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Finite-Timeless Local Observability for Linear Control Systems on Lie Groups

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Abstract

In this work, we introduce finite-timeless observability for linear control systems on Lie groups. Observability problem is to distinguish the elements of the state space by looking the images of the solutions passing through those elements in the output space. For this aim, we define almost indistinguishability and show that it is an equivalence relation. We consider linear control systems on Lie groups, where the output function is a Lie group homomorphism and study properties of the image of almost indistinguishable points from the neutral element. Thus, we relate local observability of this kind of systems to its finite-timeless local observability.

Keywords: observability, indistinguishability, linear control systems and Lie algebras.

Mathematics Subject Classification: 93B07

1 Introduction

The aim of this paper is to introduce a new approach to the observability problem in Control Theory. Observability problem is one of the classical fundamental problem in the area and it is important from the applications point of view. In this work, our approach comes from the finite-time point of view. For a general control system $\Sigma = (G, \mathcal{D}, h, V)$ on Lie groups, where G and V are Lie groups, $\mathcal{D}$ is the dynamic of the system and h is a smooth function between G and V, observability problem is to distinguish the points of the state space and the solutions of the system just by looking their image under h in V with the positive time.

In [5] and [6], the authors study observability problem of linear control systems on Lie groups. In [5], the authors characterize local and global observability with a Lie group homomorphism and in [6], the authors characterize observability with a kind of projection. In [9], the author examines the existence of observable linear control systems on Lie groups appears in [5] and [6]. Related to these articles, we give a new time apporoach for the usual local observability.

In this work, we consider linear control systems on connected Lie groups and introduce finite-timeless observability. For this new concept of observability, we define almost indistinguishability and show that it is an equivalence relation. We also study properties of the image of almost indistinguishable points from the neutral element. Finally, we relate local observability to finite-timeless local observability.

2 Linear Control Systems on Lie Groups

Let G be a connected Lie group, i.e., a smooth manifold having a group structure, with the property that the multiplication and inversion maps are smooth.

Denote by $\chi(G)$ the set of all smooth vector fields on G and let $L(G)$ be the Lie algebra of G which is the set of all smooth left invariant vector fields on G.

In general, a linear control system on a Lie group G is represented by the pair $\Sigma = (G, \mathcal{D})$, where G is a Lie group and $\mathcal{D}$ is the dynamic of the system determined by the following family of differential equations:

$$\dot{g}(t) = X(g(t)) + \sum_{j=1}^{k} u_j(t) Y^j(g(t)).$$

Here, $g \in G$, $u = (u_1, u_2, \cdots, u_k) \in \mathcal{U}$, where $\mathcal{U}$ is a family of piecewise constant real valued functions, $Y^1, Y^2, \cdots, Y^k \in L(G)$ and X is a linear vector field. Linear vector fields are generally called infinitesimal automorphisms in the context of the theory of Lie groups. An infinitesimal automorphism $X \in Ia(G)$, namely a vector field said to be linear if its flow is a one parameter group of automorphisms that means for each $t \in \mathbb{R}$, X_t is an automorphism on G.

The dynamic $\mathcal{D}$ of the system Σ is a subset of $\chi(G)$ parametrized by the control u and defined by

$$\mathcal{D} = \{X + uY^i \mid X \in Ia(G), Y^i \in L(G) \text{ and } u \text{ is control}\}.$$

There are many works on linear control systems on Lie groups and some of them can be found in [1], [2], [3] and [4]

In this work, we deal with the observability problem aspects of the linear control systems, so we use the fourtuple $\Sigma = (G, \mathcal{D}, h, H)$, where H is also a Lie group as output space and h is a smooth function as output map between G and H. We consider that G and H have Hausdorff topology in our work.

This control system Σ induces a group of diffeomorphisms

$$G_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} | Z^j \in \mathcal{D}, t_j \in \mathbb{R}, j \in \mathbb{N}\}$$

and the semi group

$$S_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} | Z^j \in \mathcal{D}, t_j \geq 0, j \in \mathbb{N}\}.$$

Besides,

$$G_\Sigma(g) = \{\varphi(g) | g \in G, \forall \varphi \in G_\Sigma\}$$

is the orbit, and

$$S_\Sigma(g) = \{\varphi(g) | g \in G, \forall \varphi \in S_\Sigma\}$$

is the positive orbit of the system at the state g.

The classical linear control system $\Sigma = (\mathbb{R}^n, \mathcal{D}, C, \mathbb{R}^s)$ on Euclidean space $\mathbb{R}^n$ is well-known, [10]. Here, the dynamic $\mathcal{D}$ is determined by the specification of the following data:

$$\dot{x}(t) = A(x(t)) + Bu(t)$$

$$h(x) = Cx = y \in \mathbb{R}^s,$$

where, A, B and C are matrices of appropriate orders.

Linear control system on a Lie group G is a generalization of linear control system on Euclidean space $\mathbb{R}^n$.

3 Finite-timeless Observability

In this section, we would like to introduce finite-timeless locally observability (almost locally observability) and finite-timeless observability (almost observability) for linear control systems on Lie groups.

3.1. Definition:

Let g_1, g_2 be two distinct elements of Lie group G. Then, they are called almost indistinguishable points of the system, if the following equation holds for all $t \in \mathbb{R}^+ \cup \{0\} - F$, where F is a subset of $\mathbb{R}$ having measure zero;

$$h \circ S_\Sigma(g_1) = h \circ S_\Sigma(g_2).$$

If we rewrite this equality by the elements of S_Σ, then we have

$$h(\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k(g_1)) = h(\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k(g_2)).$$

We denote by $g_1 \sim_a g_2$, when g_1 and g_2 are almost indistinguishable.

3.2. Remark:

We say that a pair of distinct elements (g_1, g_2) of G is almost indistinguishable points of the system Σ, if for all $t \in \mathbb{R}^+ \cup \{0\} - F$;

$$g_1 \sim_a g_2 \iff h \circ \varphi_t(g_1) = h \circ \varphi_t(g_2), \forall \varphi_t \in S_\Sigma,$$

where the subset $F \subset \mathbb{R}$ is of measure zero.

3.3. Definition:

A linear control system Σ is called finite-timeless locally observable or almost locally observable at g, if $g \in G$ has a neighborhood U such that for all $t \in \mathbb{R}^+ \cup \{0\} - F$, where $F \subset \mathbb{R}$ is of measure zero;

$$h \circ S_\Sigma(l) \neq h \circ S_\Sigma(g), \forall l \in U.$$

If this is true for all elements of G, then the system Σ is called locally finite-timeless observable or almost locally observable.

3.4. Definition:

A linear control system Σ is called finite-timeless observable or almost observable at g, if for all $t \in \mathbb{R}^+ \cup \{0\} - F$, where $F \subset \mathbb{R}$ is of measure zero;

$$\forall x \in G, h \circ S_\Sigma(g_1) \neq h \circ S_\Sigma(g_2).$$

If this is true for all elements of G, then the system Σ is called finite-timeless observable or almost observable.

3.5. Remark:

For general finite-timeless observability (almost observability), local finite-timeless observability (almost local observability) is a necessary condition.

3.6. Lemma:

Consider a linear control system $\Sigma = (G, \mathcal{D}, h, H)$, where G and H are Lie groups. Then, $\sim_a$ is an equivalence relation.

Proof: In this kind of systems, it is known that all vector fields are complete and therefore normal indistinguishability is an equivalence relationship. Here, we would like to show that almost indistinguishable relation is also an equivalence relation. The first two properties are obvious. For transitivity, we take any $g_1, g_2, g_3 \in G$ such that $g_1 \sim_a g_2$ and $g_2 \sim_a g_3$ and want to show that $g_1 \sim_a g_3$. $g_1 \sim_a g_2$ implies that for some subset $F \subset \mathbb{R}$ with measure zero and for all $\forall t \in \mathbb{R}^+ \cup \{0\} - F$, we have

$$h \circ \varphi_t(g_1) = h \circ \varphi_t(g_2), \forall \varphi_t \in S_\Sigma.$$

In the same way, for some subset $F' \subset \mathbb{R}$ with measure zero and for all $\forall t \in \mathbb{R}^+ \cup \{0\} - F'$, we have

$$g_2 \sim_a g_3 \iff h \circ \varphi_t(g_2) = h \circ \varphi_t(g_3), \forall \varphi_t \in S_\Sigma.$$

Then, we have the following two cases:

1. $F = F' \Rightarrow g_1 \sim_a g_3 \iff h \circ \varphi_t(g_1) = h \circ \varphi_t(g_3), \forall \varphi_t \in S_\Sigma,$

or

2. $F \neq F' \Rightarrow$:

a) Either F is a proper subset of F' or F' is a proper subset of F. In each cases, transitivity holds since any subset of set with measure zero is also with measure zero.

b) Either they are disjoint or they have nonempty intersection. Then, for all $t \in \mathbb{R}^+ \cup \{0\} - F \cup F'$, where $F \cup F' \subset \mathbb{R}$ is with measure zero since countable union of sets with measure zero is also with measure zero;

$$g_1 \sim_a g_3 \iff h \circ \varphi_t(g_1) = h \circ \varphi_t(g_3), \forall \varphi_t \in S_\Sigma.$$

After this point, we would like to consider Σ with a specific output function $h \in Hom(G, H)$, Lie group homomorphisms, [5]. We study the set of almost indistinguishable elements in the following lemma in order to study finite-timeless observability.

3.7. Lemma:

Let $\boldsymbol{I}_e$ be the set of images of the almost indistinguishable points from the neutral element e of G of a linear control system Σ with $h \in Hom(G, H)$. Then, $\boldsymbol{I}_e$ is a Hausdorff space.

Proof: The set of images of the almost indistinguishable points from the neutral element e of G has the following form:

$$\boldsymbol{I}_e = \{s \in H | h(g) = s, \forall g \sim_a e \text{ and } s \in h \circ \varphi_t(g)\}$$

$\boldsymbol{I}_e$ has a Hausdorff topology as a continuous image of the set of almost indistinguishable points from the neutral element e of G which has a Hausdorff topology.

3.8. Lemma:

For any linear control system Σ with $h \in Hom(G, H)$, $\boldsymbol{I}_e$ is closed.

Proof: For any element of any sequence $\{s_n\}_{n \in \mathcal{N}} \in \boldsymbol{I}_e$ which converges to s, there exists at least one element $g_n \in G$ as a pre-image satisfying $h(g_n) = s_n$ and $g_n \sim_a e$, $\forall \varphi_t \in S_\Sigma$. If K denotes the set of points in G with this property, then any element of any sequence in $\boldsymbol{I}_e$ has a pre-image in K. For any sequence g_n in K which converges to g, we have $h \circ \varphi_t(g_n) \to h \circ \varphi_t(g)$ for a fixed $t \in \mathbb{R}^+ \cup \{0\} - F$ by the continuity of $h \circ \varphi_t$. For each n, g_n is an element of K and there exists an element $s_n \in \boldsymbol{I}_e$ such that $s_n \in h \circ \varphi_t(g_n)$. Thus, $h \circ \varphi_t(g_n) \to h \circ \varphi_t(g)$, $s \in h \circ \varphi_t(g)$ and therefore $s \in \boldsymbol{I}_e$.

3.9. Corollary:

The set $\boldsymbol{I}_e$ of images of almost indistinguishable points from the neutral element e of G of a linear control system Σ with $h \in Hom(G, H)$ is a Lie subgroup of H, since any closed subgroup of a Lie group is also a Lie group, [8].

3.10. Remark:

Consider a linear control system $\Sigma = (G, \mathcal{D}, h, H)$, where G and H are Lie groups and $h \in Hom(G, H)$. Denote by $\boldsymbol{I}_e^*$ the set of all indistinguishable points from the neutral element e of G, by $L(\boldsymbol{I}_e^*)$ its Lie algebra and by $L(\boldsymbol{I}_e)$ the Lie algebra of the images of all indistinguishable points from the neutral element e of G. In [5], local observability is given by $L(\boldsymbol{I}_e^*) \equiv 0$. Then, the local observability criteria and finite-timeless local observability can be related as

$$L(\boldsymbol{I}_e^*) \equiv 0 \Rightarrow L(\boldsymbol{I}_e) \equiv 0.$$

3.11. Corollary:

For a linear control system $\Sigma = (G, \mathcal{D}, h, H)$ with $h \in Hom(G, H)$,

$L(\boldsymbol{I}_e) \equiv 0 \Rightarrow \Sigma$ is finite-timeless locally observable.

3.12. Example:

Consider Heisenberg Lie group of dimension 3, [7] :

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

with its Lie algebra

$$L(H) = span_{L.A.} \left\{ X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$[X, Y] = Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $[X, Y] = XY - YX = Z$ and the other Lie brackets vanish.

Consider $\Sigma = (H, \mathcal{D}, \pi, H/exp(\mathbb{R}X))$ with the canonic projection output. If we take drift vector field that is the infinitesimal automorphism as bZ, then the system is locally observable, [5]. Then, it is also finite-timeless locally observable.

Conclusions

In this work, we introduced finite-timeless locally and globally observability (almost locally and globally observability) for linear control systems on connected Lie groups with the output function which is a Lie group homomorphism. We showed that almost indistinguishability is an equivalence relation. We studied properties of the image of almost indistinguishable points from the neutral element and related local observability of this kind of systems to its finite-timeless local observability.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.