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On the mean curvature of spacelike hypersurfaces in spatially parabolic GRW spacetimes

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Abstract:

Several uniqueness results for complete spacelike hypersurfaces in Generalized Robertson-Walker (GRW) spacetimes whose fiber has a parabolic universal Riemannian covering are proved under boundedness assumptions on the mean curvature function and suitable geometric assumptions.

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1 Introduction

In this paper we study spacelike hypersurfaces in a GRW spacetime $I \times_f F$, i.e., a spacetime consisting as the warped product of a negative definite interval I as a base and a Riemannian manifold F as a fiber, moreover, a positive smooth function f as a warped function (see section 2). Furthermore, if the fiber F is a compact (without boundary) Riemannian manifold, then the GRW spacetime is spatially closed. On the other hand, when the fiber is complete and non-compact, the GRW spacetime is saide to be spatially open. In last case, if moreover the fiber is parabolic then we will say that the GRW spacetime is spatially parabolic (see [6]). In a geometric-analytic point of view, the study of the parabolicity of the fiber of a GRW is very wealth. Before providing further details of our work, we give a brief outline of some recent results related to ours.

In [6], Romero, Rubio and Salamanca provided some uniqueness results for the maximal case in spatially parabolic GRW spacetimes whose fiber is a parabolic Riemannian manifold. Moreover, using the generalized maximum principle, the same authors obtain in [7] new uniqueness results in other relevant spatially open GRW spacetimes for complete maximal hypersurfaces which are between two spacelike slices (time bounded) and have a bounded hyperbolic angle. Latter, in [5] some new uniqueness and non-existence results for complete maximal hypersurfaces in spatially open Robertson-Walker spacetimes whose fiber is flat. Recently, [8, 2, 3, 9, 10] proved some uniqueness results for complete spacelike hypersurfaces in GRW spacetimes.

Recall that a complete Riemannian manifold is said to be parabolic, if its only positive superharmonic functions are the constants. The main assumption along this paper is that the universal Riemannian covering of the fiber in the GRW spacetime is parabolic. In order to obtain a appropriate description of this hypothesis, we may say that a GRW spacetime is spatially parabolic covered if its universal Lorentzian covering is spatially parabolic. Notice that, for a GRW spacetime $\overline{M} = I \times_f F$, its universal Lorentzian covering is $\widetilde{M} = I \times_f \widetilde{F}$, where $\widetilde{F}$ is the universal Riemannian covering of the fiber F in $\overline{M}$. Therefore, a spatially parabolic covered GRW spacetime is, indeed, spatially parabolic.

Our paper is organized as follows. Section 2 is devoted to introduce some notions used to describe GRW spacetimes and spacelike hypersurfaces immersed in these ambient spaces. In section 3, under some suitable geometric conditions, we prove uniqueness results (Theorem 3.2, Theorem 3.5). Note that in these results, mean curvature H is not assumed to be constant. These results specialize to constant mean curvature spacelike hypersurfaces; in particular, all extend a known result in the maximal case (Corollary 3.4, Remark 3.6).

2 Preliminaries

Let (F, g_F) be an n-dimensional (connected) Riemannian manifold, I an open interval in $\mathbb{R}$ endowed with the metric $-dt^2$, and f a positive smooth function define on I. Then, the product manifold $I \times F$ can be endowed with the

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Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\pi_F^*(g_F), \tag{1}$$

where π_I and π_F denote the projections onto I and F, respectively. This spacetime, denoted by $\overline{M} = I \times_f F$, is a warped product in the sense of ([4], Chap.7), with fiber (F, g_F), base $(I, -dt^2)$ and warping function f. We will refer to M as a *generalized Robertson-Walker* (GRW) spacetime (see [1]).

The coordinate vector field $\partial_t := \partial/\partial t$ globally defined on $\overline{M}$ is (unitary) timelike, and so $\overline{M}$ is time-orientable. We will also consider on $\overline{M}$ the conformal closed timelike vector field $K := f(\pi_I)\partial_t$. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of the base and the fiber [4] (see Corollary 7.35), it follows that

$$\overline{\nabla}_X K = f'(\pi_I)X, \tag{2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of the Lorentzian metric (1). Thus, K is conformal with $\mathcal{L}_K\langle,\rangle = 2f'(\pi_I)\langle,\rangle$ and its metrically equivalent 1-form is closed.

Given an n-dimensional manifold M, an immersion $\psi : M \to \overline{M}$ is said to be spacelike if the Lorentzian metric (1) induces, via ψ, a Riemannian metric g on M. In this case, M is called a spacelike hypersurface.

Since $\overline{M}$ is time-orientable we can take, for each spacelike hypersurface M in $\overline{M}$, a unique unitary timelike vector field $N \in \mathfrak{X}^\perp(M)$ globally defined on M with the same time-orientation as $-\partial_t$, i.e., such that $\overline{g}(N, -\partial_t) < 0$. From the wrong-way Cauchy-Schwarz inequality (see [4], Proposition 5.30, for instance), we have $\overline{g}(N, \partial_t) \geq 1$, and the equality holds at a point $p \in M$ if and only if $N = -\partial_t$ at p.

For a spacelike hypersurface $\psi : M \to \overline{M}$ with Gauss map N, the hyperbolic angle φ, at any point of M, between the unit timelike vectors N and $-\partial_t$, is given by $\overline{g}(N, \partial_t) = \cosh\varphi$. By simplicity, throughout this paper we will refer to φ as the hyperbolic angle function on M.

We will denote by A and $H := -(1/n)\text{trace}(A)$ the shape operator and the mean curvature function associated to N. A spacelike hypersurface with constant H is called a constant mean curvature (CMC) spacelike surface. A spacelike hypersurface with $H = 0$ is called a maximal hypersurface.

In any GRW spacetime $\overline{M}$ there is a notable family of spacelike hypersurfaces, namely its spacelike slices $\{t_0\} \times F$, $t_0 \in I$. It can be easily seen that a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the function $\tau := \pi_I \circ \psi$ is constant. Furthermore, a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the hyperbolic angle φ vanishes identically. The shape operator of the spacelike slice $\tau = t_0$ is given by $A = (f'(t_0)/f(t_0))I$ and $H = -f'(t_0)/f(t_0)$. Therefore, the slices are totally umbilic CMC surfaces.

Denoting $\partial_t^T := \partial_t + \overline{g}(N, \partial_t)N$ the tangential component of ∂_t along an arbitrary spacelike hypersurface M, then it is easy to check that the gradient of τ on M is given by

$$\nabla\tau = -\partial_t^T. \tag{3}$$

From this equation, we have

$$|\partial_t^T|^2 = g(\nabla\tau, \nabla\tau) = \sinh^2\varphi. \tag{4}$$

On the other hand, if we denote by $\overline{\nabla}$ the Levi-Civita connection of $\overline{M}$, then the Gauss and Weingarten formulas for the immersion ψ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - g(AX, Y)N, \tag{5}$$

and

$$AX := -\overline{\nabla}_X N, \tag{6}$$

for all $X, Y \in \mathfrak{X}(\overline{M})$, where ∇ is the Levi-Civita connection on M.

From (2) and the Gauss and Weingarten formulas (5), (6), we get

$$\nabla_X K^T = -f(\tau)\overline{g}(N, \partial_t)AX + f'(\tau)X \tag{7}$$

where $K^T := f(\tau)\partial_t^T = K + \overline{g}(K, N)N$ the tangential component of K along ψ, and $f(\tau) := f \circ \tau$ and $f'(\tau) := f' \circ \tau$. From (7), we have

$$f(\tau)\text{div}(\partial_t^T) + \overline{g}(\nabla f(\tau), \partial_t^T) + f(\tau)\overline{g}(N, \partial_t)tr(A) = nf'(\tau) \tag{8}$$

where div denotes the divergence operator on M.
From (8) and (3), we get

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\{n + |\nabla\tau|^2\} - nH\bar{g}(N, \partial_t) \tag{9}$$

where $\triangle$ denotes the Laplacian operator on M.
Consequently

$$\Delta f(\tau) = -n\frac{f'(\tau)^2}{f(\tau)} + |\nabla\tau|^2 f(\tau)(\log f)''(\tau) - nHf'(\tau)\cosh\varphi \tag{10}$$

3 Mean curvature of spacelike hypersurfaces

In any GRW spacetime $\overline{M}$, there is a natural foliation whose leaves, the level surfaces of the time coordinate of $\overline{M}$, constitute a distinguished family of spacelike hypersurfaces in $\overline{M}$: its spacelike slices. This article is devoted to characterize this family from several points of view. The key starting point is the fact that on any spacelike hypersurface M, the restriction, $f(\tau)$, of the warping function f of $\overline{M}$ satisfies a differential equation, (10), which, under suitable assumptions, leads to the function $f(\tau)$ on M to be superharmonic. Therefore, provided that M is parabolic, we can conclude that $f(\tau)$ is constant on M, which implies that τ is constant on M when the GRW spacetime is proper.
Moreover, on the ambient GRW spacetimes which admit a complete parabolic spacelike hypersurface, we state the following technical result (see [6], Theorem 4.4 for the proof),

Lemma 3.1. *Let M be a complete spacelike hypersurface in a spatially parabolic covered GRW spacetime. If the hyperbolic angle of M is bounded and the warping function on M satisfies:*
(i) $\sup f(\tau) < \infty$, *and*
(ii) $\inf f(\tau) > 0$,
then, M is parabolic.

The following, considering a spacelike hypersurface M with mean curvature H in a spatially parabolic covered GRW spacetime $\overline{M}$, which obeys $(\log f)''(\tau) \leq 0$. Under a stronger assumption on H, we can derive the following uniqueness result.

Theorem 3.2. *Let M be a complete spacelike hypersurface in a spatially parabolic covered GRW spacetime $\overline{M} = I \times_f F$ which obeys $(\log f)''(\tau) \leq 0$. Suppose that $\sup f(\tau) < \infty$ and $\inf f(\tau) > 0$. If the hyperbolic angle φ of M is bounded, and the mean curvature H of M satisfies $H^2 \leq \frac{1}{\cosh^2\varphi}\frac{f'(\tau)^2}{f(\tau)^2}$, then M must be a spacelike slice.*

Proof. From (10), using $-2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}$ with $a = \frac{f'(\tau)}{f(\tau)}$, $b = H\cosh\varphi$, we get

$$\begin{aligned}\frac{1}{f(\tau)}\Delta f(\tau) &\leq -n\frac{f'(\tau)^2}{f(\tau)^2} + (\log f)''(\tau)|\nabla\tau|^2 + \frac{n}{2}(\frac{f'(\tau)^2}{f(\tau)^2} + H^2\cosh^2\varphi)\\ &= \frac{n}{2}(H^2\cosh^2\varphi - \frac{f'(\tau)^2}{f(\tau)^2}) + (\log f)''(\tau)|\nabla\tau|^2\end{aligned}$$

Thus, $f(\tau)$ is a positive superharmonic function from our hypothesis. On the other hand, M is parabolic using Lemma 3.1. Therefore $f(\tau)$ must be constant and then M is necessarily a spacelike slice. □

Consequently, we have

Corollary 3.3. *The only constant mean curvature spacelike hypersurfaces M in a spatially parabolic covered GRW spacetime $\overline{M} = I \times_f F$ which obeys $(\log f)''(\tau) \leq 0$. Suppose that $\sup f(\tau) < \infty$ and $\inf f(\tau) > 0$. If the hyperbolic angle φ of M is bounded, and the mean curvature H of M satisfies $H^2 \leq \inf_M(\frac{1}{\cosh^2\varphi}\frac{f'(\tau)^2}{f(\tau)^2})$, then M must be spacelike slices.*

In particular for the complete maximal spacelike hypersurfaces we have that:

Corollary 3.4. *The only complete maximal hypersurfaces M in a spatially parabolic covered GRW spacetime $\overline{M} = I \times_f F$ which obeys $(\log f)''(\tau) \leq 0$. Suppose that $\sup f(\tau) < \infty$ and $\inf f(\tau) > 0$. If the hyperbolic angle φ of M is bounded, then M must be spacelike slices $\tau = \tau_0$ with $f'(\tau_0) = 0$.*

Theorem 3.5. *Let M be a complete spacelike hypersurface in a spatially parabolic covered GRW spacetime $\overline{M}$ which obeys $f''(\tau) \leq 0$. Suppose that $\sup f(\tau) < \infty$ and $\inf f(\tau) > 0$. If the hyperbolic angle φ of M is bounded, and the mean curvature H of M satisfies $H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}$, then M must be a spacelike slice.*

Proof. A similar reasoning as in Theorem 3.2 gives

$$\begin{aligned}\frac{1}{f(\tau)}\Delta f(\tau) &\leq -n\frac{f'(\tau)^2}{f(\tau)^2} + (\log f)''(\tau)|\nabla\tau|^2 + \frac{n}{2}(\frac{f'(\tau)^2}{f(\tau)^2} + H^2\cosh^2\varphi)\\ &\leq \frac{n}{2}(H^2 - \frac{f'(\tau)^2}{f(\tau)^2})\cosh^2\varphi + \frac{f''(\tau)}{f(\tau)}|\nabla\tau|^2 + \frac{n-2}{2}\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\\ &\leq \frac{n}{2}(H^2 - \frac{f'(\tau)^2}{f(\tau)^2})\cosh^2\varphi + \frac{f''(\tau)}{f(\tau)}|\nabla\tau|^2\end{aligned}$$

Thus, $f(\tau)$ is a positive superharmonic function from our hypothesis. On the other hand, M is parabolic using Lemma 3.1. Therefore $f(\tau)$ must be constant and then M is necessarily a spacelike slice. □

Remark 3.6. *Taking into account that, under the assumptions of* Corollary 3.4*, the condition* $(\log f)''(\tau) \leq 0$ *necessarily implies the condition* $f''(\tau) \leq 0$*, we can also derive it as a consequence of* Theorem 3.5*. Moreover, some analogous results to* Corollary 3.3 *can be also stated from* Theorem 3.5*.*

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