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Metallic Means in Primitive Pythagorean Triples : Metallic Ratios substantiated in Pythagorean Triangles and other Right Angled Triangles

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Abstract : This paper highlights the precise correlation between metallic ratios and the different families of primitive Pythagorean triples. This paper also describes the explicit formulae those provide the mathematical relationships between different metallic means. Main purpose of this paper is to highlight the geometric substantiation of metallic means in Pythagorean triples and other right triangles, and the trigonometric expression of the metallic ratios.

Keywords : Metallic Means, Pythagorean Triples, Right Triangles

Introduction :

Metallic Mean, also called as Metallic Ratio ($\boldsymbol{\delta}$n) is the root of the simple Quadratic Equation : $x^2 - nx - 1 = 0$ where n is any positive natural number. [1] [2]

Thus, the fractional expression for the n^{th} Metallic Ratio is $\boldsymbol{\delta}n = \frac{n+\sqrt{n^2+4}}{2}$

It is also observed that each metallic mean is substantiated in a primitive Pythagorean triple. [3] [4]

Consider a Pythagorean triple (a, b, c) with positive integers $a < b < c$ those are coprime and $c - b \in \{1,2,8\}$. Such Pythagorean triangle precisely exhibits a metallic mean. The cotangent of one quarter of the smaller acute angle of such Pythagorean triangle (a, b, c) is a metallic mean $\boldsymbol{\delta}$n.

$$\boldsymbol{\delta}\mathbf{n} = \cot\left(\frac{\theta}{4}\right) = \frac{a}{c-b} + \sqrt{\frac{2c}{c-b}}$$

where θ is the smaller acute angle of the Pythagorean triangle (a, b, c) and $\mathbf{n} = 2\sqrt{\frac{c+b}{c-b}}$

For example, the primitive Pythagorean triple (20, 21, 29) gives the 5th metallic mean; (3, 4, 5) gives the 6th metallic mean; (28, 45, 53) gives the 7th metallic mean; (8, 15, 17) gives the 8th metallic mean; (5, 12, 13) gives the 10th metallic mean; and so on.

Noticeably, in any primitive Pythagorean triple, the difference between hypotenuse c and the longer cathetus b is either an $(\text{odd})^2$ or $2\times(\text{odd})^2$ or $2\times(\text{even})^2$.

And 1, 2, 8 are being the minimum values of an odd^2, $2\times\text{odd}^2$, $2\times\text{even}^2$ respectively; the primitive Pythagorean triple with (c –b) ∈ {1, 2, 8} exhibits a specific metallic mean. Such triples (c –b) ∈ {1, 2, 8} are sometimes referred as Pythagoras, Plato and Socrates families of Pythagorean triples respectively.

In such Pythagorean triangles, $\mathbf{2cot}\left(\frac{\theta}{2}\right)$ becomes an integer **n**, and $\mathbf{cot}\left(\frac{\theta}{4}\right)$ yields the precise value of $\mathbf{n^{th}}$ metallic mean **δn**.

Hence, the trigonometric expression for the $\mathbf{n^{th}}$ metallic mean **δn** is

$$\boldsymbol{\delta}_n = \cot\left(\frac{\arctan \frac{4n}{n^2-4}}{4}\right)$$

From n ≥ 5, each n^{th} metallic mean **δn** equals cotangent of $\left(\frac{\theta}{4}\right)$ of a specific Pythagorean triple as follows :

<u>**If n is odd**</u> : the n^{th} metallic mean is given by the primitive Pythagorean triple

$(4n,\ n^2 - 4,\ n^2 + 4)$.

<u>**If n is even and a multiple of 4**</u> : the n^{th} metallic mean is given by

$(n,\ n^2/4 - 1,\ n^2/4 + 1)$

<u>**If n is even but not a multiple of 4**</u> : the n^{th} metallic mean is given by

$(n/2,\ (n^2 - 4)/8,\ (n^2 + 4)/8)$

The notable exceptions are being first four metallic means.

The First Metallic Ratio i.e. Golden Ratio and the Fourth Metallic Ratio are embedded in (3, 4, 5) Pythagorean triple :

The Golden Ratio $\boldsymbol{\delta_1} = \cot\left(\frac{\theta + 90^0}{4}\right)$ where θ is the smaller acute angle of the Pythagorean triangle (3, 4, 5).

The Copper Ratio i.e. Fourth Metallic Ratio $\boldsymbol{\delta_4} = \cot\left(\frac{\phi}{4}\right)$ where ϕ is the larger acute angle of the Pythagorean triangle (3, 4, 5).

Similarly, the Bronze Ratio i.e. Third Metallic Ratio $\boldsymbol{\delta_3} = \cot\left(\frac{\phi}{4}\right)$ where ϕ is the larger acute angle of the Pythagorean triangle (5, 12, 13).

Remarkably, there is no specific Pythagorean triple for Silver Ratio i.e. the Second Metallic Ratio. However, the Second Metallic Ratio $\boldsymbol{\delta_2}$ is embedded in every right triangle as $\boldsymbol{\delta_2} = \cot\left(\frac{90^0}{4}\right)$

Moreover, in all Pythagorean triangles with $c - b \in \{1, 2, 8\}$, the cotangent of one quarter of the larger acute angles approach the Silver Ratio as the triangle sizes grow.

From the Fifth Metallic Ratio onwards, each metallic mean is precisely given by the cotangent of one quarter of the smaller acute angle of the concerned Pythagorean triangle, as mentioned above.

Metallic Ratios in Fermat Family of Pythagorean Triples :

Now, it has been observed that the Pythagorean triples belonging to the Fermat Family in which two legs differ by one **(b - a = 1)** also exhibit certain Metallic Ratios, with following couple of formulae.

(1) Consider a Pythagorean triple (a, b, c) with positive integers $a < b < c$ those are coprime and $|b - a| = 1$. Such Pythagorean triangle exhibits $(2c)^{th}$ metallic mean

$$\boldsymbol{\delta}_n = \cot\left(\frac{\arctan\left(\frac{c}{ab}\right)}{4}\right) \quad \text{where } n = 2c$$

For example, 3-4-5 triple gives 10^{th} metallic mean by above formula, 20-21-29 triple gives 58^{th} metallic mean,

119-120-169 triple gives 338^{th} metallic mean, and so on.

Such Pythagorean triple of Fermat family **(b-a=1)** can also generate another Pythagorean triple of Pythagoras family **(c-b=1)**. Consider a Pythagorean triple of Fermat family (a, b, c) with $|b-a|$ =1. We can generate a new triple from this, as

(**c**, **ab**, **ab+1**).

Noticeably, this (**c**, **ab**, **ab+1**) triple is the Pythagorean triple for $(2c)^{th}$ metallic mean i.e. the cotangent of one quarter of the smaller acute angle of (**c**, **ab**, **ab+1**) triangle gives the $(2c)^{th}$ metallic ratio.

(2) There is observed another way of deriving metallic ratios from Fermat family triples. Consider a Pythagorean triangle of Fermat family (a, b, c) with **ϕ** and **θ** as its larger and smaller acute angles respectively. The difference between these two acute angles yields the $(2a+2b)^{th}$ metallic ratio.

$$\boldsymbol{\delta}_n = \cot\left(\frac{\phi - \theta}{4}\right) \qquad \text{where } n = 2(a+b)$$

For example, 3-4-5 triple gives 14^{th} metallic mean,

20-21-29 triple gives 82^{nd} metallic mean,

119-120-169 triple gives 478^{th} metallic mean, and so on.

Above result is observed because a Pythagorean triple of Fermat family can also generate one more Pythagorean triple of Pythagoras family (c-b=1). Consider a Pythagorean triple of Fermat family (a, b, c) with $|b-a|$ =1. We can generate a new triple from this, as (**a+b**, 2**ab**, 2**ab+1**).

The smaller acute angle of this new triple equals the difference between two acute angles of original triple (a, b, c) of Fermat family. Hence, the new triple (**a+b**, 2**ab**, 2**ab+1**) is the Pythagorean triple for $(2a+2b)^{th}$ metallic mean i.e. the cotangent of one quarter of the smaller acute angle of (**a+b**, 2**ab**, 2**ab+1**) triangle gives the $(2a+2b)^{th}$ metallic ratio.

Metallic Ratios in other Pythagorean Triples :

Apart from the Pythagorean triples of above mentioned Pythagoras, Plato, Socrates and Fermat families, the metallic ratios can also be substantiated from various other primitive Pythagorean triples.

For example, consider three Pythagorean triangles (3, 4, 5), (276, 493, 565) and (396, 403, 565) with θ_1, θ_2 and θ_3 be their smaller acute angles respectively. It is observed that

the 30th Metallic Mean $\delta_{30} = \cot\left(\frac{\theta_1 - \theta_2}{4}\right) = \cot\left(\frac{\theta_3 - \theta_1}{4}\right) = \cot\left(\frac{\theta_3 - \theta_2}{8}\right)$

Such explicit relationships are observed due to the fact that a couple of primitive Pythagorean triples can be derived from any two existing Pythagorean triples, and intriguing relationship is observed between their acute angles, as follows.

Consider two primitive Pythagorean triple (a_1, b_1, c_1) and (a_2, b_2, c_2) with θ_1 and θ_2 as their smaller acute angles respectively. From these, a couple of new triples can be generated as :

Triple $(a_1a_2 + b_1b_2,\ \ a_1b_2 - b_1a_2,\ \ c_1c_2)$ with its smaller acute angle $\theta = \theta_1 - \theta_2$

and

Triple $(a_1a_2 - b_1b_2,\ \ a_1b_2 + b_1a_2,\ \ c_1c_2)$ with its acute angle $\theta = \theta_1 + \theta_2$

Hence, the triples (276, 493, 565) and (396, 403, 565) can be derived from the couple of triples (3, 4, 5) and (15, 112, 113) by above method. And the triple (15, 112, 113) is the Pythagorean triple for the 30th Metallic Mean δ_{30} i.e. the cotangent of one quarter of the smaller acute angle of (15, 112, 113) triangle gives the 30th metallic ratio. Hence, above intriguing relationship is observed with the new triples.

Consider the simple trigonometric relations :

$$\arctan\frac{a}{b} - \arctan\frac{c}{d} = \arctan\frac{ad - bc}{bd + ac}$$

and

$$\arctan\frac{a}{b} + \arctan\frac{c}{d} = \arctan\frac{ad + bc}{bd - ac}$$

Hence, if θ_1 and θ_2 are the acute angles of two distinct Pythagorean triples, there always exist the Pythagorean triples with acute angles $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$, and hence the metallic means can be substantiated with various Pythagorean triples in several such intriguing ways. Summation and subtraction of the angles of distinct Pythagorean triangles yield the precise values of various metallic ratios. For example, consider the Pythagorean triangles (3**, 4, 5**) and (**20, 21, 29**) with θ_1 and θ_2 as their smaller acute angles respectively. The difference between θ_2 and θ_1 yields the 34th metallic mean $\boldsymbol{\delta}_{34} = \cot\left(\frac{\theta_2 - \theta_1}{4}\right)$

On the other hand, the sum of smaller acute angles of the Pythagorean triangles

(3**, 4, 5**) and (**17, 144, 145**) yields the 5th metallic mean $\boldsymbol{\delta}_5 = \cot\left(\frac{\theta_1 + \theta_2}{4}\right)$

Further, consider the triples (**5, 12, 13**) and (**33, 56, 65**)

The first metallic mean i.e. Golden Ratio can be expressed as $\boldsymbol{\delta}_1 = \cot\left(\frac{\phi_1 + \phi_2}{4}\right)$

where ϕ_1 and ϕ_2 are the larger acute angles of (**5, 12, 13**) and (**33, 56, 65**) triangles respectively.

Several such pairs of Pythagorean triples yield Golden Ratio in same manner, like

(3, 4, 5) and (7, 24, 25),

(8, 15, 17) and (36, 77, 85),

(12, 35, 37) and (104, 153, 185),

(9, 40, 41) and (133, 156, 205), and so on.

Such triples' pair can be generated by multiplying a Pythagorean triple by (3, 4, 5) triple using the Bramhagupta Fibonacci Identity.

Moreover, if a Pythagorean triple is generated from the catheti values of another primitive Pythagorean triple using the Euclid's formula, an acute angle of new triple is twice that of original triple. Hence, the metallic mean represented by original Pythagorean triple can also be substantiated with the new triple.

For example, the tenth metallic mean $\boldsymbol{\delta}_{10}$ can be substantiated with various triples :

$$\boldsymbol{\delta}_{10} = \cot\left(\frac{\theta}{4}\right) = \cot\left(\frac{\phi}{8}\right)$$

where θ is the smaller acute angle of (**5, 12, 13**) triangle and ϕ i is the larger acute angle of (**119, 120, 169**) triangle.

Similarly, the fifth metallic mean $\boldsymbol{\delta}_5 = \cot\left(\frac{\theta}{4}\right) = \cot\left(\frac{\phi}{8}\right)$

where θ is the smaller acute angle of (**20, 21, 29**) triangle and ϕ is the larger acute angle of (**41, 840, 841**) triangle; and so on.

Triads of Metallic Ratios: The Mathematical Correlation between different Metallic Means, and their Pythagorean Triples :

There exists a precise mathematical correlation between different metallic means, which is given by explicit mathematical formulae. [3]

Consider three positive integers ***k, m, n*** such that $n < m < k$ and $\frac{mn+4}{m-n} = k$; then an interesting relation is observed between k^{th}, m^{th} and n^{th} metallic means $\boldsymbol{\delta}_k$, $\boldsymbol{\delta}_m$ and $\boldsymbol{\delta}_n$ which form a Triad that exhibits following intriguing relations.

$$\frac{mn+4}{m-n} = k \quad \text{and} \quad \frac{\boldsymbol{\delta m} \times \boldsymbol{\delta n}+1}{\boldsymbol{\delta m} - \boldsymbol{\delta n}} = \boldsymbol{\delta}_k$$

also
$$\frac{kn+4}{k-n} = m \quad \text{and} \quad \frac{\boldsymbol{\delta k} \times \boldsymbol{\delta n}+1}{\boldsymbol{\delta k} - \boldsymbol{\delta n}} = \boldsymbol{\delta}_m$$

and
$$\frac{km-4}{k+m} = n \quad \text{and} \quad \frac{\boldsymbol{\delta k} \times \boldsymbol{\delta m}-1}{\boldsymbol{\delta k} + \boldsymbol{\delta m}} = \boldsymbol{\delta}_n$$

All such Triads like [6, 11, 14], [4, 5, 24], [2, 4, 6] are found to observe above relationships, and every integer n forms such multiple Triads. [3]

Now, it has been observed that the number of such Triads formed by every integer **n** equals **the numbers of positive factors of (n^2 +4)**.

For example, consider n = 6 and (n^2 +4) = 40

Positive factors of 40 are **1, 2, 4, 5, 8, 10, 20** and **40**

Hence, 6^{th} metallic mean forms eight triads, as follows.

n =	6	6	6	6	6	6	6	6
m =	6-5	6-4	6-2	6-1	6+1	6+2	6+4	6+5
k =	8-6	10-6	20-6	40-6	40+6	20+6	10+6	8+6

Hence, the 6^{th} metallic mean $\boldsymbol{\delta}_6$ forms following Eight Triads :

$\boldsymbol{\delta}_n =$	$\boldsymbol{\delta}_6$	$\boldsymbol{\delta}_6$	$\boldsymbol{\delta}_6$	$\boldsymbol{\delta}_6$	$\boldsymbol{\delta}_6$	$\boldsymbol{\delta}_6$	$\boldsymbol{\delta}_6$	$\boldsymbol{\delta}_6$
$\boldsymbol{\delta}_m =$	$\boldsymbol{\delta}_1$	$\boldsymbol{\delta}_2$	$\boldsymbol{\delta}_4$	$\boldsymbol{\delta}_5$	$\boldsymbol{\delta}_7$	$\boldsymbol{\delta}_8$	$\boldsymbol{\delta}_{10}$	$\boldsymbol{\delta}_{11}$
$\boldsymbol{\delta}_k =$	$\boldsymbol{\delta}_2$	$\boldsymbol{\delta}_4$	$\boldsymbol{\delta}_{14}$	$\boldsymbol{\delta}_{34}$	$\boldsymbol{\delta}_{46}$	$\boldsymbol{\delta}_{26}$	$\boldsymbol{\delta}_{16}$	$\boldsymbol{\delta}_{14}$

If n is odd, the n^{th} Metallic mean forms comparatively lesser number of such Triads. For example, the fifth metallic ratio $\boldsymbol{\delta}_5$ forms only a couple of Triads viz. ($\boldsymbol{\delta}_5$ $\boldsymbol{\delta}_6$ $\boldsymbol{\delta}_{34}$) and ($\boldsymbol{\delta}_5$ $\boldsymbol{\delta}_4$ $\boldsymbol{\delta}_{24}$).

Moreover, an intriguing relationship is also observed between the three Pythagorean triples those represent the three metallic means in a Triad.

Consider a Triad of metallic means $\boldsymbol{\delta}_n$, $\boldsymbol{\delta}_m$ and $\boldsymbol{\delta}_k$; and the Pythagorean triples for $\boldsymbol{\delta}_n$ and $\boldsymbol{\delta}_m$ are (a_1, b_1, c_1) and (a_2, b_2, c_2) having θ_1 and θ_2 as their smaller acute angles respectively. Then the Pythagorean triple for k^{th} metallic mean $\boldsymbol{\delta}_k$ is :

$(a_1a_2+b_1b_2,\ a_1b_2-b_1a_2,\ c_1c_2)$ with its smaller acute angle $\theta_3 = \theta_1 - \theta_2$

Geometric Substantiation of Metallic Means with Right Triangles :

Apart from Pythagorean triples, other right angled triangles also provide the geometric substantiation of metallic means. [3] [5]

Following couple of right triangles provide the accurate substantiation of all metallic ratios.

(1) $1 : \frac{n}{2}$ Right Triangle :

A right angled triangle with its catheti as 1 and $\frac{n}{2}$; and

θ as its smaller acute angle provides the geometric substantiation of the n^{th} metallic mean $\boldsymbol{\delta}_n$.

$$\boldsymbol{\delta}_n = \text{Hypotenuse} + \text{Cathetus}\ \frac{n}{2} = \cot\left(\frac{\theta}{2}\right)$$

and $$\frac{1}{\delta n} = \text{Hypotenuse} - \text{Cathetus}\ \frac{n}{2}$$

And hence, the simplified trigonometric expression for the n^{th} metallic mean is

$$\boldsymbol{\delta}_n = \cot\left(\frac{\operatorname{arccot} \frac{n}{2}}{2}\right)$$

(2) $\left(\frac{n}{2}+1\right):\left(\frac{n}{2}-1\right)$ Right Triangle :

Another right angled triangle with its catheti as $\left(\frac{n}{2}+1\right)$ and $\left(\frac{n}{2}-1\right)$; and $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ as its larger and smaller acute angles respectively, provides the geometric substantiation of the n^{th} metallic mean $\boldsymbol{\delta}_{\mathrm{n}}$

$$\boldsymbol{\delta}\mathrm{n} = \cot\left(\frac{\phi-\theta}{4}\right)$$

Conclusion : Right angled triangles and the primitive Pythagorean triples are the most prototypical forms of metallic ratios. Metallic means can be geometrically substantiated with various right triangles and different families of Pythagorean triples. Moreover, the summation and subtraction of the angles of distinct Pythagorean triangles yield the precise values of various metallic ratios. Also, there exists a precise correlation between different metallic means, which is given by explicit mathematical formulae and substantiated with three correlated Pythagorean triples.

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