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Application Of Multipoint Secant-Type Method ForFinding Roots Of Nonlinear Equations

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Abstract

In this paper, we introduce a family of p_k-order iterative schemes for finding the simple root of a nonlinear algebraic equation of the function $f(x) = 0$ by using the divided difference approximation. The proposed method uses one evaluation of the function per iteration and can achieve convergence order p_k. The error equation and asymptotic convergence constant are proved theoretically and numerically. Numerical examples are included to demonstrate the exceptional convergence speed of the proposed method and thus verify the theoretical results.

Keywords: Secant-type methods; Simple root; Nonlinear algebraic equations; Root-finding; Order of convergence.

1 Introduction

The root-finding is one of the most important problems that arise in a wide variety of practical applications in science and engineering [2, 3, 11]. In this study, we consider iterative methods to find a simple root of a nonlinear algebraic equation $f(x) = 0$ where $f: D \subset R \to R$ is a scalar function on an open interval D and it is sufficiently smooth in a neighbourhood of the root. We propose a family of secant-type iterative methods to find a simple root of the nonlinear equation. It is well established that multipoint root-solvers are of great practical importance since they overcome the theoretical limits of one-point methods concerning convergence order and computational efficiency. Recently, some modifications of the secant-type methods for finding simple roots of nonlinear equations have been studied and analysed [5, 6-10]. Hence, the purpose of this paper is to demonstrate further development of the multipoint secant-type methods for obtaining a simple root of a nonlinear algebraic equation. Our objective is to design an iterative method for approximating a simple root by $n + 1$ starting points and estimating the derivative of the function by divided difference. Also, we prove the proposed iterative methods can achieve convergence order of p_k by the expanding the generating function. The proposed multipoint secant-type iterative methods are shown to have a better order of convergence than the classical secant method and other similar secant-type methods considered previously [6-10].

The structure of this paper is as follows: essential definitions relevant to the present work are stated in the section 2. In section 3, we introduce multipoint secant-type methods and prove their order of convergence. We demonstrate the performance of the proposed method in section 4, and concluding remarks are given in section 5.

2 Preliminaries

In order to verify the order of convergence of an iterative method, the following definitions are used [1-12].

Definition 1 Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α**.** The order of convergence p_k is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^{p_k}} = \sigma_k \neq 0, \tag{1}$$

where $p_k \in R^+$ and σ is the asymptotic error constant.

Definition 2 Let $e_k = x_k - \alpha$ be the error in the *k*th iteration, then the relation

$$e_{k+1} = \sigma e_k^p + O\left(e_k^{p+1}\right), \tag{2}$$

is the error equation. If the error equation exists, then *p* is the order of convergence of the iterative method.

Definition 3 Let *r* be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI\left(r, p_k\right) = \sqrt[r]{p_k}, \tag{3}$$

where p_k is the order of convergence of the method [2].

Definition 4 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of a nonlinear equation. Then the computational order of convergence [12] may be approximated by

$$COC \approx \frac{\ln\left|\left(x_{n+1}-\alpha\right)\left(x_n-\alpha\right)^{-1}\right|}{\ln\left|\left(x_n-\alpha\right)\left(x_{n-1}-\alpha\right)^{-1}\right|}. \tag{4}$$

Definition 5 Let D be an open interval of real values and let f be a function $f: D \subset R \to R$ Let $x_0, \cdots, x_n \in D$ with $x_k \neq x_n$ if $k \neq n$ The divided difference is a recursive division process based on a sequence of on $n + 1$ data points $\left(x_0, f\left(x_0\right)\right), \left(x_1, f\left(x_1\right)\right) \cdots \left(x_n, f\left(x_n\right)\right)$. Then the divided difference of f is defined recursively as

$$f\left[x_k, x_{k+1}, \cdots, x_n\right] = \frac{f\left[x_k, x_{k+1}, \cdots, x_{n-1}\right] - f\left[x_{k+1}, x_{k+2}, \cdots, x_n\right]}{x_k - x_n}. \tag{5}$$

3 Construction of the multipoint secant-type method

In this section, we define a new formula for multipoint secant-type iterative method with a convergence order of p_k. To obtain the solution of a nonlinear equation, the proposed secant-type method requires a single evaluation of a function and $n + 1$ particular starting points, ideally close to the simple root. The proposed general formula of the multipoint secant-type iterative method for determining the simple root of a nonlinear equation is based on the

$$g(x) = x + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \varphi_k(x) f^k(x), \tag{6}$$

where the unknown functions $\varphi_k(x)$, $k = 1, \ldots, m$. In fact, the unknown functions are evaluated by the recursively by

$$\varphi_1(x) = \frac{1}{f'(x)}, \quad \varphi_k(x) = \frac{\varphi'_{k-1}(x)}{f'(x)}, \quad k = 2, \ldots, m - 1. \tag{7}$$

Then the function $g(x)$ defined by (6) satisfies $g(\alpha) = \alpha$. The further details, explicit formulas and theorem are given in [1]. However, for the purpose of this paper we convert the derivative base function (7) into derivative-free function, hence, we replace the derivatives by using divided difference approximations (5). Consequently, we transform the original generating function (6) into

$$g_m(x) = x + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \omega_k(x) \prod_{v=1}^{k} \frac{f\left(x_{n-v+1}\right)}{f\left[x_{n-v+1}, x_{n-v}\right]}, \tag{8}$$

where

$$\omega_1 = 1, \quad \omega_k(x) \approx \varphi_k(x), \quad k = 2, \ldots, m. \tag{9}$$

Therefore, we present the multipoint secant-type methods for approximating the simple root of nonlinear equation $f(x) = 0$ by the following assumption

$$x_{n+1} = g_m(x), \quad m = 1, 2, \ldots, \tag{10}$$

Where x_{-k+1} are initial points. The well-known iterative method for finding a simple root of a nonlinear equation is namely, the classical secant method is obtained by $m = 1$ and is expressed as,

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f\left[x_n, x_{n-1}\right]}, \tag{11}$$

where

$f'(x) \approx f[x_n, x_{n-1}] = \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}$,

and the order of convergence is 1.618.

The unknown functions ω_k are calculated by replacing the appropriate derivatives with the divided difference approximation of multipoint secant-type methods are expressed as

$$\omega_1(x) = 1. \tag{12}$$

$$\omega_2(x) = \left(\frac{f[x_n, x_{n-1}, x_{n-2}]}{f[x_n, x_{n-1}]}\right). \tag{13}$$

$$\omega_3(x) = \left[\frac{f[x_n, x_{n-1}]f[x_n, x_{n-1}, x_{n-2}, x_{n-3}] - \left(\frac{3}{2}\right)f[x_n, x_{n-1}, x_{n-2}]^2}{f[x_n, x_{n-1}]^2}\right] \tag{14}$$

$$\omega_4(x) = [(-f[x_n, x_{n-1}]^2 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}] + 2f[x_n, x_{n-1}]f[x_n, x_{n-1}, x_{n-2}]f[x_n, x_{n-1}, x_{n-2}, x_{n-3}] - 3f[x_n, x_{n-1}, x_{n-2}]^3)f[x_n, x_{n-1}]^{-3}] \tag{15}$$

$$\omega_5(x) = [(-f[x_n, x_{n-1}]^3 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}] + [x_n, x_{n-1}]f[x_n, x_{n-1}, x_{n-2}]f[x_n, x_{n-1}, x_{n-2}, x_{n-3}] - 3f[x_n, x_{n-1}, x_{n-2}]^3)f[x_n, x_{n-1}]^{-4}] \tag{16}$$

$$\omega_6(x) = [(-\left(\frac{15}{2}\right)f[x_n, x_{n-1}, x_{n-2}]^2 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}] - \left(\frac{45}{2}\right)f[x_n, x_{n-1}, x_{n-2}]^5 + 3f[x_n, x_{n-1}]^3 f[x_n, x_{n-1}, x_{n-2}]f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}] - 15f[x_n, x_{n-1}]f[x_n, x_{n-1}, x_{n-2}]^3 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}] - f[x_n, x_{n-1}]^4 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}])f[x_n, x_{n-1}]^{-5}] \tag{17}$$

$$\omega_7(x) = [(\left(\frac{105}{2}\right)f[x_n, x_{n-1}]f[x_n, x_{n-1}, x_{n-2}]^4 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}] - \left(\frac{315}{4}\right)f[x_n, x_{n-1}, x_{n-2}]^6 - \left(\frac{7}{2}\right)f[x_n, x_{n-1}]^4 f[x_n, x_{n-1}, x_{n-2}]f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}] + \left(\frac{21}{2}\right)f[x_n, x_{n-1}]^3 f[x_n, x_{n-1}, x_{n-2}]^2 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}] - \left(\frac{105}{4}\right)f[x_n, x_{n-1}]^2 f[x_n, x_{n-1}, x_{n-2}]^3 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}] + f[x_n, x_{n-1}]^5 f[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}, x_{n-7}])f[x_n, x_{n-1}]^{-6}] \tag{18}$$

$x_{-k+1}, \ldots x_0, x_1$ are the initial points and provided that the denominators of (7) are not equal to zero. It is essential to verify our finding and prove the order of convergence of the proposed multipoint secant-type iterative methods.

4. Convergence analysis

In this section, we discuss the convergence analysis of the proposed multipoint secant-type iterative method (8). It is well established that the proof of two-to-four-point secant-type methods is given in [3, 5, 6-10] hence we will prove the next 5-to-8-point secant-type method.

Theorem 1

Let $f: D \subset R \rightarrow R$ be a sufficiently differentiable function and let for an open interval D has $\alpha \in D$ be a simple zero of $f(x)$, $f'(x) \neq 0$ in an open interval D. If the initial points $x_{-k}, \ldots, x_0, x_1$ are sufficiently close to α, then the asymptotic convergence order of the new secant-type methods defined by (8) is p_k.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha)=0$, and the errors at iteration are expressed as $e_{n-k+1}=x_{n-k+1}-\alpha,\cdots,e_{n+1}=x_{n+1}-\alpha$.

Using Taylor series expansion and taking into account that $f(\alpha)=0$, we have

$$f(x_n)=c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+\cdots \tag{19}$$

$$f(x_{n-1})=c_1e_{n-1}+c_2e_{n-1}^2+c_3e_{n-1}^3+\cdots \tag{20}$$

$$f(x_{n-2})=c_1e_{n-2}+c_2e_{n-2}^2+c_3e_{n-2}^3+\cdots \tag{21}$$

$$f(x_{n-3})=c_1e_{n-3}+c_2e_{n-3}^2+c_3e_{n-3}^3+\cdots \tag{22}$$

⋮

where

$$c_k=\frac{f^{(k)}(\alpha)}{(k!)},\qquad \text{for } k=1,2,3,4,\ldots. \tag{23}$$

Using the above series expansion, we obtain

$$f[x_n,x_{n-1}]=c_1+(e_n+e_{n-1})c_2+\cdots \tag{24}$$

$$f[x_{n-1},x_{n-2}]=c_1+(e_{n-1}+e_{n-2})c_2+\cdots \tag{25}$$

$$f[x_{n-2},x_{n-3}]=c_1+(e_{n-2}+e_{n-3})c_2+\cdots \tag{26}$$

⋮

$$f[x_n,x_{n-1},x_{n-2}]=2c_2+2(e_n+e_{n-1}+e_{n-2})c_3+2(e_n^2+e_ne_{n-2}+e_ne_{n-1}+e_{n-1}e_{n-2}+e_{n-1}^2+e_{n-2}^2)c_4+\cdots \tag{27}$$

$$f[x_n,x_{n-1},x_{n-2}]=2c_2+2(e_n+e_{n-1}+e_{n-2})c_3+2(e_{n-1}^2+e_{n-1}e_{n-2}+e_{n-1}e_{n-3}+e_{n-2}e_{n-3}+e_{n-2}^2+e_{n-3}^2)c_4+\cdots \tag{28}$$

⋮

$$f[x_n,x_{n-1},x_{n-2},x_{n-3}]=6c_3+6(e_{n-3}+e_{n-2}+e_{n-1}+e_n)c_4+\cdots \tag{29}$$

⋮

Similarly higher order divided difference approximation of $f[x_n,x_{n-1},x_{n-2},x_{n-3},x_{n-4}]$, $f[x_n,x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-5}]$, $f[x_n,x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-5},x_{n-6}]$, $f[x_n,x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-5},x_{n-6},x_{n-7}]$ are obtained by using (5). Our first scheme is based on five initial points and we set the formula as

$$g_4(x): x_{n+1}=x_n+\sum_{k=1}^{4}\frac{(-1)^k}{k!}\omega_k(x)\prod_{v=1}^{k}\frac{f(x_{n-v+1})}{f[x_{n-v+1},x_{n-v}]}. \tag{30}$$

Substituting the appropriate expressions of Taylor series expansion in (30), and simplifying the above equation, we obtain the error equation for the new multipoint secant-type iterative method, given by (30) is

$$e_{n+1} = \left(\frac{c_5}{c_1} - \frac{c_2c_4}{c_1^2} + \frac{c_2^2c_3}{c_2^3} - \frac{c_2^4}{c_1^4}\right)e_{n-4}e_{n-3}e_{n-2}e_{n-1}e_n + \cdots. \tag{31}$$

In order to prove the order of convergence of (31) and we defining positive real terms of $E_n,\ E_{n-1},\ E_{n-2},\ E_{n-3}$ and E_{n-4} as

$$E_n = \frac{|e_{n+1}|}{|e_n^m|},\ E_{n-1} = \frac{|e_n|}{|e_{n-1}^m|},\ E_{n-2} = \frac{|e_{n-1}|}{|e_{n-2}^m|},\ E_{n-3} = \frac{|e_{n-2}|}{|e_{n-3}^m|},\ E_{n-4} = \frac{|e_{n-3}|}{|e_{n-4}^m|}. \tag{32}$$

The error terms of E_{n-4} are given as

$$|e_{n-3}| = E_{n-4}|e_{n-4}^m|, \tag{33}$$

$$|e_{n-2}| = E_{n-3}|e_{n-3}^m| = E_{n-3}E_{n-4}^m|e_{n-4}^{m^2}| \tag{34}$$

$$|e_{n-1}| = E_{n-2}|e_{n-2}^m| = E_{n-2}E_{n-3}^mE_{n-4}^{m^2}|e_{n-4}^{m^3}|, \tag{35}$$

$$|e_n| = E_{n-1}|e_{n-1}^m| = E_{n-1}E_{n-2}^mE_{n-3}^{m^2}E_{n-4}^{m^3}|e_{n-4}^{m^4}|, \tag{36}$$

$$|e_{n+1}| = E_n|e_n^m| = E_nE_{n-1}^mE_{n-2}^{m^2}E_{n-3}^{m^3}E_{n-4}^{m^4}|e_{n-4}^{m^5}|. \tag{37}$$

It is obtained from (36) that

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}||e_{n-3}||e_{n-4}|} = \left|\left(\frac{c_5}{c_1} - \frac{c_2c_4}{c_1^2} + \frac{c_2^2c_3}{c_2^3} - \frac{c_2^4}{c_1^4}\right)\right|, \tag{38}$$

substituting the appropriate expressions of errors terms in (37), we get

$$\left(E_nE_{n-1}^{m-1}E_{n-2}^{m^2-m-1}E_{n-3}^{m^3-m^2-m-1}E_{n-4}^{m^4-m^3-m^2-m-1}\right)\left|e_{n-4}^{m^5-m^4-m^3-m^2-m-1}\right|. \tag{39}$$

In order to satisfy the asymptotic equation (38), the power of the error term shall approach zero, that is

$$m^5 - m^4 - m^3 - m^2 - m - 1 = 0. \tag{40}$$

The roots of the equation (39) are;

$$m = 1.965948,\ m = -0.678350 - 0.458536i,\ -0.67835 + 0.458536i$$
$$m = 0.195376 - 0.848853i\ m = 0.195376 + 0.848853i. \tag{41}$$

The order of convergence of the proposed five-point secant-type method is determined by the positive root of (40). Hence, the five-point secant-type method defined by (30) has a convergence order of 1.9659.

We repeat the procedure to prove the error equations for the six-points secant-type method and substitute the appropriate expressions in (8), we get

$$g_5(x): x_{n+1} = x_n + \sum_{k=1}^{5}\frac{(-1)^k}{k!}\omega_k(x)\prod_{v=1}^{k}\frac{f(x_{n-v+1})}{f[x_{n-v+1},x_{n-v}]}, \tag{42}$$

The error equation for the six-point (42) secant-type iterative approach is obtained by again simplifying the above equation, yielding

$$g_5(x): e_{n+1} = \sigma_5e_{n-5}e_{n-4}e_{n-3}e_{n-2}e_{n-1}e_n + \cdots. \tag{43}$$

We prove the order of convergence of (43) by using the error terms (32) and express the additional term

$$E_{n-5} = \frac{|e_{n-4}|}{|e_{n-5}^m|}, \tag{44}$$

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}||e_{n-3}||e_{n-4}||e_{n-5}|} = \left|\left(\frac{c_6}{c_1} - \frac{c_2c_5}{c_1^2} + \frac{c_2^2c_4}{c_2^3} + \frac{c_2^5}{c_1^5} - \frac{c_3c_2^3}{c_1^4}\right)\right|, \quad (45)$$

$$m^6 - m^5 - m^4 - m^3 - m^2 - m - 1 = 0. \quad (46)$$

As previously stated, the positive root of (46) determines the order of convergence of the suggested six-point secant-type approach. Therefore, the convergence order of the six-point secant-type method defined by (42) is 1.9835.

Without loss of generality, we omit the laborious expressions for the seven-point and eight-point schemes and state the essential equations, by using the error terms of (37) and express the additional terms

$$E_{n-6} = \frac{|e_{n-5}|}{|e_{n-6}^m|}, E_{n-7} = \frac{|e_{n-6}|}{|e_{n-7}^m|}. \quad (47)$$

we get

$$g_6(x): x_{n+1} = x_n + \sum_{k=1}^{6} \frac{(-1)^k}{k!}\omega_k(x) \prod_{v=1}^{k} \frac{f(x_{n-v+1})}{f[x_{n-v+1}, x_{n-v}]}, \quad (48)$$

$$g_7(x): x_{n+1} = x_n + \sum_{k=1}^{7} \frac{(-1)^k}{k!}\omega_k(x) \prod_{v=1}^{k} \frac{f(x_{n-v+1})}{f[x_{n-v+1}, x_{n-v}]}, \quad (49)$$

and substituting the appropriate expressions of errors terms of (37) and (47), we get

$$g_6(x): e_{n+1} = \sigma_6 e_{n-6} e_{n-5} e_{n-4} e_{n-3} e_{n-2} e_{n-1} e_n + \cdots, \quad (50)$$

$$g_7(x): e_{n+1} = \sigma_7 e_{n-7} e_{n-6} e_{n-5} 4e_{n-3} e_{n-2} e_{n-1} e_n + \cdots,$$

(51)

where σ_6, σ_7 are asymptotic constant of (50) and (51) respectively.

Similarly, we prove the order of convergence of (50 and (51) independently, by using the error terms (32), and must satisfy the asymptotic equation condition, that is

$$m^7 - m^6 - m^5 - m^4 - m^3 - m^2 - m - 1 = 0, \quad (52)$$

$$m^8 - m^7 - m^6 - m^5 - m^4 - m^3 - m^2 - m - 1 = 0. \quad (53)$$

It is noted that the positive roots of the equations (52) and (53) are;

$m = 1.9919$, $m = 1.9960$, respectively. (54)

Furthermore, the asymptotic equation is given by the power of the error term and, is given by the general form

$$m^{k+1} - \sum_{t=0}^{k} m^t = 0. \quad (55)$$

The order of convergence of the multipoint secant-type method is determined by the positive root of the polynomial (55), which is also shown by Sidi [4,5].

Remark

The multipoint secant-type iterative methods require a single function evaluation and have an order of convergence p_k. To determine the efficiency index of these new iterative methods, definition 3 shall be used; hence, the efficiency index of the multipoint secant-type iterative methods is the same as the order of convergence of the secant-type methods.

5 Numerical computations

In this section, we demonstrate the effectiveness of the multipoint secant-type iterative methods by testing the accuracy of the proposed iterative methods. The difference between the simple root α and the approximation x_n for the test function with starting points is displayed in tables. Furthermore, the computational order of convergence and $f(x_n)$ are displayed in tables, and we observe that this perfectly coincides with the theoretical result. The numerical computations listed in the table were performed on an algebraic system called Maple, and the errors displayed are of absolute value. We test the proposed multipoint secant-type iterative methods using the following smooth functions.

Numerical example 1

We will demonstrate the order of convergence of the new multipoint secant-type iterative methods for the following nonlinear equation

$$f(x) = \frac{x}{1-x} - 5\, ln\left[\frac{1-x}{1-1.25x}\right] + 4.45977, \tag{56}$$

having the exact value of the simple root of (56) is $\alpha = 0.757396...$. In Table 1 the errors obtained by the methods described are based on the starting points $x_{-6} = 0.762$, $x_{-5} = 0.761$, $x_{-4} = 0.760$, $x_{-3} = 0.759$, $x_{-2} = 0.758$, $x_{-1} = 0.757$, $x_0 = 0.756$, $x_1 = 0.755$.

Table 1 Errors occurring in the approximation of the simple root of nonlinear equation (56)

methods	$\|x_2 - \alpha\|$	$\|x_4 - \alpha\|$	$\|x_6 - \alpha\|$	$\|x_8 - \alpha\|$	$\|f(x_8)\|$	COC
$g_1(x_n)$	0.529e-4	0.168e-8	0.143e-20	0.275e-52	0.219e-50	1.6115
$g_2(x_n)$	0.115e-5	0.566e-16	0.172e-53	0.604e-181	0.482e-179	1.8348
	0.117e-5	0.179e-16	0.598e-58	0.547e-212	0.436e-210	1.9216
$g_3(x_n)$	0.117e-5	0.474e-16	0.186e-60	0.383e-231	0.305e-229	1.9749
$g_4(x_n)$	0.117e-5	0.475e-16	0.598e-61	0.786e-237	0.627e-235	1.9719
$g_5(x_n)$	0.117e-5	0.475e-16	0.944e-61	0.410e-239	0.327e-237	1.9897
	0.117e-5	0.475e-16	0.945e-61	0.184e-239	0.147e-237	1.9923

Numerical example 2

We will demonstrate the order of convergence of the new multipoint secant-type iterative methods for the following nonlinear equation

$$f(x) = xe^{x^{x^2}} - sin(x) + 3cos(x) + 5, \tag{57}$$

having exact value of the simple root of (57) is $\alpha =- 1.207664...$. In Table 2 the errors obtained by the methods described are based on the starting points $x_{-6} =- 1.13$, $x_{-5} =- 1.14$, $x_{-4} =- 1.15$,

$x_{-3} =- 1.16$, $x_{-2} =- 1.17$, $x_{-2} =- 1.18$, $x_0 =- 1.19$, $x_1 =- 1.20$.

Table 2 Errors occurring in the approximation of the simple root of nonlinear equation (57)

methods	$\|x_2 - \alpha\|$	$\|x_4 - \alpha\|$	$\|x_6 - \alpha\|$	$\|x_8 - \alpha\|$	$\|f(x_8)\|$	COC
$g_1(x_n)$	0.204e-3	0.716e-9	0.271e-23	0.417e-61	0.848e-60	1.6245
$g_2(x_n)$	0.634e-5	0.129e-16	0.763e-58	0.236e-197	0.480e-196	1.8366
	0.535e-5	0.624e-18	0.281e-66	0.313e-246	0.636e-245	1.9238
$g_3(x_n)$	0.576e-5	0.944e-18	0.235e-69	0.803e-269	0.163e-267	1.9708
$g_4(x_n)$	0.574e-5	0.901e-18	0.133e-71	0.224e-280	0.454e-279	1.9776
$g_5(x_n)$	0.574e-5	0.905e-18	0.779e-71	0.928e-282	0.188e-280	1.9932
	0.574e-5	0.905e-18	0.754e-71	0.329e-283	0.667e-282	1.9895

Numerical example 3

We will demonstrate the order of convergence of the new multipoint secant-type iterative methods for the following nonlinear equation

$$f(x) = x^3 - 4x^2 - 6x - 15, \tag{58}$$

having exact value of the simple root of (58) is $\alpha =- 4.595347....$ In Table 3 the errors obtained by the methods described are based on the starting points $x_{-6} =- 4.43,\ x_{-5} =- 4.44,\ x_{-4} =- 4.45,$

$x_{-3} =- 4.46,\ \ x_{-2} =- 4.47,\ x_{-1} =- 4.48,\ x_0 =- 4.49,\ x_1 =- 4.50.$

Table 3 Errors occurring in the approximation of the simple root of nonlinear equation (58)

methods	$\lvert x_2 - \alpha \rvert$	$\lvert x_4 - \alpha \rvert$	$\lvert x_6 - \alpha \rvert$	$\lvert x_8 - \alpha \rvert$	$\lvert f(x_8) \rvert$	COC
$g_1(x_n)$	0.516e-2	0.593e-6	0.192e-16	0.568e-44	0.117e-42	1.6117
$g_2(x_n)$	0.535e-3	0.217e-10	0.425e-37	0.282e-127	0.580e-126	1.8348
	0.254e-3	0.897e-13	0.985e-49	0.272e-182	0.561e-181	1.9231
$g_3(x_n)$	0.273e-3	0.503e-13	0.303e-51	0.299e-200	0.615e-199	1.9731
$g_4(x_n)$	0.271e-3	0.404e-13	0.373e-53	0.300e-210	0.618e-209	1.9759
$g_5(x_n)$	0.271e-3	0.411e-13	0.219e-53	0.851e-213	0.196e-211	1.9954
	0.271e-3	0.410e-13	0.181e-53	0.154e-214	0.318e-213	1.9839

6 Conclusion

In this study, we have introduced a general formula for the multipoint secant-type method for finding the simple root of a nonlinear algebraic equation. It is established that multipoint iterative methods improve the order of convergence and the efficiency of the iterative method [2, 3, 11]. Hence, we have proposed multipoint secant-type methods for solving nonlinear equations. The effectiveness of the new multipoint secant-type iterative methods is examined by showing the accuracy of the simple root of several nonlinear equations, the computational order of convergence, and the evaluation of the function at the approximation of the simple root. Convergence analysis and numerical experiments prove that the multipoint iterative methods preserve their order of convergence. In fact, our findings agree with the theoretical order of convergence given by Sidi [4,5]. The major advantages of the multipoint secant-type methods are that they are derivative-free, produce a better approximation of the simple root, are effective and robust, and are a valuable alternative to existing similar secant-type iterative methods. The main drawback of the proposed scheme is that the unknown function ω_k given by (9) produces extraneous terms, and therefore, in order to calculate the simple root efficiently, some of these terms must be eliminated. Hence, further investigation is essential to find an efficient generating function (9).

Conflicts of Interest

There are no conflicts of interest regarding the publication of this article.

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