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On a perfect fluid Kähler spacetime

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ABSTRACT

The object of the present paper is to study a perfect fluid Kähler spacetime. A perfect fluid Kähler spacetime satisfying the Einstein field equation with a cosmological term has been studied and the existence of killing and conformal killing vectors have been discussed. Certain results related to sectional curvature for pseudo projectively flat perfect fluid Kähler spacetime have been obtained. Dust model for perfect fluid Kähler spacetime has also been studied.

Indexing terms/Keywords

Kähler spacetime, Einstein equation, pseudo projectively curvature tensor, killing and confermal killing vector field.

Academic Discipline And Sub-Disciplines

Mathematics

SUBJECT CLASSIFICATION

[2010] 53C50, 53C15.

TYPE (METHOD/APPROACH)

Research

1. INTRODUCTION

In 1983, B. O' Neill [1] discussed the application of semi-Riemannian geometry in the theory of relativity. The curvature structure of the spacetime is studied by V. R Kaigorodov [2] in 1983. These ideas of general relativity of spacetime are extended by A. K. Roychaudhury, S. Banerji and A. Banerji [3]. M. C. Chaki and S. Roy [4] studied spacetime with covariant constant energy momentum tensor. A. A. Shaikh, Dae Won Yoon and S. K. Hui [5] studied quasi-Einstein spacetime. In 2002, B. Prasad [6] defined a tensor field on a Riemannian manifold of dimension greater than 2 and he called it pseudo projective curvature tensor. This tensor is a generalisation of projective curvature tensor. Such tensor was studied on a φ-recurrent Kenmotsu manifold by Venkatesha and C. S. Bagewadi [7]. C. S. Bagewadi, Venkatesha and N. S. Basavarajappa [8] studied such tensor on LP-Sasakian manifold while H. G. Nagaraja and G. Somashekhara [9] studied such tensor on a Sasakian manifold.

The pseudo projective curvature tensor for an n-dimensional Riemannian manifold is given by

$$P(X,Y)Z = aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y\right] - \frac{r}{n}\left(\frac{a}{n-1}+b\right)\left[g(Y,Z)X - g(X,Z)Y\right], \tag{1.1}$$

where R, S, r and g are Riemannian curvature tensor, Ricci tensor, scalar curvature and Riemannian metric of the manifold respectively. The Riemannian curvature tensor $R(X,Y)Z$ is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Contracting equation (1.1) with a vector field W, we get

$$P(X,Y,Z,W) = aR(X,Y,Z,W) + b\left[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\right] - \frac{r}{n}\left(\frac{a}{n-1}+b\right)\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right] \tag{1.2}$$

The importance of pseudo projective curvature tensor is well known in differential geometry of certain F-structures. If $a=1$ and $b=-\frac{1}{n-1}$, the pseudo projective curvature tensor becomes projective curvature tensor. A manifold whose pseudo projective curvature vanishes at every point is called a pseudo projectively flat manifold. A pseudo projectively flat

LP-Sasakian manifold was studied by C. S. Bagewadi, D. G. Prakasha and Venkatesha [10]. In this paper, we focus on the study of 4-dimensional Kählerian spacetime manifold.

In a 4-dimensional Kähler spacetime the following conditions hold:

$$\phi^2 = -X, \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X,Y), \tag{1.4}$$

$$(\nabla_X \phi)Y = 0, \tag{1.5}$$

where ϕ is $1-1$ tensor field and g is Riemannian metric. Let us denote $\phi(X)$ by $\overline{X}$. The Ricci tensor in a Kähler manifold satisfies

$$S(\overline{X}, \overline{Y}) = S(X,Y). \tag{1.6}$$

2. PERFECT FLUID KÄHLER SPACETIME

For 4-dimensional Kähler spacetime the Einstein field equation with cosmological constant is given by

$$S(X,Y) - \frac{r}{2} g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{2.1}$$

where λ is cosmological term, k is the gravitational constant and $T(X,Y)$ is the energy momentum tensor given by

$$T(X,Y) = (\mu + p) A(X) A(Y) + pg(X,Y), \tag{2.2}$$

where μ, p and A being respectively energy density, isotropic pressure and 1-form defined by $A(X) = g(X,U)$ for time-like vector field U. The time-like vector field U is called velocity of the fluid and satisfies $g(U,U) = -1.$. Also the energy density μ and the pressure p can be described in the sense that if μ vanishes then the matter of the fluid is not pure and if the pressure p vanishes then the fluid is dust.

From equation (2.1) and (2.2), we get

$$S(X,Y) - \frac{r}{2} g(X,Y) + \lambda g(X,Y) = k\left[(\mu + p) A(X) A(Y) + pg(X,Y)\right]. \tag{2.3}$$

Replacing X by $\overline{X}$ and Y and $\overline{Y}$ in equation (2.3), we get

$$S(X,Y) - \frac{r}{2} g(X,Y) + \lambda g(X,Y) = k\left[(\mu + p) A(\overline{X}) A(\overline{Y}) + pg(X,Y)\right]. \tag{2.4}$$

Subtracting equation (2.3) from (2.4), we have

$$k(\mu + p)\left[A(\overline{X}) A(\overline{Y}) - A(X) A(Y)\right] = 0. \tag{2.5}$$

Putting $Y = U$ in (2.5), we obtained

$$k(\mu + p) A(X) = 0, \tag{2.6}$$

since $k \neq 0$ and $w(X) \neq 0,$ we have

$$(\mu + p) = 0. \tag{2.7}$$

Using (2.7) in equation (2.3), we get

$$S(X,Y)=\left(\frac{r}{2}-\lambda+kp\right)g(X,Y). \tag{2.8}$$

Putting $X=Y=\{e_i\},\ 1\le i\le 4$ in equation (2.8) and taking summation over i we obtained

$$\lambda-kp=\frac{r}{4}. \tag{2.9}$$

From equation (2.8) and (2.9), we get

$$S(X,Y)=\frac{r}{4}g(X,Y). \tag{2.10}$$

Hence the perfect fluid Kähler spacetime is an Einstein manifold.

Again, using equation (2.7) in (2.2) we get

$$T(X,Y)=pg(X,Y). \tag{2.11}$$

Definition 2.1: A vector field ξ is killing vector field if and only if

$$\pounds_\xi g(X,Y)=0, \tag{2.12}$$

where $\pounds_\xi$ denotes the Lie derivative with respect to the vector ξ.

Taking Lie derivative of both sides of equation (2.11), we get

$$\pounds_\xi T(X,Y)=p\pounds_\xi g(X,Y). \tag{2.13}$$

If ξ is killing vector field, we have (2.12). In view of (2.12), equation (2.13) gives

$$\pounds_\xi T(X,Y)=0. \tag{2.14}$$

Conversely, suppose that equation (2.14) holds. Equation (2.13) gives $\pounds_\xi g(X,Y)=0$, provided $p\neq 0.$ Hence ξ is a killing vector field.

This leads to:

Theorem 2.2: In a perfect fluid Kähler spacetime admitting the Einstein field equation with a cosmological term, there exists a killing vector field ξ if the Lie derivative of the energy momentum tensor with respect to ξ vanishes. Conversely if the Lie derivative of the energy momentum tensor with respect to ξ vanishes then ξ is killing vector field provided $p\neq 0.$

Definition 2.3: A vector field ξ satisfying equation

$$\pounds_\xi g(X,Y)=2\Omega g(X,Y), \tag{2.15}$$

where Ω is a scalar function, is called a conformal killing vector field [11]. A spacetime satisfying equation (2.15) is said to admit a conformal motion.

From equations (2.13) and (2.15), we get

$$2\Omega g(X,Y)=\pounds_\xi T(X,Y). \tag{2.16}$$

Using equation (2.11) in (2.16), we get

$$\pounds_\xi T(X,Y)=2\Omega T(X,Y). \tag{2.17}$$

From equation (2.16) we can conclude that the energy momentum tensor has the symmetry inheritance property [12,13]. Thus, we can state:

Theorem 2.6: A perfect fluid Kähler spacetime manifold satisfying the Einstein field equation with a cosmological term admits a conformal killing vector if and only if the energy momentum tensor has the symmetric inheritance property.

3. PSEUDO PROJECTIVELY FLAT PERFECT FLUID KÄHLER SPACETIME

Let V_4 be the spacetime of general relativity. The pseudo projective curvature tensor for this space assumes the following form;

$$P(X,Y,Z,W) = aR(X,Y,Z,W) + b\left[S\left(Y,Z\right)g(X,W) - S\left(X,Z\right)g(Y,W)\right]$$
$$-\frac{r}{4}\left(\frac{a}{3}+b\right)\left[g\left(Y,Z\right)g(X,W) - g\left(X,Z\right)g(Y,W)\right]. \tag{3.1}$$

If $P(X,Y,Z,W)=0,$, then equation (3.1) gives

$$R(X,Y,Z,W) = \frac{b}{a}\left[S\left(X,Z\right)g(Y,W) - S\left(Y,Z\right)g(X,W\right]$$
$$+\frac{r}{4a}\left(\frac{a}{3}+b\right)\left[g\left(Y,Z\right)g(X,W) - g\left(X,Z\right)g(Y,W)\right], \tag{3.2}$$

Putting the value of Ricci tensor S from (2.10) in (3.2), we have

$$R(X,Y,Z,W) = \frac{r}{12}\left[g\left(Y,Z\right)g(X,W) - g\left(X,Z\right)g(Y,W)\right] \tag{3.3}$$

Thus we can state:

Theorem 3.1: A pseudo projectively flat perfect fluid Kähler spacetime is of constant curvature.

Let $U^{\perp}$ be the 3-dimensional distribution in a pseudo projectively flat perfect fluid spacetime orthogonal to U. Then by putting $Z=Y$ and $T=X$ in (3.3) we have

$$R(X,Y,Y,X) = \frac{r}{12}\left[g\left(Y,Y\right)g(X,X) - g\left(X,Y\right)g(Y,X)\right],$$

where $X,Y \in U^{\perp}$. Putting $Y=U$, we have

$$R(X,U,U,X) = -\frac{r}{12}g(X,X),$$

where $X \in U^{\perp}$. If we denote the sectional curvatures by $K\left(X,Y\right)$ and $K\left(X,U\right)$ determined by X,Y and X,U respectively then from the above equations, we have

$$K(X,Y) = \frac{R(X,Y,Y,X)}{g\left(X,X\right)g(Y,Y) - g\left(X,Y\right)g(X,Y)} \tag{3.4}$$
$$= \frac{r}{12},$$

and

$$K(X,U) = \frac{R(X,U,U,X)}{g\left(X,X\right)g(U,U) - g\left(X,U\right)g(X,U)}, \tag{3.5}$$
$$= \frac{r}{12}.$$

Thus we have:

Theorem 3.2: In a pseudo projectively flat perfect fluid Kähler spacetime the sectional curvature determined by $X,Y \in U^{\perp}$ and sectional curvature determined by two vectors X and U are same and equal to $\frac{r}{12}$.

A Lorentzian manifold was called by Karcher [14] infinitesimally isotropic relative to the velocity vector field U if the Riemannian curvature tensor R satisfies

$$R(X,Y,Y,X) = l\left[g\left(Y,Z\right)g(X,W) - g\left(X,Z\right)g(Y,W)\right], \tag{3.6}$$

and

$$R(X,U,U,Y) = mg(X,Y), \tag{3.7}$$

where l and m are real valued functions and $X,Y,Z,U \in U^{\perp}$. Putting $Y = Z = U$ in (3.3), we get

$$R(X,U,U,W) = -\frac{r}{12} g(X,W). \tag{3.8}$$

Hence from (3.7) and (3.8), we can state:

Theorem 3.3: A pseudo projectively flat perfect fluid Kähler spacetime obeying the Einstein field equation with a cosmological constant and having the vector field U as the velocity vector field is infinitesimally spatially isotropic relative to the unit time-like vector field U .

Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2} g(X,Y), = kT(X,Y), \tag{3.9}$$

where r be the scalar curvature of the manifold and $k \neq 0$.

Using equation (2.11), we get

$$S(X,Y) - \frac{r}{2} g(X,Y) = kpg(X,Y). \tag{3.10}$$

Putting $X = Y = \{e_i\}$ in equation (3.10) and taking the summation over $i,\ 1 \le i \le 4$, we get

$$r = -4kp. \tag{3.11}$$

Using equation (3.11) in (3.10), we obtained

$$S(X,Y) = -kpg(X,Y). \tag{3.12}$$

The Einstein field equation for purely electromagnetic distribution [15] in Kähler spacetime is given by

$$S(X,Y) = kpg(X,Y). \tag{3.13}$$

From equations (3.12) and (3.13), we get

$$p = 0.$$

Putting $p = 0$ in (2.11) gives $r = 0$.

Using $r = 0$ in equation (3.3), we get

$$R(X,Y,Z,W) = 0,$$

which shows that the spacetime is flat. Hence, we have:

Theorem 3.4: A pseudo projectively flat Kähler spacetime satisfying the Einstein field equation without cosmological term is an Euclidean space.

Note: This theorem points towards the conditions under which a Riemannian space can be reduced to a Euclidean space [16].

4. DUST MODEL FOR PERFECT FLUID KÄHLER SPACETIME

The energy momentum tensor for dust model (i.e isotropic pressure $p = 0$ and $\mu = 0$) is

$$T(X,Y) = \mu A(X)A(Y). \tag{4.1}$$

Using equations (2.11) and (4.1), we get

$$pg(X,Y) = \mu A(X)A(Y). \qquad (4.2$$

Putting $X = Y = e_i$ in (4.2) and taking the summation over $i,\ 1 \le i \le 4$, we get

$$4p = \mu. \tag{4.3}$$

Now putting $X = Y = U,$, in equation (4.2), we obtain

$$-p = \mu. \tag{4.4}$$

From equations (4.3) and (4.4), we get $\mu = 0$.

Thus, we have:

Theorem 4.1: A perfect fluid Kähler spacetime satisfying the Einstein field equation with a cosmological term represent a dust cosmological model if $\mu = 0$.

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