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Solving cubic and quartic equations by means of Vieta's formulas

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Abstract

In this paper, we will prove that with the use of $Vieta's$ formulas, it is possible to apply a unified method in solving equations of the third and fourth degree.

Keywords: ***cubic equation, quartic equation, Vieta's formulas***

1. Introduction

This is not a new idea, because this approach to solving algebraic equations has already been discussed and explained in the past, for example [1]. First, we will make a short analysis of Vieta's formulas.

Suppose that polynomials $f(x), g(x)$ and $h(x)$ are defined in the following way:

$$f(x) = x^n + a_{n-1}\,x^{n-1} + a_{n-2}\,x^{n-2} + ...a_1\,x + a_0 \tag{1}$$

$$g(x) = (x - x_0)(x - x_1)...(x - x_{n-1}) \tag{2}$$

$$h(x) = x^n - (x_0 + x_1 + ... + x_{n-1})x^{n-1} + (x_0x_1 + x_0x_2 + ... + x_{n-2}x_{n-1})x^{n-2} + (-1)^n x_0x_1...x_{n-1} \tag{3}$$

It is easy to prove that the following identity holds:

$$h(x) \equiv g(x) \tag{4}$$

If $(x_0, x_1, ..., x_{n-1})$ are solutions of polynomial $f(x)$, then it is obvious that they are also solutions of polynomial $g(x)$. Since the polynomials $g(x)$ and $h(x)$ are identical, they are also solutions of polynomial $h(x)$. This means that the polynomials $f(x)$ and $h(x)$ are identical, therefore the following equalities, Vieta's formulas, apply:

$$a_{n-1} = -(x_0 + x_1 + ... + x_{n-1}) \tag{5}$$

$$a_{n-2} = x_0x_1 + x_0x_2 + ... + x_{n-2}x_{n-1} \tag{6}$$

$$. \tag{7}$$

$$. \tag{8}$$

$$. \tag{9}$$

$$a_0 = (-1)^n x_0x_1...x_{n-1} \tag{10}$$

Now we will assume that the equations (5)-(10) hold. This means that the polynomials $f(x)$ and $h(x)$ are identical. The solutions of the polynomial $h(x)$ are $(x_0, x_1, ..., x_{n-1})$ and therefore they are the solutions of the polynomial $f(x)$. Finally, we can conclude that $(x_0, x_1, ..., x_{n-1})$ are solutions of the polynomial $f(x)$ if and only if the Vieta's formulas hold.

2. Cubic Equation

Without loss of generality a cubic polynomial in one variable is defined in the following way:

$f(x) = x^3 + bx^2 + cx + d$

Where b, c **and** d **are real numbers. The corresponding cubic equation is defined as follows:**

$$f(x) = x^3 + bx^2 + cx + d = 0 \tag{11}$$

Our goal to to solve the equation (11) for x.

First we substitute $x = \alpha + y$

$$(\alpha + y)^3 + b(\alpha + y)^2 + c(\alpha + y) + d = 0 \tag{12}$$
$$\alpha^3 + y^3 + 3\alpha^2 y + 3\alpha y^2 + b\alpha^2 + by^2 + 2b\alpha y + c\alpha + cy + d = 0 \tag{13}$$
$$y^3 + (3\alpha + b)y^2 + (3\alpha^2 + 2b\alpha + c)y + \alpha^3 + b\alpha^2 + c\alpha + d = 0 \tag{14}$$
$$y^3 + B\,y^2 + C\,y + D = 0 \tag{15}$$

$$B = 3\alpha + b \tag{16}$$
$$C = 3\alpha^2 + 2b\alpha + c \tag{17}$$
$$D = \alpha^3 + b\alpha^2 + c\alpha + d \tag{18}$$

We can easily memorize the coefficients B, C **and** D **because we have the following equalities:**

$$f(\alpha) = \alpha^3 + b\alpha^2 + c\alpha + d \tag{19}$$
$$f'(\alpha) = 3\alpha^2 + 2b\alpha + c \tag{20}$$
$$\frac{f''(\alpha)}{2} = 3\alpha + b \tag{21}$$

The parameter α **is determined so the coefficient B is equal to zero:**

$$(B = 0) \Rightarrow (3\alpha + b = 0) \Rightarrow \left(\alpha = -\frac{b}{3}\right) \tag{22}$$

Now we have that:

$$B = 0 \tag{23}$$
$$C = 3\alpha^2 + 2b\alpha + c = \frac{3c - b^2}{3} \tag{24}$$
$$D = \alpha^3 + b\alpha^2 + c\alpha + d = \frac{2b^3 - 9bc + 27d}{27} \tag{25}$$
$$y^3 + 0\,y^2 + C\,y + D = y^3 + C\,y + D = 0 \tag{26}$$

if $(C = 0)$ **then it follows that:**

$$y^3 + D = 0 \tag{27}$$

$$3\theta = Acos\left(\frac{-D}{|D|}\right),\ (|D| > 0) \tag{28}$$

$$-D = |D|\, e^{3\theta \boldsymbol{i}} \tag{29}$$

$$y^3 = -D = |D|\, e^{(3\theta)\boldsymbol{i}} \tag{30}$$

$$y_j = |D|^{\frac{1}{3}}\, e^{(\theta)\boldsymbol{i}} e^{\frac{2\pi j}{3}\boldsymbol{i}} \tag{31}$$

$$x_j = \alpha + y_j,\ j \in \{0, 1, 2\} \tag{32}$$

We will assume that $(|C| > 0)$ and the solutions of the equation (26) are determined by the following equations:

$$y_0 = \varepsilon_0\, p + \varepsilon_0\, q \tag{33}$$

$$y_1 = \varepsilon_1\, p + \varepsilon_2\, q \tag{34}$$

$$y_2 = \varepsilon_2\, p + \varepsilon_1\, q \tag{35}$$

where

$$\varepsilon_0 = 1 = e^{2\pi(k)\boldsymbol{i}} \tag{36}$$

$$\varepsilon_1 = e^{\frac{2\pi}{3}\boldsymbol{i}} = e^{2\pi(\frac{1}{3}+k)\boldsymbol{i}} \tag{37}$$

$$\varepsilon_2 = e^{\frac{4\pi}{3}\boldsymbol{i}} = e^{2\pi(\frac{2}{3}+k)\boldsymbol{i}} \tag{38}$$

and p^3 and q^3 are the two solutions of the quadratic equation

$$z^2 + b_0 z + c_0 = 0 \tag{39}$$

Where b_0, c_0 are the coefficients to be determined.

It is easy to prove that:

$$\varepsilon_0 + \varepsilon_1 + \varepsilon_2 = 0 \tag{40}$$

$$\varepsilon_0\, \varepsilon_1\, \varepsilon_2 = 1 \tag{41}$$

Applying $Vieta's$ formula to (26), we obtain the following equations:

$$-(y_0 + y_1 + y_2) = 0 \tag{42}$$

$$y_0\, y_1 + y_0\, y_2 + y_1\, y_2 = C \tag{43}$$

$$-y_0\, y_1\, y_2 = D \tag{44}$$

Equations (42) - (44) hold if and only if y_0, y_1 and y_2 are the roots of equation (26).
Applying the results of the program written in $Maxima$, we obtain the following equations:

$$y_0 + y_1 + y_2 = 0 \tag{45}$$

$$y_0\, y_1 + y_0\, y_2 + y_1\, y_2 = -3\, p\, q \tag{46}$$

$$y_0\, y_1\, y_2 = p^3 + q^3 \tag{47}$$

Now we have that:

$$(-3pq = C) \Rightarrow \left(pq = -\frac{C}{3}\right) \tag{48}$$

$$p^3\,q^3 = -\frac{C^3}{27} \tag{49}$$

$$p^3 + q^3 = -D \tag{50}$$

Knowing the sum and the product of p^3 and q^3 one can conclude that they are the two solutions of the quadratic equation (39), which will be denoted by z_0 and z_1.

$$b_0 = -(z_0 + z_1) = D \tag{51}$$

$$c_0 = z_0 z_1 = -\frac{C^3}{27} \tag{52}$$

$$(C \neq 0) \Rightarrow (z_0 \neq 0) \wedge (z_1 \neq 0) \tag{53}$$

$$\Delta = b_0^2 - 4c_0 = D^2 + \frac{4\,C^3}{27} \tag{54}$$

The solutions of equation (39) are given by the following expressions:

$$z_0 = \frac{-D + \Delta^{\frac{1}{2}}}{2} \tag{55}$$

$$z_1 = \frac{-D - \Delta^{\frac{1}{2}}}{2} \tag{56}$$

We will convert z_0 and z_1 from Cartesian Coordinates to Polar Coordinates:

$$3\theta_0 = Atan2\left(\frac{Re(z_0)}{|z_0|}, \frac{Im(z_0)}{|z_0|}\right) \tag{57}$$

$$z_0 = |z_0|\,e^{3\theta_0 \boldsymbol{i}} \tag{58}$$

$$p_0 = |z_0|^{\frac{1}{3}}\,e^{\theta_0 \boldsymbol{i}} \tag{59}$$

$$z_0^{\frac{1}{3}} \in \{p_0, p_0\,\varepsilon_1, p_0\,\varepsilon_2\} \tag{60}$$

$$3\theta_1 = Atan2\left(\frac{Re(z_1)}{|z_1|}, \frac{Im(z_1)}{|z_1|}\right) \tag{61}$$

$$z_1 = |z_1|\,e^{3\theta_1 \boldsymbol{i}} \tag{62}$$

$$q_0 = |z_1|^{\frac{1}{3}}\,e^{\theta_1 \boldsymbol{i}} \tag{63}$$

$$z_1^{\frac{1}{3}} \in \{q_0, q_0\,\varepsilon_1, q_0\,\varepsilon_2\} \tag{64}$$

$$(z_0\,z_1)^{\frac{1}{3}} \in \left\{-\frac{C}{3}\varepsilon_0, -\frac{C}{3}\,\varepsilon_1, -\frac{C}{3}\,\varepsilon_2\right\} \tag{65}$$

$$(z_0\,z_1)^{\frac{1}{3}} = (z_0)^{\frac{1}{3}}(z_1)^{\frac{1}{3}} \in \{p_0q_0\varepsilon_0, p_0(q_0\varepsilon_1), p_0(q_0\varepsilon_2), (p_0\varepsilon_1)q_0, ..., (p_0\varepsilon_2)(q_0\varepsilon_2)\} \equiv \{p_0q_0\varepsilon_0, p_0q_0\varepsilon_1, p_0q_0\varepsilon_2\} \tag{66}$$

$$\left\{-\frac{C}{3}\varepsilon_0, -\frac{C}{3}\,\varepsilon_1, -\frac{C}{3}\,\varepsilon_2\right\} \equiv \{(p_0q_0)\varepsilon_0, (p_0q_0)\varepsilon_1, (p_0q_0)\varepsilon_2\} \tag{67}$$

Since $(p_0 q_0)\varepsilon_0, (p_0 q_0)\varepsilon_1$ and $(p_0 q_0)\varepsilon_2$ are mutually different, it follows that there exists j so that the equality (68) holds:

$$(p_0 q_0)\varepsilon_j = -\frac{C}{3} \tag{68}$$

$$q_0 \varepsilon_j = -\frac{C}{3p_0} \tag{69}$$

$$p = p_0 \tag{70}$$

$$q = q_0 \varepsilon_j = -\frac{C}{3p} \tag{71}$$

it also follows that:

$$(p\,\varepsilon_1)(q\,\varepsilon_2) = (p\,\varepsilon_2)(q\,\varepsilon_1) = p\,q = -\frac{C}{3} \tag{72}$$

After we have calculated p and q, we can now easily determine y_0, y_1 and y_2, which are defined by the equations (33)-(35). It remains to prove that y_0, y_1 and y_2 are solutions of equation (26).

$$y_0 + y_1 + y_2 = 0 \tag{73}$$

$$y_0\,y_1 + y_0\,y_2 + y_1\,y_2 = -3\,p\,q = -3\frac{-C}{3} = C \tag{74}$$

$$y_0\,y_1\,y_2 = p^3 + q^3 = -D \tag{75}$$

The Equations (42) - (44) are fulfilled, what implies that y_0, y_1 and y_2 are the roots of equation (26). And finally we have that the solutions x_0, x_1 and x_2 of equation (11) are given by the following expressions:

$$x_0 = -\frac{b}{3} + \varepsilon_0 p + \varepsilon_0 q \tag{76}$$

$$x_1 = -\frac{b}{3} + \varepsilon_1 p + \varepsilon_2 q \tag{77}$$

$$x_1 = -\frac{b}{3} + \varepsilon_2 p + \varepsilon_1 q \tag{78}$$

3. Example

$$f(x) = x^3 + 2,3x^2 - 1.4x + 5.6 = 0 \tag{79}$$

$$\alpha = -0.766666667 \tag{80}$$

$$x = \alpha + y \tag{81}$$

$$g(y) = y^3 - 3.163333333y + 7.574592593 = 0 \tag{82}$$

$$z_0 = -0.1580779333 \tag{83}$$

$$z_1 = -7.4165146592 \tag{84}$$

$$p = 0.27035044421 + 0.4682607052\boldsymbol{i} \tag{85}$$

$$q = 0.975071862300888 - 1.6888740065\boldsymbol{i} \tag{86}$$

$$x_0 = \alpha + (p)(1) + (q)(1) = 0.4787556398 - 1.2206133013\boldsymbol{i} \tag{87}$$

$$x_1 = \alpha + (p)(\varepsilon_1) + (q)(\varepsilon_2) = -3.2575112796 \tag{88}$$

$$x_2 = \alpha + (p)(\varepsilon_2) + (q)(\varepsilon_1) = 0.4787556398 + 1.2206133013\boldsymbol{i} \tag{89}$$

4. Quartic Equation

Without loss of generality a quartic polynomial in one variable is defined in the following way:

$f(x) = x^4 + bx^3 + cx^2 + dx + e$

Where b, c,d **and** e **are real numbers. The corresponding quartic equation is defined as follows:**

$$f(x) = x^4 + bx^3 + cx^2 + dx + e = 0 \tag{90}$$

Our goal to to solve the equation (90) for x.

First we substitute $x = \alpha + y$

$$(\alpha + y)^4 + b(\alpha + y)^3 + c(\alpha + y)^2 + d(\alpha + y) + e = 0 \tag{91}$$

$$\alpha^4 + y^4 + 4y^3\,\alpha + 6y^2\,\alpha^2 + 4y\,\alpha^3 + b(\alpha^3 + y^3 + 3\alpha^2 y + 3\alpha y^2) + c(\alpha^2 + y^2 + 2\alpha y) + d(\alpha + y) + e = 0 \tag{92}$$

$$y^4 + (4\alpha + b)y^3 + (6\alpha^2 + 3b\alpha + c)y^2 + (4\alpha^3 + 3b\alpha^2 + 2c\alpha)y + \alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0 \tag{93}$$

$$y^4 + B\,y^3 + C\,y^2 + Dy + E = 0 \tag{94}$$

$$B = 4\alpha + b \tag{95}$$

$$C = 6\alpha^2 + 3b\alpha + c \tag{96}$$

$$D = 4\alpha^3 + 3b\alpha^2 + 2c\alpha + d \tag{97}$$

$$E = \alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e \tag{98}$$

It is actually easy to memorize the coefficients B, C, D **and** E **because we have the following equations:**

$$f(\alpha) = \alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e \tag{99}$$

$$f'(\alpha) = 4\alpha^3 + 3b\alpha^2 + 2c\alpha + d \tag{100}$$

$$\frac{f''(\alpha)}{2!} = 6\alpha^2 + 3b\alpha + c \tag{101}$$

$$\frac{f'''(\alpha)}{3!} = 4\alpha + b \tag{102}$$

The parameter α **is determined so the coefficient B is equal to zero:**

$$(B = 0) \Rightarrow (4\alpha + b = 0) \Rightarrow \left(\alpha = -\frac{b}{4}\right) \tag{103}$$

Now we have that:

$$B = 0 \tag{104}$$

$$C = 6\alpha^2 + 3b\alpha + c = -\frac{3b^2}{8} + c \tag{105}$$

$$D = 4\alpha^3 + 3b\alpha^2 + 2c\alpha + d = \frac{b^3}{8} - \frac{b\,c}{2} + d \tag{106}$$

$$E = \alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = -\frac{3b^4}{256} + \frac{b^2\,c}{16} - \frac{b\,d}{4} + e \tag{107}$$

$$y^4 + 0\,y^3 + C\,y^2 + Dy + E = y^4 + C\,y^2 + Dy + E = 0 \tag{108}$$

If $(D = 0)$ then it follows that:

$$y^4 + C\,y^2 + E = 0 \tag{109}$$

$$z = y^2 \tag{110}$$

$$z^2 + C\,z + E = 0 \tag{111}$$

We can easily solve the quadratic equation (111), then find the the roots of equation (108) and finally find the roots of equation (90).

$$z_0 = \frac{-C + \sqrt{C^2 - 4E}}{2} \tag{112}$$

$$z_1 = \frac{-C - \sqrt{C^2 - 4E}}{2} \tag{113}$$

$$y_{0,1} = \pm\sqrt{z_0} \tag{114}$$

$$y_{2,3} = \pm\sqrt{z_1} \tag{115}$$

$$x_i = \alpha + y_i,\; i \in \{0, 1, 2, 3\} \tag{116}$$

We will assume that $(D \neq 0)$ and the solutions of the equation (108) are determined by the following equations:

$$y_0 = p + q + r \tag{117}$$

$$y_1 = -p - q + r \tag{118}$$

$$y_2 = -p + q - r \tag{119}$$

$$y_3 = p - q - r \tag{120}$$

where p^4 , q^4 and r^4 are the solutions of the equation (121),

$$z^3 + b_0\,z^2 + c_0\,z + d_0 = 0 \tag{121}$$

whose coefficients we need to determine.

Applying $Vieta's$ formulas to equation (108), we obtain the following equalities:

$$-(y_0 + y_1 + y_2 + y_3) = 0 \tag{122}$$

$$y_0\,y_1 + y_0\,y_2 + y_0\,y_3 + y_1\,y_2 + y_1\,y_3 + +y_2\,y_3 = C \tag{123}$$

$$-(y_0\,y_1\,y_2 + y_0\,y_1\,y_3 + y_0\,y_2\,y_3 + y_1\,y_2\,y_3) = D \tag{124}$$

$$y_0\,y_1\,y_2\,y_3 = E \tag{125}$$

Equations (122) - (125) hold if and only if y_0, y_1, y_2 and y_3 are the roots of equation (108). Applying the results of the program written in $Maxima$ [2], we obtain the following equalities:

$$y_0 + y_1 + y_2 + y_3 = 0 \tag{126}$$

$$y_0\,y_1 + y_0\,y_2 + y_0\,y_3 + y_1\,y_2 + y_1\,y_3 + +y_2\,y_3 = -2\,(p^2 + q^2 + r^2) \tag{127}$$

$$y_0\,y_1\,y_2 + y_0\,y_1\,y_3 + y_0\,y_2\,y_3 + y_1\,y_2\,y_3 = 8\,p\,q\,r \tag{128}$$

$$y_0\,y_1\,y_2\,y_3 = p^4 + q^4 + r^4 - 2(p^2\,q^2 + p^2\,r^2 + q^2\,r^2) \tag{129}$$

$$p^2 + q^2 + r^2 = -\frac{C}{2} \tag{130}$$

$$p\,q\,r = -\frac{D}{8} \tag{131}$$

$$p^4 + q^4 + r^4 - 2(p^2\,q^2 + p^2\,r^2 + q^2\,r^2) = E \tag{132}$$

Now we are going to determine the coefficients b_0, c_0 and d_0:

$$(p^2 + q^2 + r^2)^2 = p^4 + q^4 + r^4 + 2(p^2\,q^2 + p^2\,r^2 + q^2\,r^2) = \frac{C^2}{4} \tag{133}$$

$$p^4 + q^4 + r^4 - 2(p^2\,q^2 + p^2\,r^2 + q^2\,r^2) = E \tag{134}$$

$$p^4 + q^4 + r^4 = \frac{C^2}{8} + \frac{E}{2} \tag{135}$$

$$p^2\,q^2 + p^2\,r^2 + q^2\,r^2 = \frac{C^2}{16} - \frac{E}{4} \tag{136}$$

$$(p^2\,q^2 + p^2\,r^2 + q^2\,r^2)^2 = p^4\,q^4 + p^4\,r^4 + q^2\,r^4 + 2\,p^2\,q^2\,r^2\,(p^2 + q^2 + r^2) \tag{137}$$

$$\left(\frac{C^2}{16} - \frac{E}{4}\right)^2 = p^4\,q^4 + p^4\,r^4 + q^4\,r^4 - \frac{D^2}{32}\frac{C}{2} \tag{138}$$

$$p^4\,q^4 + p^4\,r^4 + q^4\,r^4 = \left(\frac{C^2}{16} - \frac{E}{4}\right)^2 + \frac{D^2}{32}\frac{C}{2} \tag{139}$$

$$p^4 + q^4 + r^4 = \frac{C^2}{8} + \frac{E}{2} \tag{140}$$

$$p^4\,q^4 + p^4\,r^4 + q^4\,r^4 = \left(\frac{C^2}{16} - \frac{E}{4}\right)^2 + \frac{D^2}{32}\frac{C}{2} \tag{141}$$

$$p^4\,q^4\,r^4 = \frac{D^4}{8^4} \tag{142}$$

$$b_0 = -(p^4 + q^4 + r^4) = -\frac{C^2}{8} - \frac{E}{2} \tag{143}$$

$$c_0 = p^4\,q^4 + p^4\,r^4 + q^4\,r^4 = \left(\frac{C^2}{16} - \frac{E}{4}\right)^2 + \frac{D^2}{32}\frac{C}{2} \tag{144}$$

$$d_0 = -p^4\,q^4\,r^4 = -\frac{D^4}{8^4} \tag{145}$$

We can define coefficients $\varepsilon_0, \varepsilon_1, \varepsilon_2$ and ε_3 as it follows:

$$\varepsilon_0 = 1 \tag{146}$$

$$\varepsilon_1 = e^{\frac{\pi}{2}\boldsymbol{i}} = \boldsymbol{i} \tag{147}$$

$$\varepsilon_2 = e^{\frac{2\pi}{2}\boldsymbol{i}} = -1 \tag{148}$$

$$\varepsilon_3 = e^{\frac{3\pi}{2}\boldsymbol{i}} = -\boldsymbol{i} \tag{149}$$

Equations (143) - (145) hold if and only if p^4, q^4 and r^4 are the roots of equation (121). For the sake of simplicity, the solutions of equation (121) will be denoted by z_0, z_1 and z_2.

$$(D \neq 0) \Rightarrow (z_0 \neq 0) \wedge (z_1 \neq 0) \wedge (z_2 \neq 0) \tag{150}$$

We will convert z_0, z_1 and z_2 from Cartesian Coordinates to Polar Coordinates:

$$4\theta_0 = Atan2\left(\frac{Re(z_0)}{|z_0|}, \frac{Im(z_0)}{|z_0|}\right) \tag{151}$$

$$z_0 = |z_0|\, e^{4\theta_0 \boldsymbol{i}} \tag{152}$$

$$p_0 = |z_0|^{\frac{1}{4}}\, e^{\theta_0 \boldsymbol{i}} \tag{153}$$

$$z_0^{\frac{1}{4}} \in \{p_0, p_0\,\varepsilon_1, p_0\,\varepsilon_2, p_0\,\varepsilon_3\} \tag{154}$$

$$4\theta_1 = Atan2\left(\frac{Re(z_1)}{|z_1|}, \frac{Im(z_1)}{|z_1|}\right) \tag{155}$$

$$z_1 = |z_1|\, e^{4\theta_1 \boldsymbol{i}} \tag{156}$$

$$q_0 = |z_1|^{\frac{1}{4}}\, e^{\theta_1 \boldsymbol{i}} \tag{157}$$

$$z_1^{\frac{1}{4}} \in \{q_0, q_0\,\varepsilon_1, q_0\,\varepsilon_2, q_0\,\varepsilon_3\} \tag{158}$$

$$\tag{159}$$

$$4\theta_2 = Atan2\left(\frac{Re(z_2)}{|z_2|}, \frac{Im(z_2)}{|z_2|}\right) \tag{160}$$

$$z_2 = |z_2|\, e^{4\theta_2 \boldsymbol{i}} \tag{161}$$

$$r_0 = |z_2|^{\frac{1}{4}}\, e^{\theta_2 \boldsymbol{i}} \tag{162}$$

$$z_2^{\frac{1}{4}} \in \{r_0, r_0\,\varepsilon_1, r_0\,\varepsilon_2, r_0\,\varepsilon_3\} \tag{163}$$

$$\tag{164}$$

There are a total of 64 triples $(p_0\,\varepsilon_i, q_0\,\varepsilon_j, r_0\,\varepsilon_k)$ that are potential solutions to the equations (117)-(120), but we will select only those triples that satisfy the conditions (130) and (131). One can denote by (p, q, r) a triple, which satisfies equations (130) and (131).

$$p = p_0\,\varepsilon \tag{165}$$

$$q = q_0\,\zeta \tag{166}$$

$$r = r_0\,\eta \tag{167}$$

$$p^2 + q^2 + r^2 = p_0^2\,\varepsilon^2 + q_0^2\,\zeta^2 + r_0^2\,\eta^2 = p_0^2 f_0 + q_0^2 f_1 + r_0^2 f_2 \tag{168}$$

Where $(\varepsilon, \zeta, \eta) \in \{1, \boldsymbol{i}, -1, -\boldsymbol{i}\}$ and $(f_0, f_1, f_2) \in \{-1, 1\}$.

Let's define the function $u(f_0, f_1, f_2)$ in the following way:

$$u(f_0, f_1, f_2) = p_0^2\, f_0 + q_0^2\, f_1 + r_0^2\, f_2 \tag{169}$$

In order for condition (130) is fulfilled, it is necessary to find a triple (f_0, f_1, f_2) out of eight,

$$u(f_0, f_1, f_2) = -\frac{C}{2} \tag{170}$$

so that equality (170) holds.

Now we have that:

$$\varepsilon = \pm\sqrt{f_0} \tag{171}$$
$$\zeta = \pm\sqrt{f_1} \tag{172}$$
$$\eta = \pm\sqrt{f_2} \tag{173}$$

We can fix the values of the ε and ζ.

$$\varepsilon = \sqrt{f_0} \tag{174}$$
$$\zeta = \sqrt{f_1} \tag{175}$$

Condition (131) is satisfied if and only if equation (176) holds.

$$p\,q\,r = (p_0\,\varepsilon)\,(q_0\,\zeta)\,(r_0\,\eta) = -\frac{D}{8} \tag{176}$$
$$(r_0\,\eta) = -\frac{D}{8(p_0\,\varepsilon)(q_0\,\zeta)} \tag{177}$$
$$r = -\frac{D}{8pq} \tag{178}$$

If the equation (176) is satisfied for the triple (p, q, r) then it also applies to the triples $(p, -q, -r)$, $(-p, q, -r)$ and $(-p, -q, r)$.
We have calculated p, q and r and now we able to determine y_0, y_1, y_2 and y_3 that are defined by the equations (117)-(120). We still need to prove that y_0, y_1, y_2 and y_3 are solutions of equation (108).

$$y_0 + y_1 + y_2 + y_3 = 0 \tag{179}$$
$$y_0\,y_1 + y_0\,y_2 + y_0\,y_3 + y_1\,y_2 + y_1\,y_3 + y_2\,y_3 = -2\,(p^2 + q^2 + r^2) = -2\left(-\frac{C}{2}\right) = C \tag{180}$$
$$y_0\,y_1\,y_2 + y_0\,y_1\,y_3 + y_0\,y_2\,y_3 + y_1\,y_2\,y_3 = 8\,p\,q\,r = 8\left(-\frac{D}{8}\right) = -D \tag{181}$$

$$y_0\,y_1\,y_2\,y_3 = p^4 + q^4 + r^4 - 2(p^2\,q^2 + p^2\,r^2 + q^2\,r^2) \tag{182}$$
$$p^4 + q^4 + r^4 = \frac{C^2}{8} + \frac{E}{2} \tag{183}$$
$$(p^2 + q^2 + r^2)^2 = \frac{C^2}{4} \tag{184}$$
$$p^4 + q^4 + r^4 - 2(p^2\,q^2 + p^2\,r^2 + q^2\,r^2) = 2(p^4\,q^4 + p^4\,r^4 + q^4\,r^4) - (p^2 + q^2 + r^2)^2 \tag{185}$$
$$2(p^4\,q^4 + p^4\,r^4 + q^4\,r^4) - (p^2 + q^2 + r^2)^2 = 2\left(\frac{C^2}{8} + \frac{E}{2}\right) - \frac{C^2}{4} = E \tag{186}$$
$$y_0\,y_1\,y_2\,y_3 = E \tag{187}$$

The Equations (122) - (125) are fulfilled, what implies that y_0, y_1, y_2 and y_3 are the roots of equation (108).

And finally we have that the solutions x_0, x_1, x_2 and x_3 of equation (90) are given by the following equalities:

$$x_0 = -\frac{b}{4} + y_0 \tag{188}$$

$$x_1 = -\frac{b}{4} + y_1 \tag{189}$$

$$x_2 = -\frac{b}{4} + y_2 \tag{190}$$

$$x_3 = -\frac{b}{4} + y_3 \tag{191}$$

5. Example

$$f(x) = x^4 - 2x^3 + 6x^2 + 7x + 5 = 0 \tag{192}$$

$$\alpha = 0.5 \tag{193}$$

$$x = \alpha + y \tag{194}$$

$$g(y) = y^4 + 4.5y^2 + 12y + 9.8125 = 0 \tag{195}$$

$$(z_0 = 5.5111073125821) \Rightarrow (p_0 = 1.532179) \tag{196}$$

$$(z_1 = 0.8675523524299) \Rightarrow (q_0 = 0.96510357 \tag{197}$$

$$(z_2 = 1.05884033498797) \Rightarrow (r_0 = 1.01439621) \tag{198}$$

$$u(-1,-1,1) = -2.25 \tag{199}$$

$$(p_0 q_0 r_0)(\boldsymbol{i}\ \boldsymbol{i}\ 1) = -1.5 \tag{200}$$

$$p = (p_o)(\boldsymbol{i}) = 1.53217974550053\boldsymbol{i} \tag{201}$$

$$q = (q_o)(\boldsymbol{i}) = 0.965103571821579\boldsymbol{i} \tag{202}$$

$$r = (r_o)(1) = 1.01439621295865 \tag{203}$$

$$x_0 = \alpha + (p)(1) + (q)(1) + (r)(1) = 1.5143962 + 2.4972833\boldsymbol{i} \tag{204}$$

$$x_1 = \alpha + (p)(-1) + (q)(-1) + (r)(1) = 1.5143962 - 2.4972833\boldsymbol{i} \tag{205}$$

$$x_2 = \alpha + (p)(-1) + (q)(1) + (r)(-1) = -0.5143962 - 0.5670761\boldsymbol{i} \tag{206}$$

$$x_3 = \alpha + (p)(1) + (q)(-1) + (r)(-1) = -0.5143962 + 0.5670761\boldsymbol{i} \tag{207}$$

6. Conflict of interest

The author is not aware of any conflict of interest associated with this work.

References

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