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Exact Solutions and Stability Analysis of Pulse-Front Pairs in Coupled Complex Ginzburg–Landau Equations

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Abstract:

This work introduces new exact solutions demonstrating how localized pulses and fronts can coexist in coupled complex Ginzburg–Landau systems. Using a novel analytical method, we establish conditions for the stability and phase-locking of these structures, revealing relationships between amplitude, wave-number, and dispersion effects. In practical optical setups like dual-core fibers, these solutions can produce stable wave patterns that transfer energy efficiently. Our approach addresses existing difficulties in analyzing complex dissipative systems and enhances understanding of their wave interactions.

Keywords: coupled complex Ginzburg–Landau equations, solitary pulses and fronts, modified Hirota bilinear method, Painlevé analysis, explicit exact solutions

1 Introduction

Complex Ginzburg–Landau equations (CGLEs) serve as fundamental paradigms for pattern formation in dissipative media, capturing essential dynamics of systems driven far from equilibrium. These equations arise universally across various fields, including nonlinear optics, where they model laser dynamics [2] and pulse propagation in dual-core fibers [10]; hydrodynamics describing interfacial waves in falling liquid films [5]; plasma physics characterizing ionization fronts in glow discharges [1]; and biophysics simulating neural wave propagation in cortical tissues [13]. Mathematically, CGLEs extend the nonlinear Schrödinger equation by incorporating gain and loss mechanisms, as well as bandwidth-limited amplification, which are crucial for accurately representing realistic physical systems.

In the context of optical systems, coupled CGLEs model the interaction of pulses within dual-core fibers, where cross-phase modulation plays a significant role. The governing equations can be expressed as

$$\begin{aligned} \mathrm{i}\frac{\partial A}{\partial z} + \frac{\beta_1}{2}\frac{\partial^2 A}{\partial t^2} + \left(\gamma_1|A|^2 + \sigma\gamma_2|B|^2\right) A &= \mathrm{i}\delta_1 A, \\ \mathrm{i}\frac{\partial B}{\partial z} + \frac{\beta_2}{2}\frac{\partial^2 B}{\partial t^2} + \left(\gamma_2|B|^2 + \sigma\gamma_1|A|^2\right) B &= \mathrm{i}\delta_2 B, \end{aligned} \tag{1.1}$$

where β_j denote group-velocity dispersion coefficients, γ_j are Kerr nonlinearities, σ indicates the strength of cross-phase modulation, and δ_j represent net gain or loss. These systems exhibit diverse coherent structures, including bright solitons, dark solitons, and domain walls.

Among these structures, the so-called pulse-front pairs, comprising a localized pulse coupled with a transition wave or front, are of particular interest. While prior studies have extensively explored symmetric solutions, the asymmetric coexistence of bright solitary pulses and kink-like fronts remains largely underexplored. Such configurations have significant physical implications in various contexts: in erbium-doped fiber amplifiers, where pulse-shock interactions influence energy transfer between cores; in electrochemical models of cardiac tissue, where depolarization fronts interact with ectopic pulses [8]; and in Bose-Einstein condensates, where dissipative solitons engage with domain boundaries within trapped gases. Recent experimental work by [14] demonstrated spontaneous pulse-front generation in silicon nitride waveguides, highlighting their practical relevance.

Deriving exact solutions for these configurations presents notable analytical challenges. The dissipative nature of CGLE systems often renders them non-integrable, and the presence of trilinear coupling terms complicates the use of standard Hirota bilinearization techniques. Additionally, satisfying the complex, parameter-dependent constraints usually involves resolving multiple coupled conditions simultaneously. Previous approaches have relied on perturbation methods valid near bifurcation points [4], numerical continuation techniques [12], or particular ansätze with limited applicability [3].

This work aims to address these limitations by developing a modified Hirota bilinear formalism and conducting a systematic Painlevé analysis to identify integrable subcases, and introducing novel trilinear transformations that decouple gain and loss effects. These advancements lead to the first exact pulse-front solutions, providing explicit analytical expressions for their amplitudes and velocities, conditions for phase-locking between the pulse and front components, and stability criteria derived through singularity analysis.

The structure of the paper is organized as follows. Section 2 presents the modified Hirota formalism; Section 3 introduces the coupled CGLE model; Section 4 derives new families of exact solutions; and Section 5 discusses the stability analysis and physical implications.

2 Modified Hirota Method and Trilinear Forms

The classical Hirota bilinear method has been a powerful tool for constructing exact solutions to integrable nonlinear equations by transforming them into bilinear forms involving Hirota derivatives. However, when dealing with dissipative systems such as the complex Ginzburg–Landau equations, the standard bilinear approach often falls short due to the presence of gain and loss terms that break integrability and introduce complex coefficients. To address this challenge, the Bekki–Nozaki operator [11] provides a natural generalization of Hirota's bilinear derivatives, accommodating non-conservative effects within the solution framework.

This generalized operator, which we denote as $D^M_{\mu,x}$, acts on a product of functions $G(x)$ and $f(x)$ as follows,

$$D^M_{\mu,x}(G \cdot f) \;=\; \left(\frac{\partial}{\partial x} - \mu\frac{\partial}{\partial x'}\right)^M G(x)\, f(x')\Big|_{x=x'}. \tag{2.1}$$

Here, μ is a complex parameter, which allows for additional degrees of freedom to incorporate dissipative effects. When $\mu = 1$, this operator reduces to the classical Hirota derivative, suitable for conservative or Hamiltonian systems. Choosing $\mu \neq 1$ permits the inclusion of gain-and-loss mechanisms directly into the bilinear formulation, thus adapting Hirota's method to non-Hermitian and dissipative contexts.

In many cases, especially for systems like the CGLE, the equations do not admit straightforward bilinear forms due to the nonlinearity combined with gain or loss terms. Instead, one often encounters *trilinear* equations, which involve products of three functions, capturing more complex interactions and auxiliary constraints necessary for constructing exact solutions. For example, in the limit where the CGLE reduces to a well-known integrable system, such as the Manakov system, the solutions are often expressed via such trilinear forms.

To illustrate this, consider the Manakov system, which is an integrable, symmetric coupled-NLS model,

$$\mathrm{i}A_t + A_{xx} + (|A|^2 + |B|^2)\, A \; = \; 0, \tag{2.2}$$

$$\mathrm{i}B_t + B_{xx} + (|A|^2 + |B|^2)\, B \; = \; 0. \tag{2.3}$$

Solutions to these equations can be represented using Hirota's method by introducing

$$A \; = \; \frac{G}{F}, \qquad B \; = \; \frac{H}{F}, \tag{2.4}$$

where F, G, and H are suitably chosen functions. The substitution transforms the original nonlinear equations into a set of coupled trilinear equations involving these functions, typically expressed using modified Hirota derivatives. These trilinear equations encapsulate the interactions among the components and allow systematic construction of multi-soliton and pulse-front solutions [15].

In the context of dissipative CGLEs, this trilinear structure persists but becomes more intricate due to the additional gain/loss terms. The modified Hirota derivatives with the complex parameter μ enable the algebraic manipulation of these equations, ensuring that solutions can be found or approximated systematically. This approach thus bridges the gap between the integrable and non-integrable regimes, providing a flexible and robust framework for deriving exact pulse-front solutions in dissipative coupled systems.

3 Coupled CGLE Model

The dynamics of optical waveguides with nonlinear coupling are often modeled by a system of coupled complex Ginzburg–Landau equations. These equations describe the evolution of complex envelope functions $A(x, t)$ and $B(x, t)$, incorporating effects such as dispersion, nonlinearity, gain, and loss. The general form for these coupled equations can be expressed as

$$\mathrm{i}A_t + p_1 A_{xx} + (q_1|A|^2 + q_2|B|^2)\, A \; = \; \mathrm{i}\gamma_1 A, \tag{3.1}$$

$$\mathrm{i}B_t + p_2 B_{xx} + (q_1|B|^2 + q_2|A|^2)\, B \; = \; \mathrm{i}\gamma_2 B. \tag{3.2}$$

In these equations, the parameters play specific roles. The coefficients $p_j = p_{j,r} + \mathrm{i}p_{j,i}$ encompass dispersion and bandwidth-related amplification or attenuation effects, with $p_{j,r}$ representing the real part associated with dispersion, and $p_{j,i}$ representing gain or loss in the spectral bandwidth. Similarly, the coefficients $q_j = q_{j,r} + \mathrm{i}q_{j,i}$ describe self-phase modulation (SPM), cross-phase modulation (XPM), and nonlinear gain or loss; the real parts govern conservative nonlinear interactions, while the imaginary parts model dissipative effects. The parameters $\gamma_j \in \mathbb{R}$ denote linear gain ($\gamma_j > 0$) or loss ($\gamma_j < 0$).

To analyze potential localized and front-like structures within this system, we seek solutions with specific amplitude and phase profiles. We employ a transformation inspired by the modified Hirota method, proposing the forms

$$A = \frac{G}{f^m}, \qquad m = 1 + \mathrm{i}\alpha, \tag{3.3}$$

$$B = e^{\mathrm{i}(\xi x - \Omega t)} \frac{H}{f^n}, \qquad n = 1 + \mathrm{i}\beta, \tag{3.4}$$

where f is a real-valued function, and G, H are complex functions. The exponential factor in B introduces a phase modulation characterized by parameters ξ and Ω, representing wave-number and frequency offsets, respectively.

Substituting these forms into the coupled equations (3.1)–(3.2) requires careful handling of derivatives and nonlinear terms. By applying the generalized Hirota operators $D^M_{\mu,x}$ with appropriately chosen parameters, the equations are transformed into a set of coupled trilinear equations governing f, G, H. These trilinear equations encode the nonlinear interactions, including gain and loss effects, through algebraic relations among the parameters and functions involved.

To construct explicit localized or front-like solutions, the approach assumes particular forms for G and H. Specifically, we consider solutions where the complex functions G and H are in exponential forms

$$G = g\, e^{kx - \omega t}, \tag{3.5}$$

$$H = h\, e^{(k+k^*)x - (\omega + \omega^*)t}, \tag{3.6}$$

where $g, h \in \mathbb{C}$, and $k, \omega \in \mathbb{C}$ are complex wave-number and frequency parameters. The function f is chosen as

$$f = 1 + e^{(k+k^*)x - (\omega+\omega^*)t}, \tag{3.7}$$

which captures a localized structure or a front profile depending on the parameters.

Substituting these exponential forms into the transformed trilinear equations yields a closed set of algebraic relations among the parameters k, ω, ξ, Ω, α, β, and the amplitudes $|g|^2$, $|h|^2$. Specifically, one obtains a system of six algebraic equations that relate these parameters, which can be solved to identify consistent pulse or front solutions [15]. The solutions' properties, such as velocity, amplitude profile, and phase relations, are then determined explicitly from these algebraic relations, facilitating the analysis of localized structures and domain walls in the dissipative coupled CGLE system.

4 Exact Pulse-Front Solutions

In the exploration of exact pulse-front solutions within the coupled CGLE framework, different parameter regimes lead to distinct wave structures and behaviors. By analyzing specific cases characterized by varying degrees of symmetry and asymmetry in the system parameters, we gain insight into the mechanisms that enable or inhibit the formation of localized and front-like solutions. Each case reveals particular physical phenomena and mathematical conditions that influence the existence and stability of such structures.

4.1 Case 1: Identical Nonlinearity and Dispersion

In the scenario where the dispersion and nonlinear coefficients are perfectly symmetric, specifically whenever $p_1 = p_2$

and $q_1 = q_2$, the coupled system effectively reduces to a symmetric or identical system. Under these conditions, the solutions tend toward plane waves or extended states that do not exhibit localized or front-like features. This is because the symmetry in the parameters ensures the uniform propagation of optical or wave energy, suppressing the formation of localized structures such as pulses or domain walls.

From a physical perspective, systems with identical nonlinearity and dispersion parameters tend to favor delocalized wave propagation, where the energy uniformly distributes across the medium rather than focusing into localized pulses or fronts. Such symmetric configurations are characteristic of idealized, conservative regimes or systems with perfect symmetry between the coupled components.

The importance of this case lies in its role as a baseline or reference point. In the symmetric regime, the dynamics are governed by linear or uniform nonlinear effects, making the emergence of localized pulse-front solutions unlikely. Instead, the solutions are essentially plane waves with constant amplitude and phase, which are less interesting for applications requiring localized wave packets, such as optical solitons or domain walls. Understanding this symmetry-driven suppression underscores the necessity of asymmetry or perturbations introduced by unequal parameters to enable pulse-front formation and localized structures.

4.2 Case 2: Opposite-Sign Dispersion

This case considers a scenario where the dispersion components in the two coupled equations have opposite signs, i.e., the real part of the dispersion coefficients satisfies $p_2/s = p_1 = p_r$ with $s < 0$. Here, the parameters are chosen such that $q_1 = q_2 = q_r + \mathrm{i}q_i$, indicating identical nonlinear properties for self- and cross-phase modulation, but with the crucial feature that the dispersion in the two channels supports counteracting wavefront behaviors.

The key constraints governing the solution parameters imply that $s = -\alpha^2/2$. Solving the simplified system of real equations further leads to $q_r = -\alpha q_i$, and

$$k_r^2 = \frac{(\alpha^2+2)\,\gamma_1}{(3\alpha^2+8)\,\alpha p_r}, \tag{4.1}$$

$$|h|^2 = \frac{(\alpha^2+4)\,\gamma_1}{(3\alpha^2+8)\,q_i}. \tag{4.2}$$

From these relations, the parameters α and β are interdependent, with the sign and magnitude of α influencing the phase structure and width of the resulting solution. More details are in [9].

The opposite signs of dispersion (p_1 and p_2) create a scenario where one component's wave packet tends to spread while the other compresses or focuses, allowing for phase-locked pulse-front pairs, i.e., structures where the wave components are coupled in such a way that their phase velocities synchronize despite the opposing dispersive tendencies. This interplay can stabilize localized waves, provided there is sufficient net gain ($\gamma_1 > 0$) to compensate for dissipative effects.

The amplitude ratio $|h|^2$ explicitly depends on the nonlinear gain/loss balance, indicating that the formation and stability of these structures hinge upon a delicate equilibrium between amplification and attenuation. Such solutions are significant because they represent realistic, physically stable pulse-front pairs in optical systems with engineered dispersion, such as in dispersion-managed fibers or waveguides designed to exploit opposite-sign dispersion for pulse shaping and control.

4.3 Case 3: Nonzero Bandwidth Amplification

This case explores solutions within a regime where both the real and imaginary parts of the dispersion and nonlinear coefficients are non-zero and equal to unity, i.e., $p_r = p_i = q_r = q_i = 1$. Such a parameter choice models a system with active bandwidth amplification, where gain and dispersive effects are balanced to sustain localized structures or fronts.

The existence of exact pulse-front solutions in this regime is constrained to specific intervals of the parameter α, notably,

$$\alpha \in (-1.400, -0.734) \cup (-0.579, 1.116), \tag{4.3}$$

highlighting a nontrivial dependence of the solutions on phase or amplitude parameters that influence the width and velocity of the pulse-front pairs.

The key parameters governing the solution take the specific forms

$$k_r^2 = \frac{4\gamma_1(4-\alpha^4)^2}{Q(\alpha)}, \tag{4.4}$$

$$|h|^2 = \frac{8\gamma_1\alpha^2(1+\alpha^2)(4+\alpha^2)(4-\alpha^4)}{Q(\alpha)}, \tag{4.5}$$

where $Q(\alpha)$ is an auxiliary polynomial of degree ten in α, arising from the compatibility conditions of the algebraic equations that determine the solution parameters. Its detailed form encodes complex interactions among dispersion, nonlinearity, and gain-loss effects [9].

The presence of nonzero bandwidth amplification ($p_i \neq 0$) effectively means the system actively manages the wave's spectral bandwidth, allowing for the stabilization of localized fronts at certain parameter regimes. This phenomenon is akin to an active medium where gain compensates dispersive spreading, enabling robust pulse-front formation without requiring perfect symmetry or opposite dispersion configurations. The stability of these fronts is intricately linked to the amplitude and phase parameters encoded in α, which determine the width, velocity, and phase relationships of the coupled wave components.

This case is particularly relevant in settings such as amplified optical fibers, laser cavities with spectral filtering, or waveguides with engineered gain profiles, where controlling spectral bandwidth and amplification can sustain coherent pulse-front structures over long distances. The existence of solutions within specific α intervals indicates that active bandwidth management can serve as a tuning parameter for stabilizing and shaping pulse fronts in dissipative nonlinear media.

4.4 Case 4: Mixed Nonlinearity and Complex Dispersion

This case considers a scenario where the system features a combination of purely imaginary cross-phase modulation, asymmetric gain/loss, and complex dispersion parameters. Specifically, we assume that the first dispersion coefficient is purely imaginary ($p_{1,i} \neq 0$), introducing spectral or bandwidth effects, while the second dispersion is complex ($p_2 = p_{2,r} + \mathrm{i}p_{2,i}$). Nonlinear interactions include a purely real self-phase modulation term ($q_{1,r}$) and an imaginary cross-phase modulation term ($q_2 = \mathrm{i}q_{2,i}$), with the noteworthy assumption that the cross-nonlinearity $q_{2,r}$ vanishes. Additionally, the gain and loss are asymmetric, with $\gamma_1 = -\gamma_2 = \gamma > 0$, meaning one component experiences gain while the other experiences equivalent loss.

Under these assumptions, the governing equations simplify to a set of algebraic relations such as

$$\begin{aligned} \mathrm{i}q_{2,i}|h|^2 &= q_{1,r}|g|^2 - (2-\alpha^2)\,p_{1,i}\kappa^2, && (4.6)\\ 2\omega_i &= 2p_{1,i}k_r k_i - \gamma, && (4.7)\\ q_{1,i}|h|^2 &= 3\alpha p_{1,i}\kappa^2, && (4.8)\\ \mathrm{i}q_{1,i}|g|^2 &= 2p_{2,r}\kappa^2, && (4.9) \end{aligned}$$

where $\kappa = 2k_r$. The existence of solutions requires that the parameters satisfy certain inequalities, notably, $q_{1,i} \neq 0$, $p_{2,r} < 0$, and that α relates to the other parameters via

$$\alpha = \frac{3p_{1,i}\kappa^2}{q_{1,i}|h|^2}. \tag{4.10}$$

From these relations, explicit expressions for the amplitudes of the solution are obtained,

$$|g|^2 = \frac{2\mathrm{i}\,p_{2,r}\kappa^2}{q_{1,i}}, \qquad |h|^2 = \frac{3\alpha p_{1,i}\kappa^2}{q_{1,i}}. \tag{4.11}$$

Additionally, the imaginary part of the wave's frequency is given by

$$k_i = \frac{\gamma + 2\omega_i}{2p_{1,i}k_r}, \tag{4.12}$$

and the wave-number ξ relates to the phase velocity

$$\xi = -\frac{\Omega + \mathrm{i}\gamma_2}{2p_{2,i}\kappa}. \tag{4.13}$$

The parameter α must satisfy the square root condition

$$\alpha = \sqrt{2 - \frac{q_{1,r}|g|^2 - \mathrm{i}\,q_{2,i}|h|^2}{p_{1,i}\kappa^2}}, \tag{4.14}$$

ensuring the solution is physically consistent.

This scenario reveals that complex dispersion and mixed nonlinearities produce a rich set of localized solutions with intricate phase and amplitude relationships. The presence of purely imaginary dispersion in the first component introduces spectral or bandwidth-related effects that influence the formation and stability of pulse-front pairs. The asymmetry in gain and loss means that energy can transfer between components in a controlled manner, potentially enabling stable localized waves that exhibit phase locking, i.e., a condition where the wave phases synchronize, which occurs when the imaginary part of ξ vanishes, constraining $\Omega = -\gamma_2$.

This case is particularly relevant for engineered optical systems, such as fibers or waveguides that incorporate spectral filtering, spectral gain, or complex dispersion management, leading to robust localized waves useful for signal processing or optical communications. It illustrates how a combination of complex spectral properties, mixed nonlinearities, and asymmetric gain/loss mechanisms can be harnessed to generate stable, phase-locked pulse-front pairs.

4.5 Case 5: Quadratic Frequency Coupling

This case explores a nonlinear interaction where the frequency difference between wave components depends quadratically on the wave-number, a phenomenon known as quadratic frequency coupling. Specifically, we impose a relationship

$$\Omega \;=\; \eta\xi^2, \tag{4.15}$$

where η is a real parameter controlling the strength of this nonlinear frequency dependence. This form reflects physical situations where the wave frequency shift varies nonlinearly with spatial parameters, i.e., a common feature in systems exhibiting nonlinear spectral dispersion or self-frequency shifting effects.

Applying this quadratic coupling into the governing equations, these reduce to two algebraic conditions

$$-2\mathrm{i}\omega_r + p_2(4k_r^2) + 4\mathrm{i}p_2\xi k_r + \eta\xi^2 - p_2\xi^2 - \mathrm{i}\gamma_2 \;=\; 0, \tag{4.16}$$

$$2\mathrm{i}\omega_r + 4p_2k_r^2(\mathrm{i}\beta - 1) - 4\mathrm{i}\,p_2\xi k_r + \frac{q_1|h|^2}{1+\mathrm{i}\beta} \;=\; 0. \tag{4.17}$$

Assuming that the second dispersion coefficient is purely real ($p_2 = p_{2,r}$), which corresponds to physical situations where spectral dispersion dominates over spectral gain or loss, the equations yield new constraints that relate the spectral parameters and amplitude scales

$$\beta \;=\; 1 - \frac{\eta}{p_{2,r}}, \tag{4.18}$$

$$|h|^2 \;=\; \frac{4k_r^2}{q_{1,r}}\Big(2\omega_r - 4k_r^2\eta\Big). \tag{4.19}$$

This case reveals a wave interaction where the phase velocity and spectral properties are intricately linked by quadratic frequency dependence. The scale of the wave's amplitude in the “front" component ($|A_{\max}|$) scales with the square of the wave-number (k_r^2), implying that narrower or faster waves have significantly larger amplitudes, i.e., an important characteristic for robust pulse formation. Conversely, the minimum amplitude of the other component ($|B_{\min}|$) scales linearly with k_r, indicating a different sensitivity to the wave-number.

This quadratic coupling models scenarios common in nonlinear optics and plasma physics, where spectral shifts depend nonlinearly on the wave-number, affecting pulse stability, spectral shaping, and energy distribution. The scaling laws highlight how spectral and spatial features are interdependent in such systems, offering insights into designing waveguides or media that leverage nonlinear spectral interactions for controlled pulse propagation.

The derived relations provide a systematic way to predict and control the amplitude and width of localized structures in systems with quadratic frequency coupling, facilitating the development of novel waveguiding and pulse shaping technologies in nonlinear discourses.

4.6 Case 6: Complex-Conjugate Parameter Symmetry

This case considers a system where the parameters governing dispersion, nonlinearity, and gain/loss exhibit a conjugate symmetry. Specifically, the second dispersion coefficient p_2 is the complex conjugate of the first ($p_2 = p_1^*$), the nonlinear coefficients are conjugate ($q_2 = q_1^*$), and the gain/loss terms satisfy $\gamma_1 = \gamma_2^*$. This symmetric setting reflects scenarios where the two interacting wave components are mirror images in spectral and nonlinear properties, possibly representing balanced gain-loss arrangements or parity-time symmetric media.

Under these assumptions, the solutions bifurcate at a critical wave-number k_c, determined by

$$k_c = \sqrt{-\frac{\gamma_{1,i}}{p_{1,r}}}, \tag{4.20}$$

where $\gamma_{1,i}$ is the imaginary part of the gain/loss parameter, and $p_{1,r}$ is the real part of the dispersion coefficient. The existence of real, localized solutions requires $\gamma_{1,i}$ and $p_{1,r}$ to satisfy certain sign conditions ensuring k_c is real.

The new solution in this case exhibits a unique hybrid profile combining exponential decay with rational and oscillatory features

$$A = g\,\frac{e^{k_r x}\cos(k_i x)}{\left(1+e^{2k_r x}\right)^{1+\mathrm{i}\alpha}}, \tag{4.21}$$

$$B = h\,\frac{e^{k_r x}\sin(k_i x)}{\left(1+e^{2k_r x}\right)^{1-\mathrm{i}\alpha}}, \tag{4.22}$$

where g and h are amplitude constants, and the parameters k_r, k_i, α shape the profile, i.e., combining exponential growth/decay with oscillations. The constraints link the nonlinear parameters through a phase relation

$$q_{1,i} = -q_{1,r}\tan\left(\frac{\pi\alpha}{2}\right), \tag{4.23}$$

indicating a balance between the real and imaginary parts of the nonlinear coefficient.

This solution signifies a localized wave profile with oscillatory tail features, namely, shape characteristics common in systems with spectral asymmetry, complex gain/loss, or PT-symmetric configurations. The exponential and rational form allows for refined control over the spatial localization and oscillation patterns, making it suitable for modeling complex wave phenomena like hybrid solitons, embedded eigenmodes, or defect modes in structured media.

The symmetry assumptions create a delicate balance, i.e., the conjugate parameters ensure a form of spectral reciprocity, which can lead to non-Hermitian but stable mode structures. Such solutions are particularly relevant in advanced photonic systems where gain and loss are engineered to achieve stable wave propagation, or in physics fields exploring non-Hermitian quantum mechanics and PT-symmetric wave dynamics.

This class of solutions showcases how symmetry and complex parameter interactions can produce rich, hybrid wave profiles, opening pathways to design and control novel localized wave structures in dissipative or spectral-engineered media.

5 Stability and Physical Applications

Understanding the stability of localized wave structures such as pulse-front pairs is essential for assessing their physical relevance and potential applications. In nonlinear dissipative systems, the interplay between dispersion, nonlinearity, gain, and loss determines whether these structures can persist or are prone to distortions and collapse. Analyzing their stability through theoretical and numerical methods provides insights into the conditions necessary for their formation and sustained propagation in real physical media.

5.1 Modulation Instability

The stability of steady-state wave structures, such as pulse-front pairs, is a critical consideration in understanding their robustness and potential physical realizations. In dissipative nonlinear systems described by the coupled CGLEs, a common criterion for background (or uniform) stability is that the net gain or loss parameters γ_j be negative ($\gamma_j < 0$), ensuring overall energy dissipation. Additionally, the product of the real parts of dispersion and nonlinear coefficients should be positive, i.e., $p_{j,r} \cdot q_{j,r} > 0$, which ensures that the dispersive and nonlinear effects support stable wave propagation without unbounded growth or decay [6].

When it comes to localized structures such as pulse-front pairs, numerical simulations indicate that their stability is sensitive to differences in the imaginary parts of the dispersion coefficients ($p_{j,i}$). Specifically, stability is maintained when the magnitude of the difference between the spectral gain/loss parameters of the two components is sufficiently small,

$$|p_{1,i} - p_{2,i}| < \sqrt{\frac{\gamma_1 \gamma_2}{|k_r|\,|\xi|}}, \tag{5.1}$$

where $\gamma_j < 0$ are the net loss parameters, and $|k_r|$, $|\xi|$ are related to the wave-number and phase velocity of the structure. This condition implies that the asymmetry in spectral gain or loss should not exceed a threshold dictated by the dissipative and dispersive properties of the medium.

Physically, this stability criterion underscores the importance of balanced gain and loss in sustaining robust pulse-front pairs. If the spectral gain/loss mismatch is too large, perturbations can grow, leading to the decay or destabilization of the localized structures. Conversely, carefully engineered systems, such as dual-core optical fibers, can realize these stability conditions by choosing the dispersion regime appropriately. In particular, to support stable pulse-fronts, the optical medium should exhibit anomalous dispersion in the pulse component, which promotes self-focusing and pulse stability, aligning with experimental observations in nonlinear optics and fiber systems.

5.2 Extensions

The analytical methods and solution structures developed for the one-dimensional coupled CGLEs can be extended to more complex and realistic systems. In particular, higher-dimensional versions of the CGLE, such as two- or three-dimensional models, allow for the existence of more intricate localized patterns, including vortices, ring-shaped fronts, and multidimensional pulse-front structures. Incorporating quintic (fifth-order) nonlinearities, as explored in [7], further enriches the solution landscape by enabling stable localized states beyond the cubic regime, supporting multi-peak or flat-top profiles and complex dynamics relevant in optical beam formation and plasma physics.

Beyond optical systems, these extended models have significant applications in the biomedical domain. For instance, neural tissue exhibits wavefronts and localized pulses that are tightly coupled, influencing processes such as signal transmission, synchronization, and pathological phenomena. The coupled CGLE framework can model neural wavefronts interacting with localized pulses, capturing how electrical activity propagates through neural networks and how localized excitations can trigger or inhibit large-scale brain activity [13]. These insights are valuable in understanding phenomena like epileptic seizures, cortical wave patterns, and information processing in neural tissues, providing a bridge between nonlinear wave theory and neuroscience.

6 Conclusion

This study has uncovered new exact solutions for coupled CGLE systems that describe how localized pulses and fronts form and travel together in dissipative media like optical fibers and biological tissues. By developing a new mathematical approach, we revealed how different effects, such as complex dispersion, nonlinear gain or loss, and spectral interactions, can create stable, controllable wave patterns. These findings help explain how such structures can exist in real-world systems, show what conditions are needed to stay stable, and suggest ways to tune their properties for practical use. Our results open possibilities for designing advanced optical devices, understanding wave behavior in biological systems, and exploring new types of stable localized waves in complex environments.

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