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Solving cubic equation using Cardano's method

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Abstract
A cubic equation is solvable by radicals. This means that the solutions of the cubic equation can be obtained using four basic arithmetic operations which include addition, subtraction, multiplication and division, and taking the square and cube roots.
Keywords: *Cubic equation*, *Cardano's method*

1. Introduction

Without loss of generality a cubic polynomial in one variable is defined in the following way:

$f(x) = x^3 + bx^2 + cx + d$

Where b, c and d are real numbers. The corresponding cubic equation is defined as follows:

$$f(x) = x^3 + bx^2 + cx + d = 0 \tag{1}$$

Our goal to to solve the equation (1) for x.

First we substitute $x = \alpha + y$

$$(\alpha + y)^3 + b(\alpha + y)^2 + c(\alpha + y) + d = 0 \tag{2}$$
$$\alpha^3 + y^3 + 3\alpha^2 y + 3\alpha y^2 + b\alpha^2 + by^2 + 2b\alpha y + c\alpha + cy + d = 0 \tag{3}$$
$$\alpha^3 + y^3 + (3\alpha + b)y^2 + 3\alpha^2 y + b\alpha^2 + 2b\alpha y + c\alpha + cy + d = 0 \tag{4}$$

The parameter α is determined in the following way:

$$(3\alpha + b = 0) \Rightarrow \left(\alpha = -\frac{b}{3}\right) \tag{5}$$
$$y^3 + (3\alpha^2 + 2b\alpha + c)y + \alpha^3 + b\alpha^2 + c\alpha + d = 0 \tag{6}$$
$$f(\alpha) = \alpha^3 + b\alpha^2 + c\alpha + d \tag{7}$$
$$y = p + q \tag{8}$$
$$p^3 + q^3 + 3pq(p + q) + (3\alpha^2 + 2b\alpha + c)(p + q) + f(\alpha) = 0 \tag{9}$$
$$(p + q)\left(3pq + 3\alpha^2 + 2b\alpha + c\right) + p^3 + q^3 + f(\alpha) = 0 \tag{10}$$

Now we will assume that p and q are chosen so that the following equality holds

$$\left(3pq + 3\alpha^2 + 2b\alpha + c = 0\right) \Rightarrow \left(pq = -\frac{3\alpha^2 + 2b\alpha + c}{3} = \frac{b^2 - 3c}{9}\right) \tag{11}$$

$$M = \frac{b^2 - 3c}{9} \tag{12}$$

$$F = f(\alpha) = \alpha^3 + b\alpha^2 + c\alpha + d = \frac{2b^3 - 9bc + 27d}{27} \tag{13}$$

$$p^3 + q^3 + F = 0 \tag{14}$$

$$(p^3 + q^3 + F = 0) \Rightarrow (p^3 + q^3 = -F) \tag{15}$$

$$p^3 q^3 = M^3 \tag{16}$$

Knowing the sum and the product of p^3 and q^3 one can conclude that they are the two solutions of the quadratic equation, which will be denoted by z_1 and z_2.

$$z_1 = p^3 \tag{17}$$

$$z_2 = q^3 \tag{18}$$

$$-(z_1 + z_2) = F \tag{19}$$

$$z_1 z_2 = M^3 \tag{20}$$

$$z^2 + Fz + M^3 = 0 \tag{21}$$

$$\Delta = F^2 - 4M^3 \tag{22}$$

Considering the value of the discriminant Δ, we will analyze two cases.

Case $\Delta \geqslant 0$

$$z_1 = \frac{-F + \sqrt{F^2 - 4M^3}}{2} \tag{23}$$

$$z_2 = \frac{-F - \sqrt{F^2 - 4M^3}}{2} \tag{24}$$

$$p_i^3 = z_1 \; e^{(2\pi i)\boldsymbol{i}} \tag{25}$$

$$q_j^3 = z_2 \; e^{(2\pi j)\boldsymbol{i}} \tag{26}$$

$$p_i = \sqrt[3]{z_1 \; e^{(2\pi i)\boldsymbol{i}}} = \sqrt[3]{z_1} \; e^{\frac{2\pi i}{3}\boldsymbol{i}} \tag{27}$$

$$q_j = \sqrt[3]{z_2 \; e^{(2\pi j)\boldsymbol{i}}} = \sqrt[3]{z_2} \; e^{\frac{2\pi j}{3}\boldsymbol{i}} \tag{28}$$

$$where \;\; i, j \in \{0, 1, 2\} \tag{29}$$

$$p_i q_j = \left(\sqrt[3]{z_1} \; e^{\frac{2\pi i}{3}\boldsymbol{i}}\right)\left(\sqrt[3]{z_2} \; e^{\frac{2\pi j}{3}\boldsymbol{i}}\right) = \sqrt[3]{z_1 \, z_2} \; e^{\frac{2\pi(i+j)}{3}\boldsymbol{i}} = M \; e^{\frac{2\pi(i+j)}{3}\boldsymbol{i}} \tag{30}$$

$$(p_i q_j = M) \Rightarrow \left(e^{\frac{2\pi(i+j)}{3}\boldsymbol{i}} = 1\right) \tag{31}$$

We can find the roots of the cubic equation (1) as it follows.

$$(i=0,j=0) \Rightarrow \left(e^{\frac{2\pi(0+0)}{3}\boldsymbol{i}}=1\right) \tag{32}$$

$$x_0=-\frac{b}{3}+p_0+q_0=-\frac{b}{3}+\sqrt[3]{z_1}+\sqrt[3]{z_2} \tag{33}$$

$$(i=1,j=2) \Rightarrow \left(e^{\frac{2\pi(1+2)}{3}\boldsymbol{i}}=1\right) \tag{34}$$

$$x_1=-\frac{b}{3}+p_1+q_2=-\frac{b}{3}+\sqrt[3]{z_1}\,e^{\frac{2\pi}{3}\boldsymbol{i}}+\sqrt[3]{z_2}\,e^{\frac{4\pi}{3}\boldsymbol{i}}=-\frac{b}{3}-\frac{1}{2}(\sqrt[3]{z_1}+\sqrt[3]{z_2})+\frac{\sqrt{3}}{2}(\sqrt[3]{z_1}-\sqrt[3]{z_2})\boldsymbol{i} \tag{35}$$

$$(i=2,j=1) \Rightarrow \left(e^{\frac{2\pi(2+1)}{3}\boldsymbol{i}}=1\right) \tag{36}$$

$$x_2=-\frac{b}{3}+p_2+q_1=-\frac{b}{3}+\sqrt[3]{z_1}\,e^{\frac{4\pi}{3}\boldsymbol{i}}+\sqrt[3]{z_2}\,e^{\frac{2\pi}{3}\boldsymbol{i}}=-\frac{b}{3}-\frac{1}{2}(\sqrt[3]{z_1}+\sqrt[3]{z_2})-\frac{\sqrt{3}}{2}(\sqrt[3]{z_1}-\sqrt[3]{z_2})\boldsymbol{i} \tag{37}$$

Case $\Delta < 0$

$$(\Delta<0) \Longrightarrow \left(F^2-4M^3<0\right) \Longrightarrow (M>0) \tag{38}$$

$$\sqrt{\Delta}=\sqrt{F^2-4M^3}=\sqrt{\boldsymbol{i}^2(-1)(F^2-4M^3)}=\pm\boldsymbol{i}\sqrt{4M^3-F^2} \tag{39}$$

$$z_1=\frac{-F+\boldsymbol{i}\sqrt{4M^3-F^2}}{2} \tag{40}$$

$$z_2=\frac{-F-\boldsymbol{i}\sqrt{4M^3-F^2}}{2} \tag{41}$$

We convert complex numbers z_1 and z_2 from Cartesian Coordinates to Polar Coordinates.

$$\varrho^2=\left(\frac{F}{2}\right)^2+\frac{4M^3-F^2}{4}=M^3 \tag{42}$$

$$\varrho=\sqrt{M^3} \tag{43}$$

$$3\theta=Atan2\left(-\frac{F}{2\varrho},\frac{\sqrt{4M^3-F^2}}{2\varrho}\right) \tag{44}$$

$$\cos(3\theta)=-\frac{F}{2\varrho} \tag{45}$$

$$\sin(3\theta)=\frac{\sqrt{4M^3-F^2}}{2\varrho} \tag{46}$$

$$z_1=\varrho\left(\cos(3\theta)+\boldsymbol{i}\sin(3\theta)\right)=\varrho\;e^{(3\theta+2\pi i)\boldsymbol{i}} \tag{47}$$

$$z_2=\varrho\left(\cos(3\theta)-\boldsymbol{i}\sin(3\theta)\right)=\varrho\;e^{(-3\theta+2\pi j)\boldsymbol{i}} \tag{48}$$

$$p_i^3=z_1=\varrho\;e^{(3\theta+2\pi i)\boldsymbol{i}} \tag{49}$$

$$q_j^3=z_2=\varrho\;e^{(-3\theta+2\pi j)\boldsymbol{i}} \tag{50}$$

$$where\quad i,j\in\{0,1,2\} \tag{51}$$

$$p_iq_j=\sqrt[3]{\varrho\;e^{(3\theta+2\pi i)\boldsymbol{i}}}\sqrt[3]{\varrho\;e^{(-3\theta+2\pi j)\boldsymbol{i}}}=\sqrt[3]{\varrho^2}\;e^{\frac{2\pi(i+j)}{3}\boldsymbol{i}}=M \tag{52}$$

$$(p_iq_j=M) \Rightarrow \left(e^{\frac{2\pi(i+j)}{3}\boldsymbol{i}}=1\right) \tag{53}$$

We can find the roots of the cubic equation (1) as it follows.

$$(i=0, j=0) \Rightarrow \left(e^{\frac{2\pi(0+0)}{3}\boldsymbol{i}} = 1\right) \tag{54}$$

$$p_0 = \sqrt[3]{\varrho}\ e^{\frac{3\theta}{3}\boldsymbol{i}} = \sqrt{M}\ e^{\theta\boldsymbol{i}} = \sqrt{M}\ \left(\cos(\theta) + \boldsymbol{i}\sin(\theta)\right) \tag{55}$$

$$q_0 = \sqrt[3]{\varrho}\ e^{-\frac{3\theta}{3}\boldsymbol{i}} = \sqrt{M}\ e^{-\theta\boldsymbol{i}} = \sqrt{M}\ \left(\cos(\theta) - \boldsymbol{i}\sin(\theta)\right) \tag{56}$$

$$x_0 = -\frac{b}{3} + p_0 + q_0 = -\frac{b}{3} + 2\sqrt{M}\cos(\theta) = -\frac{b}{3} + \frac{2}{3}\sqrt{b^2 - 3c}\ \cos(\theta) \tag{57}$$

$$(i=1, j=2) \Rightarrow \left(e^{\frac{2\pi(1+2)}{3}\boldsymbol{i}} = 1\right) \tag{58}$$

$$p_1 = \sqrt[3]{\varrho}\ e^{\left(\frac{2\pi}{3}+\theta\right)\boldsymbol{i}} = \sqrt{M}\ e^{(\theta)\boldsymbol{i}}\ e^{\left(\frac{2\pi}{3}\right)\boldsymbol{i}} \tag{59}$$

$$q_2 = \sqrt[3]{\varrho}\ e^{\left(\frac{4\pi}{3}-\theta\right)\boldsymbol{i}} = \sqrt{M}\ e^{(-\theta)\boldsymbol{i}}\ e^{\left(\frac{4\pi}{3}\right)\boldsymbol{i}} = \sqrt{M}\ e^{(-\theta)\boldsymbol{i}}\ e^{\left(-\frac{2\pi}{3}\right)\boldsymbol{i}} = \overline{p_1} \tag{60}$$

$$p_1 + q_2 = \sqrt{M}(2\ Re(p_1)) = 2\sqrt{M}\cos\left(\theta + \frac{2\pi}{3}\right) = -\frac{\sqrt{b^2-3c}}{3}\left(\cos(\theta) + \sqrt{3}\sin(\theta)\right) \tag{61}$$

$$x_1 = -\frac{b}{3} + p_1 + q_2 = -\frac{b}{3} - \frac{\sqrt{b^2-3c}}{3}\left(\cos(\theta) + \sqrt{3}\sin(\theta)\right) \tag{62}$$

$$(i=2, j=1) \Rightarrow \left(e^{\frac{2\pi(2+1)}{3}\boldsymbol{i}} = 1\right) \tag{63}$$

$$p_2 = \sqrt[3]{\varrho}\ e^{\left(\frac{4\pi}{3}+\theta\right)\boldsymbol{i}} = \sqrt{M}\ e^{(\theta)\boldsymbol{i}}\ e^{\left(\frac{4\pi}{3}\right)\boldsymbol{i}} \tag{64}$$

$$q_1 = \sqrt[3]{\varrho}\ e^{\left(\frac{2\pi}{3}-\theta\right)\boldsymbol{i}} = \sqrt{M}\ e^{(-\theta)\boldsymbol{i}}\ e^{\left(\frac{2\pi}{3}\right)\boldsymbol{i}} = \sqrt{M}\ e^{(-\theta)\boldsymbol{i}}\ e^{\left(-\frac{4\pi}{3}\right)\boldsymbol{i}} = \overline{p_2} \tag{65}$$

$$p_2 + q_1 = \sqrt{M}(2\ Re(p_2)) = 2\sqrt{M}\cos\left(\theta + \frac{4\pi}{3}\right) = \frac{\sqrt{b^2-3c}}{3}\left(\sqrt{3}\sin(\theta) - \cos(\theta)\right) \tag{66}$$

$$x_2 = -\frac{b}{3} + p_2 + q_1 = -\frac{b}{3} + \frac{\sqrt{b^2-3c}}{3}\left(\sqrt{3}\sin(\theta) - \cos(\theta)\right) \tag{67}$$

Thus, in the case that the discriminant is negative, we calculated the roots of the cubic equation using trigonometric functions. Now we will prove that it is not possible to express the function $\cos(\theta)$ as a finite sequence in terms of coefficients b, c and d.

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) \tag{68}$$

$$4\cos^3(\theta) - 3\cos(\theta) - \cos(3\theta) = 0 \tag{69}$$

$$\cos^3(\theta) - \frac{3}{4}\cos(\theta) - \frac{\cos(3\theta)}{4} = 0 \tag{70}$$

$$b = 0,\ c = -\frac{3}{4},\ d = -\frac{\cos(3\theta)}{4} \tag{71}$$

$$M = \frac{b^2 - 3c}{9} = \frac{1}{4} \tag{72}$$

$$F = \frac{2b^3 - 9bc + 27d}{27} = -\frac{\cos(3\theta)}{4} \tag{73}$$

$$\Delta_0 = F^2 - 4M^3 = \left(-\frac{\cos(3\theta)}{4}\right)^2 - 4\left(\frac{1}{4}\right)^3 = \frac{\cos^2(3\theta) - 1}{16} = -\frac{\sin^2(3\theta)}{16} \leqslant 0 \tag{74}$$

If the discriminant Δ_0 were positive, then we could solve Equation (70), but since Δ_0 is negative, it means that we cannot solve the Equation (70) without using trigonometric functions.

2. Examples

Case $\Delta = 0$

$$f(x) = x^3 - 4x^2 + 5x - 2 = 0 \tag{75}$$
$$\alpha = 1.33333333 \tag{76}$$
$$F = f(\alpha) = -0.074074074 \tag{77}$$
$$M = 0.111111111 \tag{78}$$
$$M^3 = 0.001371742 \tag{79}$$
$$\Delta = 0 \tag{80}$$
$$x_0 = 2 \tag{81}$$
$$x_1 = 1 \tag{82}$$
$$x_2 = 1 \tag{83}$$
$$\tag{84}$$

Case $\Delta > 0$

$$f(x) = x^3 + 5.2x^2 + 7.5x - 2.7 = 0 \tag{85}$$
$$\alpha = -1.733333333 \tag{86}$$
$$F = f(\alpha) = -5.284592593 \tag{87}$$
$$M = 0.504444444 \tag{88}$$
$$M^3 = 0.128363051 \tag{89}$$
$$\Delta = 27.41346667 \tag{90}$$
$$x_0 = 0.2959 \tag{91}$$
$$x_1 = -2.747929 + 1.254942\boldsymbol{i} \tag{92}$$
$$x_2 = -2.747929 - 1.254942\boldsymbol{i} \tag{93}$$
$$\tag{94}$$

Case $\Delta < 0$

$$x^3 + 8.1x^2 - 17.2x + 6.4 = 0 \tag{95}$$

$$\alpha = -2.7 \tag{96}$$

$$F = f(\alpha) = 92.206 \tag{97}$$

$$M = 13.02333333 \tag{98}$$

$$M^3 = 2208.851246 \tag{99}$$

$$\Delta = -333.4585481 \tag{100}$$

$$\varrho = 46.99841748 \tag{101}$$

$$\cos(3\theta) = -0.980947923 \tag{102}$$

$$\sin(3\theta) = 0.194270875 \tag{103}$$

$$\theta = 0.982026181 \tag{104}$$

$$x_0 = 1.308196337 \tag{105}$$

$$x_1 = -9.902248954 \tag{106}$$

$$x_2 = 0.494052617 \tag{107}$$

3. Comment

This article is for educational and informational purposes only.

4. Conflict of interest

The author is not aware of any conflict of interest associated with this work.