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Introduction to The Classical Spiral Electrodynamics: The" Spiral-Spin"

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Abstract

This paper demonstrates the existence of analytical solutions of the Lorentz equation for charged particles in "uniform pilot time-varying magnetic fields". These analytical solutions represent a temporal generalization of the Larmor's orbits and are expressed through a Schwarz-Christoffel spiral mapping or in spiral coordinates.

The concepts of "spiral-spin" moment and "polar-spiral" angular momentum are then presented, the existence of a subclass of solutions for which these two angular moments are conserved is demonstrated.

It is also shown that under the action of the "pilot fields," there exist particular trajectories for which the charged particles have a "spiral-spin" momentum constant proportional to +1/2 (solution named "spiral-spin-up ↑ ") and -1/2 (solution named "spiral-spin-down ↓ "), respectively.

The results are in full agreement with the ideas of L.DeBroglie and A. Einstein on the possible existence of pilot fields able to describe the physical reality deterministically.

Finally, the solution of the Lorentz equation is discussed with the WKB (Wentzel-Kramers-Brillouin) method for a superposition of two uniform magnetic fields with the same direction, the first constant and the second time-varying.

Keywords: Larmor Frequency; Lorentz Equation; Maxwell Equations; Chandrasekhar Equation; Newton's Laws; Schwarz-Christoffel Conformal Mapping; Spiral Coordinates; Wentzel-Kramers-Brillouin (WKB) Method.

Subject Classification: QA1-939, QA1-43, QA299.6-433, QA440-699, QC1-999, QC1-75, QC170-197, QC501-766, QC669-675.8, QC750-766, QC770-798, QC793-793.5.

Introduction

It is well known that the angular momentum of a point-like mass under the action of the central force fields is conserved ([1], p. 30, [2] p. 289). The Kepler's problem (see [1] p. 35) is an example of deterministic motion of a point-like mass in a central field for which the angular momentum is conserved. Despite the numerous efforts made at the beginning of the last century to use the Kepler's planetary model to explain atomic phenomena [3,4], deterministic models in this field have not been very successful.

In 1954 Max Born in his Nobel Lecture [5] wrote on the formalism of quantum mechanics:

"While we were still discussing this point, there came the second dramatic surprise, the appearance of Schröedinger's famous papers. He took up quite a different line of thought which had originated from Louis de Broglie .... The exciting dissertation by De Broglie was well known to us in Göttingen."

Following his line of thought, Louis de Broglie (see [6] p. 240-241) proposed instead the idea of a pilot wave capable of guiding the charged particles and that would justify their corpuscular and deterministic nature by completing the theory of quantum mechanics, as later claimed by A.Einstein [7].

This deterministic idea is taken up in this article even if in a different way "pilot wave" indicates an external electromagnetic field able to guide the charge as it happens inside the atom. In other words, it is shown that under the action of a certain class of uniform time-varying magnetic fields, the charged particle follows the spiral trajectories with constant "spiral-spin" angular moment.

The fascinating idea of being able to interact with charged particles and determine their motion pervades all the science, from particle accelerators [8] to free-electron lasers [9,10], nuclear magnetic resonance [11], new spintronics and microelectronics devices [12,14,15,16] and devices that produce ionizing radiation for oncology (see for example [17]).

The possibility of driving charged particles on spiral trajectories can lead to the development of new scientific models, new types of devices, new electromagnetic signals for chemistry, cancer therapy, and nuclear physics.

Recalling Schopenhauer's famous dichotomy: "phenomenon" and "thing-in-itself":

"... the ancient wisdom of the Indian philosophers declares, It is Mâyâ, the veil of deception, which blinds the eyes of mortals, and makes them behold a world of which cannot say either that it is or that it is not...."(see [18] p. 9)

we could say that once the veil of Mâyâ of the "spiral-spin" ("phenomenon") theoretically envisaged in this article is torn apart, we can find the equation of the trajectory of charged particle in the pilot field which is a rational mathematical ("thing-in-itself or noumenon"). In the same way, the "spiral-spin" phenomenon is in full agreement with the dualistic and deterministic theory of De Broglie [6], and we could say that there is a potential "pilot wave" capable of driving a charged particle with a constant "spiral-spin" moment in a deterministic way.

In 1913 Niels Bohr [4] criticizing the planetary model of the Rutherford atom, wrote:

"The inadequacy of the classical electrodynamics in accounting for the properties of atoms from an atom-model as Rutherford's, will appear very clearly if we consider a simple system consisting of a positively charged nucleus of very small dimensions and an electron describing closed orbits around it",

highlighting from the point of view of the classical electrodynamics, the instability of the electron orbits around the nucleus. Moreover, it is worth remembering that, in a draft sent to Rutherford in 1912 (Memorandum [19]), Bohr demonstrated in polar coordinates that the radiating electron would lose energy and collapse on the nucleus within nanoseconds following a spiral orbit.

More than a hundred years after that historic moment that marked the beginning of quantum mechanics, it might be time to begin to review Bohr's "spiral" critique of the Rutherford atom model in light of the new concepts of classical electrodynamics in spiral differential geometry, in short, "spiral electrodynamics".

Materials and Methods

The Lorentz-Chandrasekhar equation for the pilot time-varying magnetic field.

The non-relativistic equation of motion of a particle in an electromagnetic field is given by (see [24,22] in C.G.S.)

$$m\frac{d^2\vec{r}}{dt^2} = e\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right), \qquad (1)$$

Where m is the mass of the particle, and e is its charge.

It is assumed that the magnetic field is uniform in space but as a function of time of the type

$$\vec{B} = B(t)e_z, \tag{2}$$

where e_z is a constant unit vector in the z-direction? On the assumption that there are no net space charges the electric field $\vec{E}$ satisfies the following Maxwell's equations

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0, & Coulomb \quad law, \\ \vec{\nabla}\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t}, & Faraday \quad law, \end{cases} \tag{3}$$

According to the vector identities (see for example, [21, 22]), the solution of eqs. (3) is readily found to be

$$\begin{cases} \vec{E} = -\dfrac{1}{2c}(\vec{n}\times\vec{r})\dfrac{dB}{dt}, \\ \vec{\nabla}\times\left(-\dfrac{1}{2c}(\vec{n}\times\vec{r})\dfrac{dB}{dt}\right) = -\dfrac{1}{c}\dfrac{d\vec{B}}{dt}, \end{cases} \tag{4}$$

where the partial derivative of $\vec{B}$ has been replaced by the total derivative since $\vec{B}$ depends only on time, and we have set $e_z \equiv \vec{n}$, $|\vec{B}| = B$, [23,22].

The magnetic field can be derived from a vector potential $\vec{A}$ and eq. (2) defined by (see [24] p.179)

$$\vec{B} = \vec{\nabla}\times\vec{A} = e_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right), \tag{5}$$

Inserting the vector potential $\vec{A}$ in Faraday's law we obtain the following relation

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \tag{6}$$

Where Φ is the scalar potential.

We denote the Larmor spiral frequency by

$$\omega(t) = \frac{eB(t)}{mc}, \tag{7}$$

Substituting for $\vec{E}$ in according with eq. (7) into eq. (3) yields

$$\dot{\vec{v}} = \omega(t)\vec{v}\times e_z + (1/2)\dot{\omega}(t)\vec{r}\times e_z. \tag{8}$$

For the problem at hand, the motion of the particle along the lines of force is of no particular interest $\vec{B}\perp\vec{n}$, therefore we set $e_z\cdot\vec{v} = 0$, and we investigate the motion of the particle on the x-y plane.

The x and y components of the equation of motion can therefore be written in the form

$$\begin{cases} \vec{r}(t) = x(t)e_x + y(t)e_y, \\ \vec{v}(t) = \dot{\vec{r}}(t) = \dot{x}e_x + \dot{y}e_y, \\ \vec{v}\times\vec{B} = \dot{y}B(t)e_x - \dot{x}B(t)e_y, \\ e_z \times \vec{r} = -ye_x + xe_y. \end{cases} \tag{9}$$

By introducing the complex variable $\xi = x + iy,$ (see [22], p. 39 and [9] p. 5), S. Chandrasekhar obtained the following equation

$$\ddot{\xi} + i\omega\dot{\xi} + \frac{i}{2}\dot{\omega}\xi = 0, \tag{10}$$

If the magnetic field is constant in time, then $\dot{\omega} = 0,$ and eq. (10) reduces to the classical case studied by Larmor [20,22], the solution of which is given by

$$\xi(t) = \xi_0 + \rho e^{i\omega t + \phi}. \tag{11}$$

where ξ_0 represents the center of the circular motion and ρ ϕ are the radius of the gyration of the particle and ϕ its phase, respectively.

Now, let's consider the following time-varying magnetic "pilot field.

$$\vec{B}(t) = \frac{mc}{eg}\frac{\Omega}{1+\Omega(t-t_*)}e_z, \tag{12}$$

where $\Omega[1/s], g, t_*[s],$ are arbitrary constants.

It is worth noting that if $\Omega\Delta t \ll 1,$ eq. (12) is reduced to the classic case studied by Larmor ([20] p.503).

The vector magnetic potential $\vec{A}$ and the electric potentials Φ can be written in the form

$$\begin{cases} \Phi = -xy\dfrac{m}{2e}\dot{\omega}(t) + F(x,y,t), \\ \vec{A} = \dfrac{mc}{e}\omega(t)xe_y + \vec{G}(x,y,t), \\ \vec{\nabla}\cdot\vec{A} = 0, \\ \omega(t) = \dfrac{1}{g}\dfrac{\Omega}{1+\Omega(t-t_*)}. \end{cases} \tag{13}$$

where $F(x,y,t), \vec{G}(x,y,t),$ represent the Coulomb gauge functions (see [24]).

The non-relativistic Lagrangian L of the system becomes (see for example [24] p. 408)

$$\begin{cases} L = m\dfrac{v^2}{2} - e\Phi + e\dfrac{\vec{v}\cdot\vec{A}}{c}, \\ L(x,\dot{x},y,\dot{y},t) = \dfrac{m}{2}(\dot{x}^2+\dot{y}^2) + \dfrac{m}{2}xy\dot{\omega}(t) + mx\dot{y}\omega(t), \end{cases} \tag{14}$$

which is analogous to that proposed by L.DeBroglie (see [6] p.237). A more precise comparison with the Lagrangian proposed by L.DeBroglie requires further studies and the introduction of new spiral conformal mappings.

It is easy to verify that the Euler-Lagrange equations

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x} = 0 = m\ddot{x} - \dfrac{m}{2}y\dot{\omega}(t) - m\dot{y}\omega(t), \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{y}} - \dfrac{\partial L}{\partial y} = 0 = m\ddot{y} + \dfrac{m}{2}x\dot{\omega}(t) + m\dot{x}\omega(t), \end{cases} \tag{15}$$

are equal to the equations of motion of Newton (10) found by Chandrasekhar ([22], p. 39).

To solve the eq. (10) for the "pilot fields" of eq. (12), the following transformation is introduced

$$\begin{cases} \xi = ze^{\gamma}, \qquad \gamma(t) = \left(\dfrac{1}{2}+i\chi\right)\ln\left[1+\Omega(t-t_*)\right], \\ \dot{\xi} = \dot{z}e^{\gamma} + z\left(\dfrac{1}{2}+i\chi\right)\dfrac{\Omega e^{\gamma}}{1+\Omega(t-t_*)}, \\ \ddot{\xi} = \ddot{z}e^{\gamma} + 2\dot{z}\left(\dfrac{1}{2}+i\chi\right)\dfrac{\Omega e^{\gamma}}{1+\Omega(t-t_*)} + z\left(\dfrac{1}{2}+i\chi\right)\left(-\dfrac{1}{2}+i\chi\right)\dfrac{\Omega^2 e^{\gamma}}{\left[1+\Omega(t-t_*)\right]^2}, \end{cases} \tag{16}$$

where χ is a constant to be determined.

Substituting eq. (16) into eq. (10) we obtain the following differential equation

$$\begin{aligned} &\ddot{z} + 2\dot{z}\left(\frac{1}{2}+i\chi\right)\frac{\Omega}{1+\Omega(t-t_*)} + z\left(\frac{1}{2}+i\chi\right)\left(-\frac{1}{2}+i\chi\right)\frac{\Omega^2}{\left[1+\Omega(t-t_*)\right]^2} + \\ &+ i\frac{\Omega}{g}\frac{1}{1+\Omega(t-t_*)}\left[\dot{z} + z\left(\frac{1}{2}+i\chi\right)\frac{\Omega}{1+\Omega(t-t_*)}\right] - \frac{i}{2g}\frac{\Omega^2}{\left[1+\Omega(t-t_*)\right]^2}z = 0, \end{aligned} \tag{17}$$

Let's choose χ so that

$$\left(\frac{1}{2}+i\chi\right)\left(-\frac{1}{2}+i\chi\right) + \frac{i}{g}\left(\frac{1}{2}+i\chi\right) - \frac{i}{2g} = 0, \tag{18}$$

in this way the z component of eq. (17) vanishes.

Since there exist two equivalent possibilities, i.e. $\chi_{\pm}=\frac{1}{2g}\left(-1\pm\sqrt{1-g^2}\right)$, let's proceed with χ_+ (the result is the same with χ_-.

Eq. (17) will become

$$\ddot{z}+\dot{z}\left[1+2i\chi_{+}+\frac{i}{g}\right]\frac{\Omega}{1+\Omega(t-t_*)}=0, \quad (19)$$

which can be solved analytically.

The spiral angular momentum.

In spiral coordinates [25,26,27], the motion of an ideal point-like particle of mass m is described by

$$\begin{cases} x=e^{\frac{\delta}{g}-g\theta}\cos(\delta+\theta+\psi), \\ y=e^{\frac{\delta}{g}-g\theta}\sin(\delta+\theta+\psi), \\ \dot{x}=e^{\frac{\delta}{g}-g\theta}\left[\left(\frac{\dot{\delta}}{g}-g\dot{\theta}\right)\cos(\delta+\theta+\psi)-(\dot{\delta}+\dot{\theta}+\dot{\psi})\sin(\delta+\theta+\psi)\right], \\ \dot{y}=e^{\frac{\delta}{g}-g\theta}\left[\left(\frac{\dot{\delta}}{g}-g\dot{\theta}\right)\sin(\delta+\theta+\psi)+(\dot{\delta}+\dot{\theta}+\dot{\psi})\cos(\delta+\theta+\psi)\right]. \end{cases} \quad (20)$$

therefore two types of spiral angular moments must be defined, the "spiral-spin" moment $\vec{S}$ and the "polar-spiral" moment $\vec{L}$, their sum is the total spiral angular moment $\vec{J}$.

$$\begin{cases} \vec{J}=m\vec{v}\times\vec{r}=m(\dot{x}y-\dot{y}x)e_z=-me^{2\frac{\delta}{g}-2g\theta}(\dot{\delta}+\dot{\theta}+\dot{\psi})e_z=\vec{S}+\vec{L}, \\ \vec{S}=-me^{2\frac{\delta}{g}-2g\theta}(\dot{\delta}+\dot{\theta})e_z, \\ \vec{L}=-me^{2\frac{\delta}{g}-2g\theta}(\dot{\psi})e_z. \end{cases} \quad (21)$$

The sign - is due to the definition of the spiral coordinates.

Results and Discussion

The final solution of eq. (10) is

$$\begin{cases} \boxed{A} \quad \xi(t)=\left[1+\Omega(t-t_*)\right]^{\frac{1}{2}-\frac{i}{2g}}\left[K_1\left(1+\Omega(t-t_*)\right)^{-\frac{\sqrt{1-g^2}}{2g}}+K_2\left(1+\Omega(t-t_*)\right)^{\frac{\sqrt{1-g^2}}{2g}}\right], \quad |g|\neq 1, \\ \boxed{B} \quad \xi(t)=\left[1+\Omega(t-t_*)\right]^{\frac{1}{2}-\frac{i}{2g}}\left[K_3\ln\left(1+\Omega(t-t_*)\right)+K_4\right], \quad |g|=1, \end{cases} \quad (22)$$

where $K_1, K_2, K_3, K_4 \in$ � are complex constants to be determined.

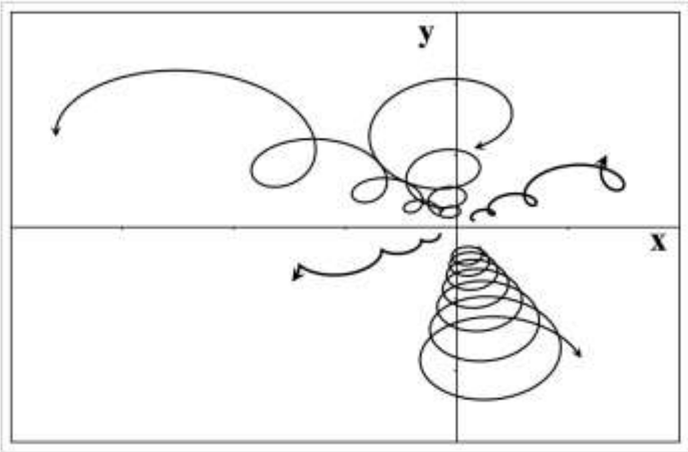

Figure 1. Trajectories of charged particles for various $|g|<1$.

The corresponding velocities are

$$\begin{cases} \boxed{A} \quad \dot{\xi}(t)=\dfrac{\Omega}{2}\left[1+\Omega(t-t_*)\right]^{-\frac{1}{2}-\frac{i}{2g}} \left[\begin{array}{l} K_1\left(\dfrac{1}{2}-\dfrac{i}{2g}-\dfrac{\sqrt{1-g^2}}{2g}\right)\left(1+\Omega(t-t_*)\right)^{-\frac{\sqrt{1-g^2}}{2g}}+ \\ +K_2\left(\dfrac{1}{2}-\dfrac{i}{2g}+\dfrac{\sqrt{1-g^2}}{2g}\right)\left(1+\Omega(t-t_*)\right)^{\frac{\sqrt{1-g^2}}{2g}} \end{array}\right], & |g|\neq 1, \\ \boxed{B} \quad \dot{\xi}(t)=\dfrac{\Omega}{2}\left[1+\Omega(t-t_*)\right]^{-\frac{1}{2}-\frac{i}{2g}}\left[\left(1-\dfrac{i}{g}\right)\left(K_3\ln\left(1+\Omega(t-t_*)\right)+K_4\right)+2K_3\right], & |g|=1, \end{cases}$$

(23)

$|g|=1$ represents a threshold beyond which the trajectories become unstable, i.e. trajectories with the same initial conditions (position, momentum) are much wider and less curvilinear as shown in Fig. 2.

Similar results were reported by Akcasu et al. (see [23]).

Kinetic energy as a function of time is determined by eq. $K.E.=m\dot{\xi}\dot{\bar{\xi}}/2$, i.e.

$$K.E. = \begin{cases} \dfrac{m\Omega^2}{4\left[1+\Omega(t-t_*)\right]} \left\{ \begin{aligned} & |K_1|^2 \left(1+\frac{\sqrt{g^2-1}}{g}\right)\left(1+\Omega(t-t_*)\right)^{\frac{\sqrt{g^2-1}}{g}} + \\ & + \frac{2|K_1||K_2|}{g^2}\cos\left(\arg(K_1)-\arg(K_2)+\arctan\left(\sqrt{g^2-1}\right)\right) + \\ & + |K_2|^2\left(1-\frac{\sqrt{g^2-1}}{g}\right)\left(1+\Omega(t-t_*)\right)^{-\frac{\sqrt{g^2-1}}{g}} \end{aligned} \right\}, & |g|>1, \\ \dfrac{m\Omega^2}{4\left[1+\Omega(t-t_*)\right]} \left\{ \begin{aligned} & |K_3|^2 \ln^2\left(1+\Omega(t-t_*)\right) + \\ & + \left(2|K_3||K_4|\cos\left[\arg(K_3)-\arg(K_4)\right]\right)\ln\left(1+\Omega(t-t_*)\right) + \\ & + 2|K_3|^2 \ln\left(1+\Omega(t-t_*)\right) \\ & + 2|K_3||K_4|\cos\left[\arctan\left(\frac{1}{g}\right)+\arg(K_3)-\arg(K_4)\right] + \\ & + 2|K_3|^2 + |K_4|^2 \end{aligned} \right\}, & |g|=1, \\ \dfrac{m\Omega^2}{4\left[1+\Omega(t-t_*)\right]} \left\{ \begin{aligned} & \frac{|K_1|^2}{g^2}\left(1+\sqrt{1-g^2}\right) + \frac{|K_2|^2}{g^2}\left(1-\sqrt{1-g^2}\right) + \\ & + \frac{2|K_1||K_2|}{g^2}\cos\left[\begin{aligned} & \frac{\sqrt{1-g^2}}{g}\ln\left(1+\Omega(t-t_*)\right)-\arg(K_1) + \\ & + \arg(K_2)+\arctan\left(\frac{\sqrt{1-g^2}}{g}\right) \end{aligned}\right] \end{aligned} \right\}, & |g|<1. \end{cases} \quad (24)$$

The instability effect could be exploited in applications such as ion thrusters [30] or in the field of charged particle accelerators [8].

Instead, temporal sequences of "pilot magnetic signals" with $|g| \le 1$ overturn the "spiral-spin" moment of the charged particles to a value of "+1/2" from an initial "-1/2" or "up" from "down" and vice versa. These "spiral-spin" effects could be used to design new devices similar to spin-valve transistors [12] or for perpendicular magnetic recording [13]. A more detailed analysis of this classic analytical phenomenon requires the use of matrix formalism and a dedicated paper.

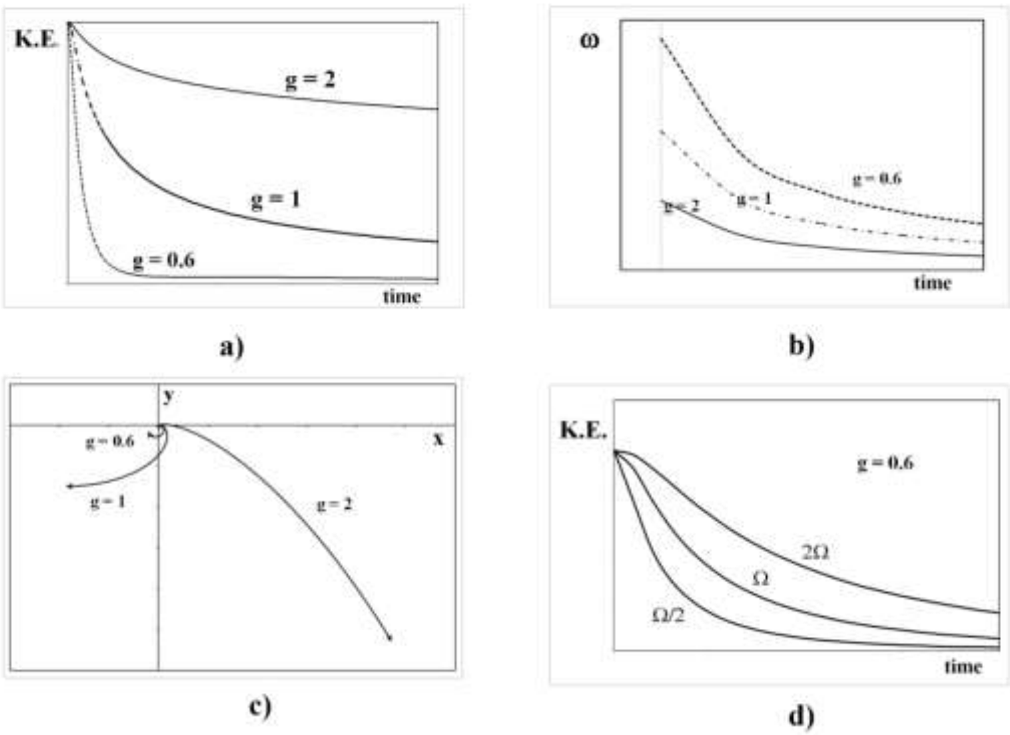

Figure 2. a) Kinetic energies $K.E.$, b) magnetic function $\omega(t)$, and c) trajectories of the charged particles for $|g|<1, |g|=1, |g|>1,$ d) K.E. for various Ω and $g=0.6$.

Constant "spiral-spin" momentum.

For $K_3=0,$ eq. $\left(22\boxed{B}\right)$ can be rewritten as follows

$$\begin{cases} \xi(t)=|K_4|\sqrt{1+\Omega(t-t_*)}\begin{pmatrix}\cos\left(-\frac{1}{2g}\ln\left(1+\Omega(t-t_*)\right)+\arg(K_4)\right)+\\ +i\sin\left(-\frac{1}{2g}\ln\left(1+\Omega(t-t_*)\right)+\arg(K_4)\right)\end{pmatrix}= \\ =e^{\frac{\delta}{g}-g\theta}\left(\cos\left(\delta(t)+\theta(t)+\psi(t)\right)+i\sin\left(\delta(t)+\theta(t)+\psi(t)\right)\right), \\ \theta(t)=-\frac{1}{2g}\ln\left(1+\Omega(t-t_*)\right)-\frac{g}{2}\ln(K_4)=\mu_S\left[\ln\left(1+\Omega(t-t_*)\right)-\ln(K_4)\right], \qquad \mu_S=\pm\frac{1}{2}, \\ \delta(t)=\frac{g}{2}\ln(K_4), \qquad |g|=1, K_3=0, \\ \dot{\xi}(t)=\frac{|K_4|\Omega}{2}\frac{1}{1+\Omega(t-t_*)}\begin{pmatrix}\cos\left(-\frac{1}{2g}\ln\left(1+\Omega(t-t_*)\right)+\arg(K_4)-g\frac{\pi}{4}\right)+\\ +i\sin\left(-\frac{1}{2g}\ln\left(1+\Omega(t-t_*)\right)+\arg(K_4)-g\frac{\pi}{4}\right)\end{pmatrix}, \end{cases}$$

(25)

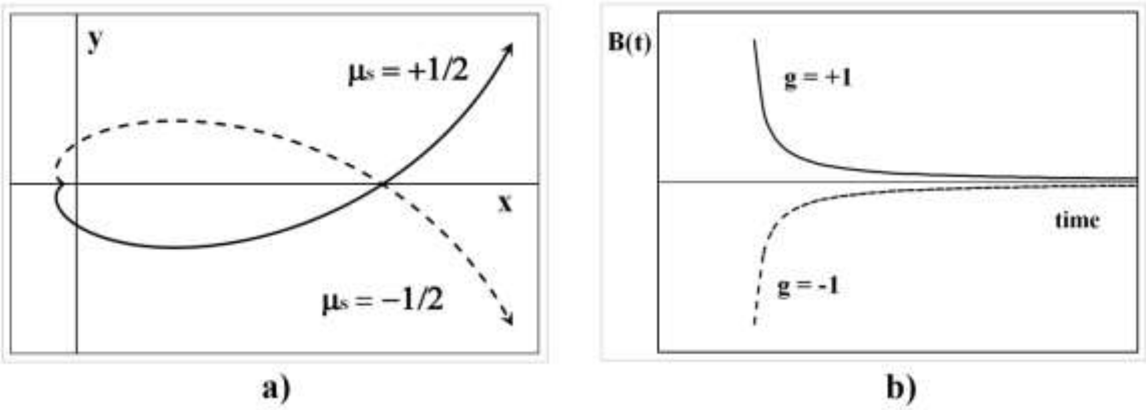

Figure 3. a) "Spiral-spin" trajectories of the charged particle for $\mu_S = +1/2$ and $\mu_S = -1/2$ and b) time-varying magnetic fields with $\Omega > 0, |g| = 1, t_0 = t_*$.

In order to obtain a pure "spiral-spin" solution of eq. (25), eq.(10) of Lorentz-Maxwell-Chandrasekhar must be associated with the following initial (boundary) condition of Cauchy-Robin (see [28] p. 1070 and [29])

$$\dot{\xi}(t_0) - \frac{\omega(t_0)}{2}(g - i)\xi(t_0) = 0, \qquad (26)$$

The spiral motions of eq. (25) are particularly important because according to eq. (21) their angular momentum "polar-spiral" is zero $\vec{L} = 0(\dot{\psi} = 0)$ and their "spiral-spin" moment is constant, i.e.

$$\vec{S} = m|K_4|^2 \Omega \mu_S e_z, \qquad \mu_S = \pm\frac{1}{2}, \qquad (27)$$

Since $g = \pm 1$, the angular momentum "polar-spiral" is proportional to a characteristic parameter μ_S which can be $+1/2$ or $-1/2$, or "spiral-spin up $\uparrow$ " and "spiral-spin down $\downarrow$.

Constant spiral-polar momentum.

Observing eq. $(22\boxed{A})$ it is possible to recognize two other pure "spiral-spin" motions, i.e. $|g| < 1, K_1 \neq 0, K_2 = 0,$ and $K_1 = 0, K_2 \neq 0$. These two solutions of eq. (10) are obtained for the following initial conditions of Cauchy-Robin

$$\begin{cases} \dot{\xi}(t_0) - \frac{\omega(t_0)}{2}\left(g - i + i\sqrt{1-g^2}\right)\xi(t_0) = 0, & \Rightarrow \quad K_1 = 0, \\ \dot{\xi}(t_0) - \frac{\omega(t_0)}{2}\left(g - i - i\sqrt{1-g^2}\right)\xi(t_0) = 0, & \Rightarrow \quad K_2 = 0, \end{cases} \qquad (28)$$

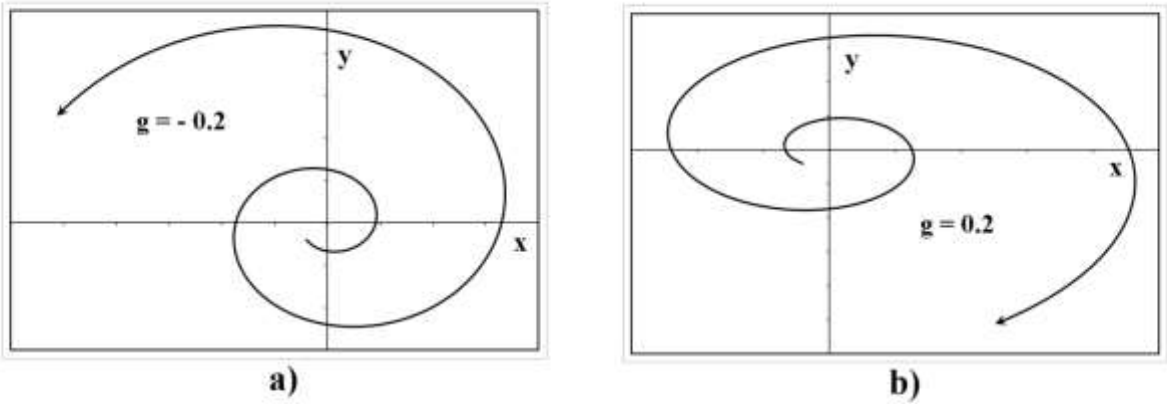

Figure 4. "Spiral-spin" trajectories of the charged particle for $\Omega>0,|g|<1$ **a)** $g=-0,2$ **b)** $g=0,2$.

and can be rewritten as follows

$$\xi(t)=\begin{cases}=|K_1|\sqrt{1+\Omega(t-t_0)}\left(\begin{array}{l}\cos\left[\left(-\dfrac{1+\sqrt{1-g^2}}{2g}\right)\ln\left(1+\Omega(t-t_0)\right)+\arg(K_1)\right]+\\+i\sin\left[\left(-\dfrac{1+\sqrt{1-g^2}}{2g}\right)\ln\left(1+\Omega(t-t_0)\right)+\arg(K_1)\right]\end{array}\right), & \boxed{A_1}\\ \theta(t)=-\dfrac{1}{2g}\ln\left(1+\Omega(t-t_0)\right), \\ |g|<1, K_2=0, K_1\neq 0, \delta=g\ln|K_1|, \\ \psi(t)=\arg(K_1)-\dfrac{\sqrt{1-g^2}}{2g}\ln\left(1+\Omega(t-t_0)\right)-g\ln|K_1|, \end{cases} \tag{29}$$

$$\xi(t)=\begin{cases}=|K_2|\sqrt{1+\Omega(t-t_0)}\left(\begin{array}{l}\cos\left[\left(-\dfrac{1+\sqrt{1-g^2}}{2g}\right)\ln\left(1+\Omega(t-t_0)\right)+\arg(K_2)\right]+\\+i\sin\left[\left(-\dfrac{1+\sqrt{1-g^2}}{2g}\right)\ln\left(1+\Omega(t-t_0)\right)+\arg(K_2)\right]\end{array}\right), & \boxed{A_2}\\ \theta(t)=-\dfrac{1}{2g}\ln\left(1+\Omega(t-t_0)\right), \\ |g|<1, K_2\neq 0, K_1=0, \delta=g\ln|K_2|, \\ \psi(t)=\arg(K_2)+\dfrac{\sqrt{1-g^2}}{2g}\ln\left(1+\Omega(t-t_0)\right)+g\ln|K_2|, \end{cases} \tag{30}$$

According to eq. (21) the total spiral angular momentum $\vec{J}$, which is the sum of $\vec{L}$ and $\vec{S}$, is given by

$$\begin{cases} \vec{J} = \vec{S} + \vec{L}, \\ \vec{J} = \frac{\Omega}{|g|}|K_{1,2}|^2 \mu_J e_z, \\ \vec{S} = \frac{\Omega}{|g|}|K_{1,2}|^2 \mu_S e_z, \\ \vec{L} = \frac{\Omega}{|g|}|K_{1,2}|^2 \mu_L e_z, \\ \mu_S = \pm\frac{1}{2}, \\ \mu_L = \pm\frac{\sqrt{1-g^2}}{2}, \\ \mu_J = \mu_L + \mu_S = \pm\frac{1\pm\sqrt{1-g^2}}{2}, \qquad \boxed{A_{1,2}} \end{cases} \tag{31}$$

A precise comparison between the spiral angular moments and the moments of quantum mechanics requires the introduction of additional three-dimensional mathematical tools that go beyond the context of this paper.

For $|g| > 1$ the spiral coordinates developed so far cannot be used to represent these trajectories, a new type of Schwarz-Christoffel spiral conformal mapping is required.

The potential $\Phi(x(t), y(t), t)$ is given by

$$\Phi(x(t), y(t), t) = \begin{cases} \frac{A}{1+\Omega(t-t_0)}\sin\left(\ln\left[1+\Omega(t-t_0)\right]+\vartheta_1\right), & |g|=1, K_3 \neq 0, K_4 = 0, \\ \frac{A}{1+\Omega(t-t_0)}\sin\left(\left(\frac{1}{g}+\frac{\sqrt{1-g^2}}{g}\right)\ln\left[1+\Omega(t-t_0)\right]+\vartheta_2\right), & |g|<1, K_1 = 0, K_2 \neq 0. \end{cases} \tag{32}$$

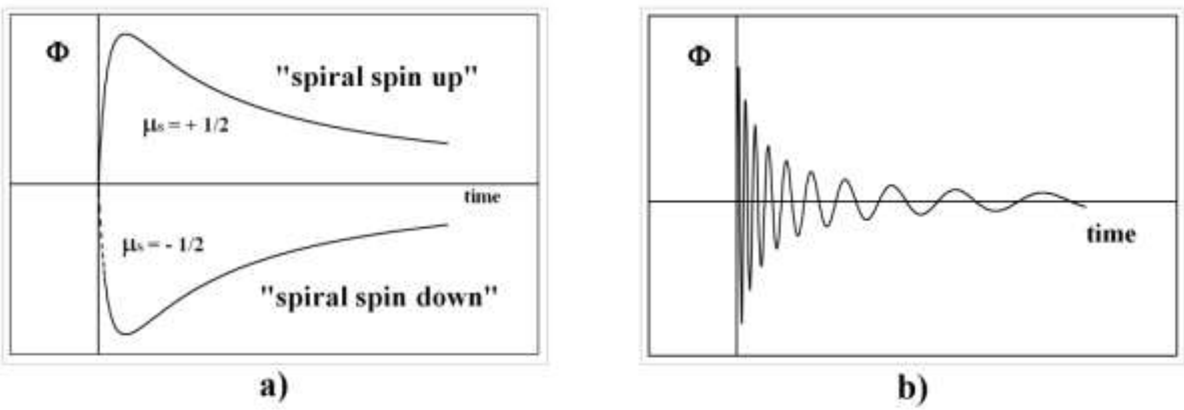

Figure 5. Spiral potentials seen by the charged particle at time t for a) $g = \pm 1, \mu_S = \pm 1/2$ **and b) for** $|g| < 1$.

Larmor's spiral motion.

Now consider the case of a constant and uniform magnetic field $\vec{B} = B_0 e_z$ superimposed on the time-varying magnetic field of eq. (12).

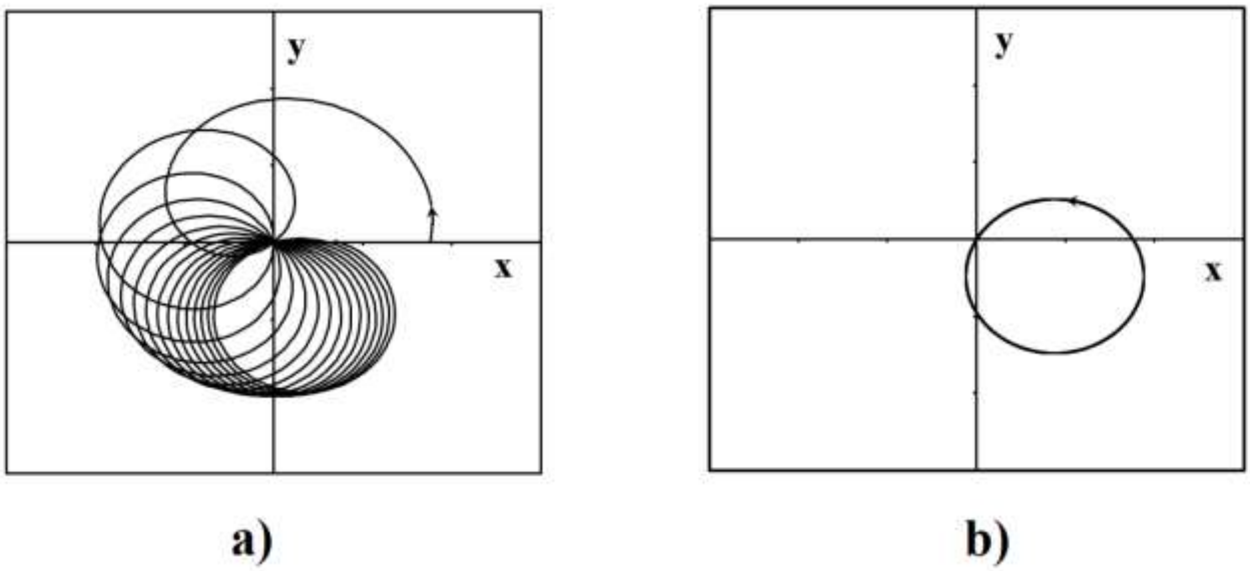

Figure 6. Larmor's spiral motion of a charged particle a) spiral $\Omega = 5$, b) circular $\Omega = 0$.

$$\vec{B}_{LarS}(t) = \left(B_0 + \frac{mc}{eg} \frac{\Omega}{1+\Omega(t-t_*)} \right) e_z, \tag{33}$$

$\vec{B}_{LarS}(t)$ is defined as the spiral magnetic field extension of Larmor's one.

Eq. (10) of Lorentz-Chandrasekhar becomes

$$\begin{cases} \ddot{\xi} + i(\omega(t)+\omega_0)\dot{\xi} + \frac{i}{2}\dot{\omega}\xi = 0, \\ \omega(t) = \frac{\Omega}{g} \frac{1}{1+\Omega(t-t_*)} + \omega_0, \\ \omega_0 = \frac{eB_0}{mc}, \end{cases} \tag{34}$$

where ω_0 is the typical Larmor's frequency (see [20] p.503, [22] p.38).

To solve the eq. (34) the following Chandrasekhar's transformation ([22] p.40) is introduced

$$\begin{cases} \xi = z e^{\gamma}, \\ \gamma(t) = -\frac{i}{2}\int \omega dt = -\frac{i}{2}\left(\omega_0 t + \frac{1}{g}\ln\left[1+\Omega\left(t-t_*\right)\right] + K_\omega \right), \\ \dot{\xi} = \dot{z}e^{\gamma} - \frac{i}{2}\omega z e^{\gamma}, \\ \ddot{\xi} = \ddot{z}e^{\gamma} - i\dot{z}\omega e^{\gamma} - z\frac{i}{2}\dot{\omega}e^{\gamma} - z\frac{1}{4}\omega^2 e^{\gamma}. \end{cases} \tag{35}$$

Where K_ω is a constant of integration.

Eq. (34) can be written as

$$\begin{cases} \varepsilon^2 \ddot{z} + z = 0, \\ \varepsilon = \frac{2}{\left|\omega(t)\right|}. \end{cases} \tag{36}$$

This is the Riccati differential equation which we shall solve with the WKB (Wentzel-Kramers-Brillouin) method.

Under the following condition

$$\begin{cases} \left|\frac{1}{\omega}\frac{d\omega}{dt}\right| \square \left|\frac{\omega}{2}\right|, \\ \omega_0 \square \frac{\Omega}{|g|}\frac{\sqrt{2|g|}-1}{1+\Omega\left(t-t_*\right)}, \end{cases} \tag{37}$$

the asymptotic solution of eq. (36) is given by (see [31] p. 487)

$$z(t) \square c_1 e^{i\frac{\Omega t}{2\left[1+\Omega(t-t_*)\right]}+i\frac{\omega_0}{2}t-i\frac{\pi}{4}} + c_2 e^{-i\frac{\Omega t}{2\left[1+\Omega(t-t_*)\right]}-i\frac{\omega_0}{2}t-i\frac{\pi}{4}}, \tag{38}$$

and consequently the solution of eq. (34) becomes

$$\xi(t) = e^{-\frac{i}{2}\left(\omega_0 t + \frac{1}{g}\ln\left[1+\Omega(t-t_*)\right]+K_\omega\right)}\left(c_1 e^{i\frac{\Omega t}{2\left[1+\Omega(t-t_*)\right]}+i\frac{\omega_0}{2}t-i\frac{\pi}{4}} + c_2 e^{-i\frac{\Omega t}{2\left[1+\Omega(t-t_*)\right]}-i\frac{\omega_0}{2}t-i\frac{\pi}{4}} \right) \tag{39}$$

Typical trajectories of the charged particles are shown in Fig. 6.

It is easy to verify that eq. (39) is reduced to the classical circular case of Larmor for $\Omega = 0$.

Conclusions

The analytical solutions of the Lorentz equation for charged particles have been studied for a particular type of uniform time-varying magnetic fields.

Exploiting the mathematical formalism in the complex plane developed by Chandrasekhar, the analytical solution representing the trajectories of charged particles was determined.

These solutions were expressed using the spiral coordinates, and the concepts of "spiral-spin" and "polar-spiral" moments were introduced.

It has been shown that, for particular Cauchy-Robin conditions, such solutions have a constant angular momentum.

Moreover, the Lagrangian and the electromagnetic potential of these magnetic fields were determined, and a full agreement was reached with the original ideas of De Broglie on the possible existence of a pilot field.

Finally, the solution to the Lorentz equation was studied for a superposition of two uniform magnetic fields, one constant and one time-varying, the characteristics of the Larmor motion extended to the spiral case were found.

Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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