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A new generalization on metric space and its metrizability.

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Abstract;

In this paper, our particular scope is to give a new generalization for the metric function and after that a proof of the metrizability of generalized φ-metric space. This new approach is influenced by the Chittenden's metrization theorem.

Keywords: generalized φ-metric space, metrizability, metric space.

Introduction

Metric spaces are the most generalized spaces, with lots of new concepts and a huge number of articles, exploring properties and new theories about them. In 1993, Czerwik [4] presented the b-metric function, generalizing the metric function. Later in 1998, [5] he altered this concept by replacing the coefficient $K\geq 1$ instead of the coefficient 2 in the triangle inequation, presenting the definition bellow:

Definition 1.1. [5] Let X be a nonempty set and $d: X\times X\to R\geq 0$ be a function for all $x, y, z\in X$ which fulfills the following conditions:

1) $d(x, y) = 0\Leftrightarrow x = y$;
2) $d(x, y) = d(y, x)$;
3) $d(x, z)\leq K[d(x, y) + d(y, z)]$, $K\geq 1$.

Then *d* is called a *b-metric function* and the pair (X, d) is called a *b-metric space*.

After that, definitions of *strong b-metric* and *S-metric* were given as generalizations of the b-metric function. In [6] a new approach of generalizing metric function was given with the definition below:

Definition 1.2. [6] The function $d_\phi: X\times X\to R\geq 0$, is called a ϕ-*metric* if it satisfies the following conditions:

a) $d_\phi(x, y) = 0\Leftrightarrow x = y$;
b) $d_\phi(x, y) = d_\phi(y, x)$;
c) $d_\phi(x, z) \leq d_\phi(x, y) + d_\phi(y, z) + \phi(x, y, z)$, $\forall x, y, z\in X$, with $\phi: X\times X\times X\to R\geq 0$ a function fulfilling:

1) $\phi(x, y, z) = 0$ if $x = z$ or $y = z$;
2) $\phi(x, y, z) = \phi(y, x, z)$;
3) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\phi(x, y, z) < \varepsilon$, whenever $d_\phi(x, y) < \delta$ or $d_\phi(y, z) < \delta$, $\forall x, y, z\in X$.

The ordered pair (X, d_ϕ) is called a ϕ-*metric space*.

Aimar, H, [1] in 1998, proved the metrizability of the b-metric space, the generalized notion presented by Czverwik in [5]. Later in 2015, An. V, [2] gave a proof of the metrizability of b-metric spaces with a constant $K > 0$, with the expectation that the distance function is continuous in one variable.

Theorem 1.3. [2] Let (X, D, K) be a b-metric space. If D is continuous in one variable, then every open cover of X has an open refinement which is both locally finite and σ −discrete.

Corollary 1.4. [2] Let (X, D, K) be a b-metric space. If D is continuous in one variable, then X is metrizable.

Also, for the definition of the ϕ-metric space in [6] it is given a metrizability proof for this space by a Stone-type theorem.

Now let us recall metrizability theorems due to Niemytski and Wilson [8] and Chittenden's metrization theorem [3].

Theorem 1.5. [8] Let X be a topological space and $F: X\times X\to[0,\infty)$ be a distance function on X. If the distance function *F* satisfies:

i. $F(x,y) = 0 \Leftrightarrow x = y$ for all $x, y\in X$
ii. $F(x,y) = F(y,x)$ for all $x, y\in X$, and one of the following conditions:
iii. Given a point $a\in X$ and a number $\epsilon > 0$, there exists $\phi(a,\epsilon) > 0$ such that if $F(a,b) < \phi(a,\epsilon)$ and $F(b,c) < \phi(a,\epsilon)$ then $F(a,c) < \varepsilon$;
iv. If $a\in X$ and $\{a_n\}_{n\in N}$, $\{b_n\}_{n\in N}$ are two sequences in X such that $F(a_n, a)\to 0$ and $F(a_n, b_n)\to 0$ as $n\to\infty$ then $F(b_n, a)\to 0$ as $n\to\infty$.
v. For each point $a\in X$ and a positive number k, there is a positive number *r* such that if $b\in X$ for which $F(a,b)\geq k$, and *c* is any point then $F(a,c) + F(b,c)\geq r$;

then the topological space *X* is metrizable.

Theorem 1.6. [3] Let *X* be a topological space and $F: X\times X\to[0,\infty)$ be a distance function on X. If the distance function *F* satisfies the following conditions:

i. $F(x,y) = 0 \Leftrightarrow x = y$ for all $x, y\in X$
ii. $F(x,y) = F(y,x)$ for all $x, y\in X$
iii. (Uniformly regular) For every $\varepsilon > 0$ there exists $\phi(\varepsilon) > 0$ such that for all $x, y, z\in X$, $F(x,y) < \phi(\varepsilon)$ and $F(y,z) < \phi(\varepsilon)$ imply $F(x,z) < \varepsilon$,

then the topological space *X* is metrizable.

Main Results

In this part, we present a new generalization for the ϕ-metric function. After that, two alternative proofs for the metrizability of this new space are listed, by using Niemytski and Wilson [8] theorem for the metrizability and Chittenden's metrization theorem [3].

Definition 2.1. The function $G: X\times X\to R\geq 0$, is called a *generalized φ-metric function* if satisfies the following conditions:

A. $G(x,y) = 0 \Leftrightarrow x = y$;
B. $G(x,y) = G(y,x)$;

C. $G(x,z)\leq K[G(x,y) + G(y,z)] + \varphi(x,y,z)$, $\forall x, y, z\in X$, and $K\geq 1$, with $\varphi: X\times X\times X\to R\geq 0$ a function fulfilling:
D. $\varphi(x,y,z) = 0$ if $x = z$ or $y = z$;
E. $\varphi(x,y,z) = \varphi(y,x,z)$;
F. $\forall\varepsilon > 0, \exists\delta > 0$ such that $\varphi(x,y,z) < \varepsilon$, whenever $G(x,y) < \delta$ or $G(y,z) < \delta$, $\forall x, y, z\in X$.

The ordered pair (X, G) is called a *generalized φ-metric space*.
Theorem 2.2. Let (X, G) be a *generalized φ-metric space*. Then *X* is metrizable.

Proof. Let (X, G) be a *generalized φ-metric space*. By the definition of the *generalized φ-metric function* $G: X\times X\to R\geq 0$, it satisfies the first two conditions of Niemytski and Wilson's metrization theorem:

i. $G(x,y) = 0 \Leftrightarrow x = y$, for all $x, y\in X$;
ii. $G(x,y) = G(y,x)$ for all $x, y\in X$;

To prove the third condition, we are going to use condition (v) of the theorem 1.5. Let $a\in X$ and s be a positive real number such that $s > \varphi(a,b,c)$. Suppose that $b\in X$ such that $G(a,b)\geq s$. If *c* is any point in *X* then by the definition of a *generalized φ-metric space* we have:

$$G(a,b)\leq K[G(a,c) + G(c,b)] + \varphi(a,b,c)$$

$$\Rightarrow G(a,c) + G(c,b) \geq \frac{s-\varphi(a,b,c)}{K} = r > 0$$

This shows that the *generalized φ-metric space* satisfies the locally regular condition, and X is metrizable.

Theorem 2.3. Let (X, G) be a *generalized φ-metric space*. Then X is metrizable.

Proof 1. Let X be a generalized φ-metric space then the function $G: X\times X\to R\geq 0$ satisfies the first two conditions of Chittenden's metrization results:

i. $G(x,y) = 0 \Leftrightarrow x = y$ for all $x, y\in X$;
ii. $G(x,y) = G(y,x)$ for all $x, y\in X$;

Now to prove the third condition, let $\varepsilon > 0$, for $\varepsilon - \varphi(x,z,y)\in R$, there exists $\delta > 0$ such that $0 < \delta < \varepsilon - \varphi(x,z,y)$. Let us choose $\phi(\varepsilon) = \frac{\delta}{2K}$. If $G(x,y) < \frac{\delta}{2K}$ and $G(y,z) < \frac{\delta}{2K}$ then $K[G(x,y) + G(y,z)] + \varphi(x,z,y) < \delta$. Then by the third condition of the G function the following will hold:

$$G(x,z) \leq K[G(x,y) + G(y,z)] + \varphi(x,z,y)$$
$$< \delta < \varepsilon - \varphi(x,z,y) + \varphi(x,z,y) = \varepsilon$$

Thus: For $G(x,y) < \phi(\varepsilon), G(y,z) < \phi(\varepsilon) \Longrightarrow G(x,z) < \varepsilon$.

So, the third condition of the Chittenden's metrization theorem holds, which implies that (X, G) is metrizable.

Proof 2. An alternative way to prove the metrizability of the generalized φ-metric space is by using point (iii) and (iv) of theorem 1.5.

Let $a\in X$ and $\{a_n\}_{n\in N}$, $\{b_n\}_{n\in N}$ are two sequences in X such that $G(a_n, a)\to 0$ and $G(a_n, b_n)\to 0$ as $n\to\infty$. Let $\varepsilon > 0$, for $\varepsilon - \varphi(a, b_n, a_n)\in R$, there exists $\delta > 0$ such that $0 < \delta < \varepsilon - \varphi(a, b_n, a_n)$. Let us choose $\phi(\varepsilon) = \frac{\delta}{2K}$, there exists $k_1, k_2\in N$ such that:

$$G(a_n, a) < \frac{\delta}{2K},\ \forall n\geq k_1 \text{ and } G(a_n, b_n) < \frac{\delta}{2K},\ \forall n\geq k_2.$$

If $n\geq max\{k_1, k_2\}$ and $a\neq b_n$, then by the definition of the generalized φ-metric space, we have:

$$G(a, b_n)\leq K[G(a, a_n) + G(a_n, b_n)] + \varphi(a, b_n, a_n)$$
$$\leq K[\frac{\delta}{2K} + \frac{\delta}{2K}] + \varphi(a, b_n, a_n) = \varepsilon.$$

This shows that $G(b_n, a)\to 0$ as $n\to\infty$. So, by the metrization criteria of the Niemytski and Wilson, we conclude that the generalized φ-metric space is metrizable.

Conclusions

The generalized φ-metric space is metrizable.

References

[1] Aimar, H., Iaffei, B. and Nitti, L., *On the Macias-Segovia metrization of quasi-metric spaces*, Rev. Un. Mat. Argentina, 41 (1998), No. 2, 67–75.

[2] An, V. T., Tuyen, Q. L. and Dung, V. N., *Stone-type theorem on b-metric spaces and applications*, Topology Appl., 185-186 (2015), 50–64.

[3] Chittenden, E. W., *On the equivalence of Ecart and voisinage*, Trans. Amer. Math. Soc., 18 (1917), 161–166.

[4] Czerwik, S., *Contraction mappings in b-metric spaces*, Acta Math. Univ. Ostraviensis, 1 (1993), 5–11.

[5] Czerwik, S., *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Semin. Mat. Fis. Univ. Modena, 46 (1998), 263–276.

[6] Das, A., Kundu, A., Bag, T,. *A new approach to generalize metric functions*. Int. J. Nonlinear Amal. Appl. 14(2023) 3, 279-298.

[7] Jleli, M. and Samet, B., *On a new generalization of metric spaces*, J. Fixed Point Theory Appl., (2018), 20:128.

[8] Niemytski, V. W., *On the third axiom of metric space*, Trans. Amer. Math. Soc., 29 (1927), 507–513.