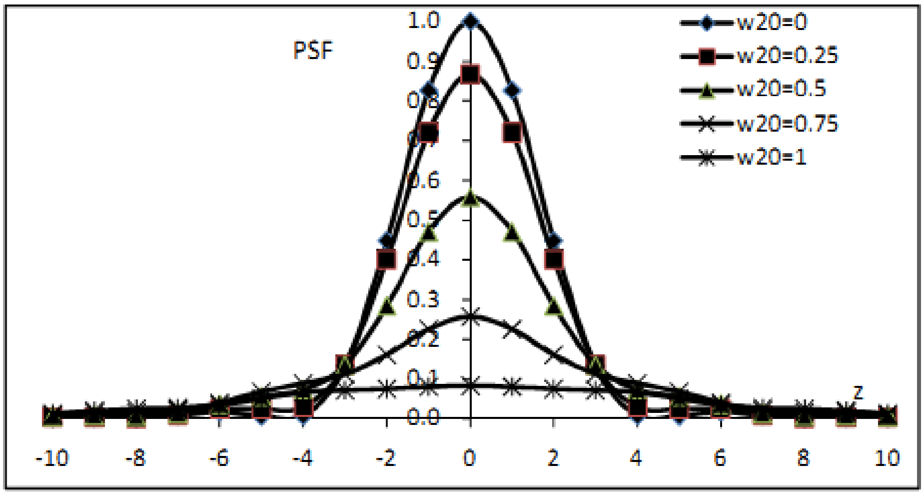

Fig. 4: PSF for inclined ellipse in x- axis with different values of focus error w20.

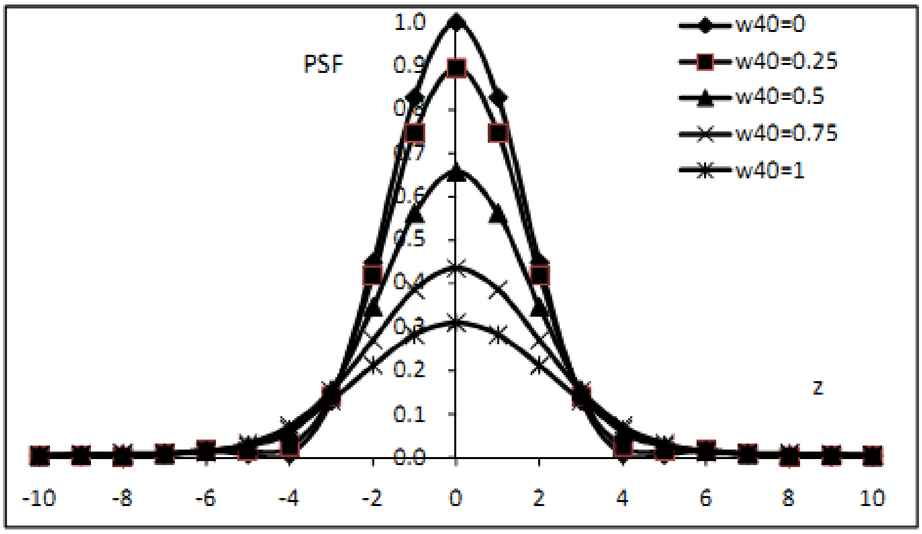

Fig. 5: PSF for inclined ellipse in x- axis with different values of spherical error w40.

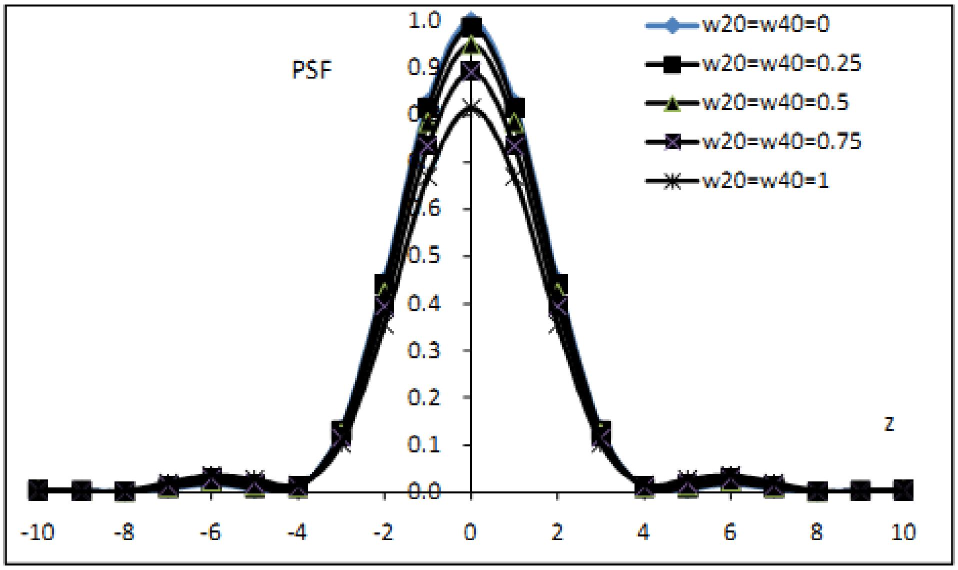

Fig. 6: PSF for inclined ellipse in x- axis with different values w20= -w40.

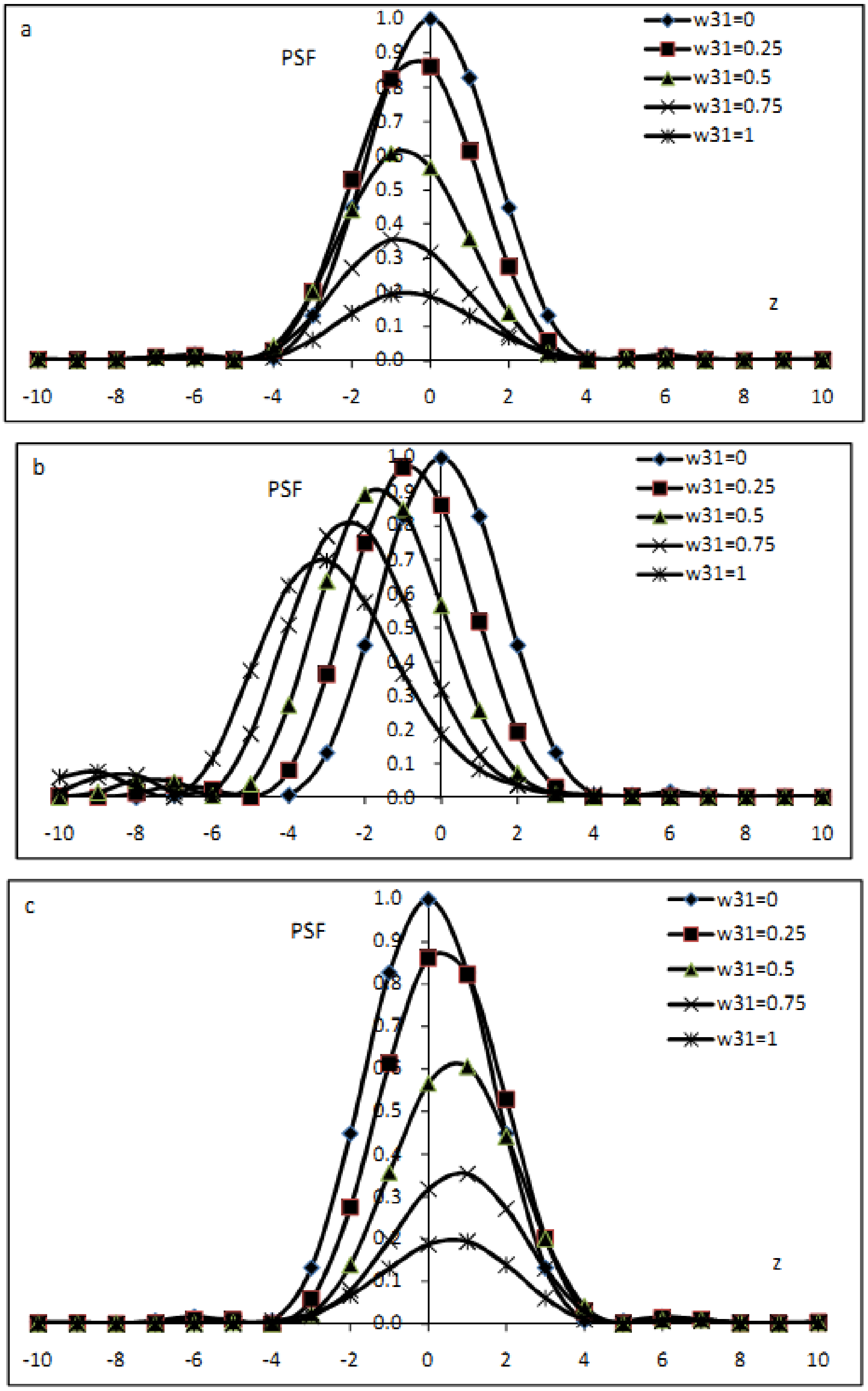

Fig. 7: PSF for inclined ellipse in x- axis with different values of coma.

a) ψ=0 b) ψ=π/2 c) ψ=π

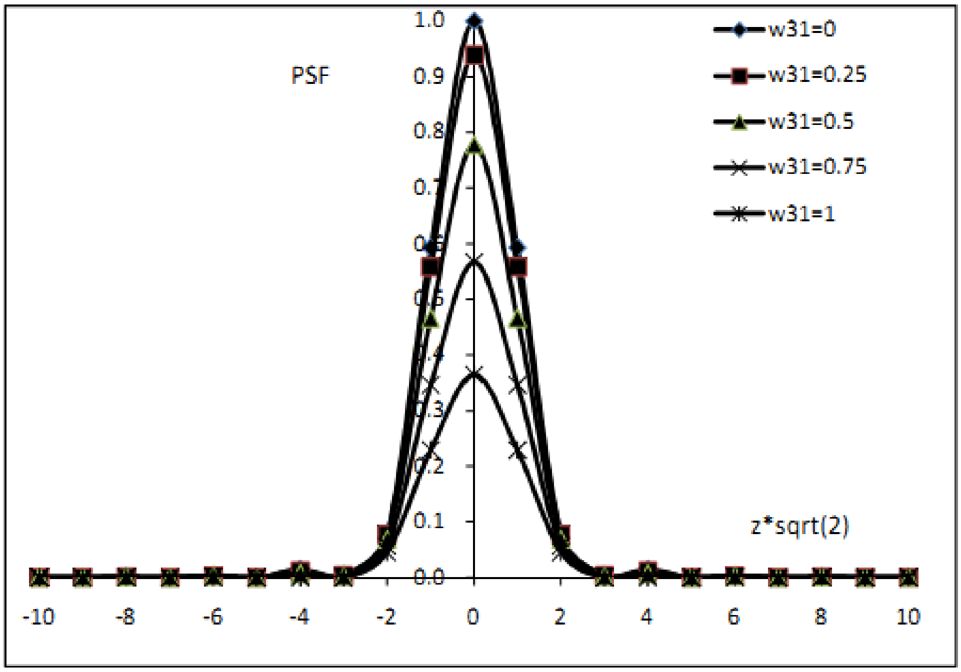

Fig. 8: PSF for inclined ellipse in m- axis with different values of coma. ψ=3π/4

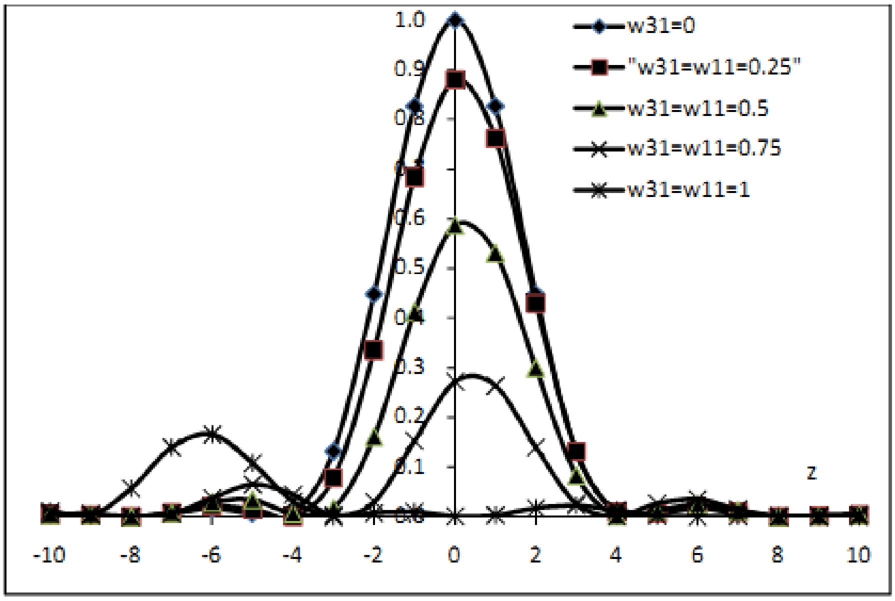

Fig. 9: PSF for inclined ellipse in x- axis with different values of coma w31= -w11. ψ=0,