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A New Quantitative Characterization Of The Monster Group M And The Baby Monster Group B

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Abstract

Assuming G is a finite group, $\pi(G)$ is the set of prime factors of the order of G, and p_m is the largest element in $\pi(G)$ and $|S_{p_m}(G)|$ denotes the order of the Sylow p_m-subgroups of G. In this paper, we will give a quantitative characterization of the Monster group M and the Baby Monster group B via the even-order components of the group and $|S_{p_m}(G)|$.

Keywords: Finite groups; Finite Simple groups; Order components; Order of Sylow subgroups.

Mathematics Subject Classification: 20D34, 20E45

1 Introduction

Given a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the set of order of elements of G is denoted by $\pi_e(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices p and q are adjacent if and only if pq in $\pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_1, for $i = 1, 2\cdots, t(G)$. If G is of even order, it is stipulated that $2 \in \pi_1(G)$. If $\pi_1, \pi_2\cdots, \pi_{t(G)}$ are all the connected components of G's prime graph, then $|G| = m_1m_2\cdots m_{t(G)}$, where the prime factor set of m_i is π_i for $i = 1, 2\cdots, t(G)$. For convenience, we denote the even-order component of G as $m_1(G)$(seesee[1]). However, in the way, some of groups have been characterized by this method. For example, in [4] and [5], the authors prove that Janko simple groups, Alternating group, by this method can be characterized.

Let p_m is the largest element in $\pi(G)$ and $|S_{p_m}(G)|$ denotes the order of the Sylow p_m-subgroups of G. In this paper, we will give a quantitative characterization of the Monster group M and the Baby Monster group B via the even-order components of the group and $|S_{p_m}(G)|$.

2 Preliminaries

In this paper, $\pi(G)$ denotes the set of prime factors of the order of G; p_m stands for the largest element of $\pi(G)$.The symbol $|\pi(G)|$ refers to the number of prime factors of the order of G. For $p_i \in \pi(G)$, S_{p_i} denotes a Sylow p_i-subgroup of G. All other symbols not explicitly defined are standard and can be found in [3].

The following theorem provides a characterization of the structure of finite groups when $t(G) \geq 2$.

Lemma 2.1 *[6, Corollary] Let G be a finite group with disconnecting prime graph. Then the structure of G is as follows:*

(1) G is a Frobenius group or a 2-Frobenius group.

(2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent $\pi_1(G)$-group, G/K is soluble $\pi_1(G)$-group, K/H is a non-abelian simple group, and $|G/K|$ divides $|Out(K/H)|$.

Remark 2.1 *Let G be a finite group. G is called a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are Frobenius groups with kernels K/H and H, respectively (see[1]).*

The following two lemmas respectively provide characterizations of the structure of even-order Frobenius groups and even-order 2-Frobenius groups.

Lemma 2.2 *[1, Theorem 1] Let G be an even-order Frobenius group with Frobenius kernel H and Frobenius complement K. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$ Moreover, the structure of G is one of the following:*

1) If $2 \in \pi(H)$, then the Sylow subgroups of K are cyclic.

2) If $2 \in \pi(K)$, then H is an abelian group. When K is soluble, the odd-order Sylow subgroups of K are cyclic and the Sylow 2-subgroup is either a cyclic group or a generalized quaternion group. When K is insoluble, there exists $K_0 \leq K$ such that $|K : K_0| \leq 2$ and $K_0 \simeq Z \times SL(2, 5)$, where $(|Z|, 30) = 1$ and the Sylow subgroups of Z are cyclic.

Lemma 2.3 *[1, Theorem 2] Let G be an even-order 2-Frobenius group. Then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, $|G/K|$ divides $|Aut(K/H)|$, and both $|G/K|$ and $|K/H|$ are cyclic groups. In particular, $|G/K| \leq |K/H|$ and G is soluble.*

Lemma 2.4 *[1, Lemma 1]Suppose that a finite group G contains a minimal normal subgroup N and N is nonsolvable. Then $G/(C_G(N)N$ is isomorphic to a subgroup of $Out(N)$. In particular if $C_G(N)$=1, G is isomorphic to a subgroup of $Aut(N)$.*

3 Proof of Main Theorem

Given a subgroup K of a group G, it is obvious that $m_1(K)$ divides $m_1(G)$. In the following proof, we will frequently use this fact without further explanation.

Theorem 3.1 *Let G be a finite group and M be one of the Monster group. Then G is isomorphic to M if and only if:*

(1) $m_1(G) = m_1(M)$.

(2) $|S_{p_m}(G)| = |S_{p_m}(M)|$.

Proof: The necessity of the theorem is obvious, so we only need to prove the sufficiency.

As all knows that the Monster group M is the biggest finite simple group, its order is $2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47 \cdot 59 \cdot 71$, and $m_1(G) = m_1(M) = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$, and $|S_{p_m}(G)|$=$|S_{p_m}(M)|$

=71. Since $m_1(G) = m_1(M) = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$, it follows that $t(G) \geq 2$. Therefore, by Lemma 2.1, G has the following structure:

1) G is a Frobenius group or a 2-Frobenius group.

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, we have $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$ by Lemma 2.2 .

If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7 \times S_{11} \times S_{13} \times S_{17} \times S_{19} \times S_{23} \times S_{29} \times S_{31} \times S_{47}$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$ and 47. Therefore, $|K| \big| |\mathrm{Aut}(S_{47})|$. However, since $71 \big| |K|$, we have a contradiction with $71 \big| |\mathrm{Aut}(S_{47})| = 47 - 1 = 46$.

If $2 \in \pi(K)$, then the Sylow 71-subgroup S_{71} of G is normal in G and its order is 71. Let g be an element of order 47, the action of g on S_{71} yields an element of order 47×71 in G, which contradicts the fact that $\pi_1(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 47\}$. Therefore, G is not a Frobenius group.

We claim that G is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$ by Lemma 2.3. Since $m_1(G) = m_1(M) = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$, we have $71 \in \pi(K/H) = \pi_2(G)$. If $|H| = 2^k \cdot 5^l (0 \leq k \leq 46$, and $0 \leq l \leq 9$, and k and l cannot be simultaneously equal to 0), since K/H is the kernel of G/H, the Sylow 71-subgroup S_{71} of K/H is normal in G/H. The acting of an element of order 47 of G/H on S_{71} can get a contradiction. If 3 or 7 or 11 or 13 or 17 or 19 or 23 or 29 or 31 or $47 \in \pi(H)$, by acting an element of order 71 of K on the Sylow 3-subgroup or the Sylow 7-subgroup or the Sylow 11-subgroup or the Sylow 13-subgroup or the Sylow 17-subgroup or the Sylow 19-subgroup or the Sylow 23-subgroup or the Sylow 29-subgroup or the Sylow 31-subgroup or the Sylow 47-subgroup of H, we can also obtain a contradiction. Hence, G is not a 2-Frobenius group.

Thus, by Lemma 2.1(2), the structure of G is as follows: there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, G/K is a soluble $\pi_1(G)$-group, and K/H is a nonabelian simple group. Since $m_1(G) = m_1(M) = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$, we have $t(G) \geq 2$, which implies that $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 47\}$ and $71 \in \pi(K/H)$. Since $\{2, 71\} \subseteq \pi(K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71\}$, by examining Table 2 to Table 4 in [2], we get that K/H can only be the sporadic simple group $L_2(71)(2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71)$ or M. If $K/H \cong L_2(71)$ and $|H| = 2^k \cdot 5^l (0 \leq k \leq 43$, and $0 \leq l \leq 8$, and k and l cannot be simultaneously equal to 0), we have $11 \big| |G/K| \big| |\mathrm{Out}(L_2(71))| = 2$, which is a contradiction. If $K/H \cong L_2(71)$ and H is non-trivial, and 3 or 7 or 11 or 13 or 17 or 19 or 23 or 29 or 31 or $47 \in \pi(H)$, by acting a 71th-order element of K on the Sylow 3-subgroup or the Sylow 7-subgroup or the Sylow 11-subgroup or the Sylow 13-subgroup or the Sylow 17-subgroup or the Sylow 19-subgroup or the Sylow 23-subgroup or the Sylow 29-subgroup or the Sylow 31-subgroup or the Sylow 47-subgroup of H, we can also obtain a contradiction. Therefore we get that $H = 1$. If $K/H \cong M$, then $m_1(K/H) = m_1(M) = m_1(G)$, we can also obtain that $H = 1$ too. Thus we have that $H = 1$ in all cases.

Now G has a normal simple subgroup K such that $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 47\}$. Considering that $m_1(G) = m_1(M) = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$, and $|S_{p_m}(G)| = |S_{p_m}(M)| = 71$, we have $t(K) \geq 2$. By examining Table 2 and Table 4 in [2] and using the condition $|S_{p_m}(G)| = |S_{p_m}(M)| = 71$, we conclude that K can only be $L_2(71)$ or M.

If $K \cong L_2(71)$, then $\pi(K) = \{2, 3, 5, 7, 71\}$, and $m_1(G) = m_1(M) = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$, $m_1(K) = 2^3 \cdot 3^2$. By Lemma 2.1(2), $2^{43} \cdot 3^{28} \big| |G/K| \big| |\mathrm{Out}(L_2(71))| = 2$, a contradiction. Hence $K \ncong L_2(71)$.

Therefore, we have $K \cong M$, and thus $1 \trianglelefteq M \trianglelefteq G$. It is clear that $C_G(M) = 1$ and $|\mathrm{Out}(M)| = 2$. By Lemma 2.4, we have either $G \cong M$, or $G \cong \mathrm{Aut}(M)$. If $G \cong \mathrm{Aut}(M)$, then we have $m_1(G) \neq m_1(M)$, obviously a contradiction. Hence $G \cong M$.

Theorem 3.2 *Let G be a finite group and B be the Baby Monster simple group B. Then G is isomorphic to B if and only if:*

(1) $m_1(G) = m_1(B)$.

(2) $|S_{p_m}(G)| = |S_{p_m}(B)|$.

Proof: The necessity of the theorem is obvious, so we only need to prove the sufficiency.

Since $m_1(G) = m_1(B) = 2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, and $|S_{p_m}(G)| = |S_{p_m}(B)| = 47$, it follows that $t(G) \geq 2$. Therefore, by Lemma 2.1, G has the following structure:

1) G is a Frobenius group or a 2-Frobenius group.

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, we have $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$ by Lemma 2.2.

If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7 \times S_{11} \times S_{13} \times S_{17} \times S_{19} \times S_{23}$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7, 11, 13, 17, 19$ and 23. Therefore, $|K| \big| |\mathrm{Aut}(S_{23})|$. However, since $47 \big| |K|$, we have a contradiction with $47 \big| |\mathrm{Aut}(S_{23})| = 23 - 1 = 22$.

If $2 \in \pi(K)$, then the Sylow 47-subgroup S_{47} of G is normal in G and has order 47. Let g be an element of order 19, the action of g on S_{47} yields an element of order 19×47 in G, which contradicts the fact that $\pi_1(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$. Therefore, G is not a Frobenius group.

We claim that G is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$ by Lemma 2.3. Since $m_1(G) = m_1(B) = 2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, we have $47 \in \pi(K/H) = \pi_2(G)$, If $|H| = 2^k \cdot 3^l (0 \leq k \leq 41$, and $0 \leq l \leq 31$, and k and l cannot be simultaneously equal to 0), since K/H is the kernel of G/H, the Sylow 47-subgroup S_{47} of K/H is normal in G/H. The acting of an element of order 19 of G/H on S_{47} can get a contradiction. If 5 or 7 or 11 or 13 or 17 or 19 $23 \in \pi(H)$, By acting an elements of order 47 of K on the Sylow 5-subgroup or the Sylow 7-subgroup or the Sylow 11-subgroup or the Sylow 13-subgroup or the Sylow 17-subgroup or the Sylow 19-subgroup or the Sylow 23-subgroup of H, we can also obtain a contradiction. Hence, G is not a 2-Frobenius group.

Thus, by Lemma 2.1(2), there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, and K/H is a nonabelian simple group. Since $m_1(G) = m_1(B) = 2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, and $|S_{p_m}(G)| = |S_{p_m}(B)| = 47$, we have $t(G) \geq 2$, which implies that $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ and $47 \in \pi(K/H)$. Since K/H is a nonabelian simple group, we have 4 divides $|K/H|$. Therefore, the largest possible order of the Sylow 2-subgroup of H is 2^{39}. If H is non-trivial, then $\{2, 47\} \subseteq \pi(K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 37, 41, 43, 47\}$, by examining Table 2 to Table 4 in [2], we get that K/H can only be the sporadic simple group $L_2(47)(2^4 \cdot 3 \cdot 23 \cdot 47)$

or B. If $K/H \cong L_2(47)$ and $|H| = 2^k \cdot 3^l (0 \le k \le 37$, and $0 \le l \le 30$, and k and l cannot be simultaneously equal to 0), we have $5^6 \big| |G/K| \big| |\mathrm{Out}(L_2(71))| = 2$, which is a contradiction. If $K/H \cong L_2(47)$ and H is non-trivial, and 5 or 7 or 11 or 13 or 17 or 19 or $23 \in \pi(H)$, by acting an elements of order 47 of K on the Sylow 5-subgroup or the Sylow 7-subgroup or the Sylow 11-subgroup or the Sylow 13-subgroup or the Sylow 17-subgroup or the Sylow 19-subgroup or the Sylow 23-subgroup of H, we can also obtain a contradiction. Therefore we get that $H = 1$. If $K/H \cong B$, then $m_1(K/H) = m_1(B) = m_1(G)$, we can also obtain that $H = 1$. Thus we get that $H = 1$ in all cases .

Now G has a normal simple subgroup K such that $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$. Considering that $m_1(G) = m_1(B) = 2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, and $|S_{p_m}(G)| = |S_{p_m}(B)| = 47$, we have $t(K) \ge 2$. By examining Table 2 and Table 4 in [2] and using the condition $|S_{p_m}(G)| = |S_{p_m}(B)| = 47$, we conclude that K can only be $L_2(47)$ or B.

If $K \cong L_2(47)$, then $\pi(K) = \{2, 3, 23, 47\}$, and $m_1(G) = m_1(B) = 2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, $m_1(K) = 2^4 \cdot 3$. By Lemma 2.1(2), $2^{37} \cdot 3^{30} \big| |G/K| \big| |\mathrm{Out}(L_2(47))| = 2$, a contradiction. Hence $K \ncong L_2(47)$.

Therefore, we have $K \cong M$, and thus $1 \trianglelefteq M \trianglelefteq G$. It is clear that $C_G(M) = 1$ and $|\mathrm{Out}(M)| = 2$. By Lemma 2.4, we have either $G \cong M$, or $G \cong \mathrm{Aut}(M)$. If $G \cong \mathrm{Aut}(M)$, then we have $m_1(G) \ne m_1(M)$, obviously a contradiction. Hence $G \cong M$.

4 Conclusion

From the above proof, we can see that the even-order components and the order of some given Sylow subgoups are effective in characterizing simple groups, and they have a profound impact on the structure of finite groups. However, due to limitations in length and time, this paper has not verified whether this method is equally effective for more simple groups. Currently, this series of work is progressing in an orderly manner.

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Conflicts of Interest

The authors declare no conflict of interest.