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An Engineering Boundary Eigenvalue Problem Studied by Functional-Analytic Methods

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Abstract

In this paper, we take up a boundary value problem (BVP) from the area of engineering that is described in a book by L. Collatz. Whereas there, the BVP is cast into a boundary eigenvalue problem (BEVP) having complex eigenvalues, here the original BVP is transformed into a BEVP that has positive simple eigenvalues and real eigenfunctions. Further, unlike there, we derive the inverse $T = G$ of the differential operator L associated with the BEVP, show that $T = G$ is compact in an appropriate real Hilbert space H, expand $Tu = Gu$ and u for all $u \in H$ in a respective series of eigenvectors, and obtain max-, min-, min-max, and max-min-Rayleigh-quotient representation formulas of the eigenvalues. Specific examples for generalized Rayleigh quotients illustrate the theoretical findings. The style of the paper is expository in order to address a large readership.

Keywords: Compact operator with simple eigenvalues; Boundary eigenvalue problem; Boundary value problem; Expansion in series of eigenvectors; Generalized Rayleigh-Quotient; Real parts of eigenvalues

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1 Introduction

In the author's opinion, it is most important to illustrate general results by applications. Therefore, in this paper, we take up a boundary value problem described in a book of L. Collatz, namely the damped vibration of a string with the following characteristic data: length l, string force S, density ϱ, and cross-section area A. However, we make a series of changes, precisions, and extensions such as follows:

- We introduce different notations
- Instead of $l = 1$, we admit a general length l
- We formulate the BEVP such that it can be treated by functional-analytic methods
- In the derivation of the solution to the BEVP, we first merely demand that the occurring partial derivatives be continuous, but later we introduce appropriate function spaces, norms, and scalar products as well as operators in these spaces.
- Thereby, it will be possible to determine the inverse $T = G$ of the differential operator L pertinent to the BEVP. And it can be shown that the operator $T = G$ is compact having positive (and thus real) simple eigenvalues converging to zero.
- This opens the way to obtain expansions of $Tu = Gu$ and $u = Pu$ in series of eigenvectors, where H is an appropriate Hilbert space and where P is the projection operator onto the geometric eigenspace of $T = G$
- Additionally, one can derive expansions for $(Tu, v) = (Gu, v)$ and $(u, v) = (Pu, v)$ for $u, v \in H$

- Based on this, generalized Rayleigh-quotient formulas are derived and tested on specific examples
- In the same way as damped vibrations of a string, also damped torsional vibrations of rods and shafts as well as the telegraph equation can be treated

Next, we give a table of Contents that contains the sections and subsections handling the above items.

Contents

Now, we make some comments on the individual sections.

For the readers convenience, Sections 2 and 3 recapitulate corresponding sections in [20] and [21], but without proofs. Section 4 is devoted to the conditions $(C1_L)$ - $(C5_L)$ for the linear operator L replacing the conditions $(C1_d)$ - $(C5_d)$

in [21] that guarantee the conditions $(C1)$ - $(C4)$ for the inverse operator $T = G$ of L. Sections 5 and 6 make up the core of the present paper. Whereas in Section 5 the BEVP from the area of engineering is treated in an informal way, Section 6 shows how the general results of the preceding sections can be applied by using precise conditions and functional-analytic methods. Finally, Section 7 contains the conclusion. The non-cited references [1], [3], [5], [7] - [15], [17], [19], [22], [23], [25], [27] - [29], [31], [32], [34], and [35] are taken from [21]. They are included here since the author thinks that they could be of interest to the reader in the context of this paper.

2 Expansion of a Linear Compact Operator and of a Pertinent Projection Operator in Hilbert Space

Together with Section 3, this section forms a basis for what follows. The statements are taken over from [20], but most of the proofs are omitted.

(i) The Conditions (C1) - (C4)

We assume the following conditions:

(C1) $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$

(C2) $0 \neq T \in B(H)$ is compact having countably many simple non-zero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ with $\lim_{k\to\infty} \lambda_k = 0$ pertinent to the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$. Further, $0 \notin \sigma(T)$.

(C3) The eigenvectors of the adjoint T^* of T with the eigenvalues $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \cdots$ are $\psi_1, \psi_2, \psi_3, \cdots$

(C4) $\lambda_i \neq \lambda_j,\ i \neq j,\ i, j = 1, 2, 3 \cdots$

One has the following theorem.

Theorem 2.1 (Biorthonormality relations for $\lambda_j \neq \lambda_k,\ j \neq k$)

Let the conditions (C1) - (C4) be fulfilled. Then, with appropriate normalization, the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$ and $\psi_1, \psi_2, \psi_3, \cdots$ are biorthonormal, that is,

$$(\chi_j, \psi_k) = \delta_{jk},\ j, k \in J. \tag{2.1}$$

Proof: See [20, Theorem 3.1]. ⋄

Furthermore, we obtain the following theorem.

Theorem 2.2 (Expansion of Tu as well as of Pu in a series of eigenvectors)

Let the conditions (C1) - (C4) be fulfilled. Then,

$$Tu = \sum_{j\in J} \lambda_j (u, \psi_j) \chi_j,\ u \in H \tag{2.2}$$

as well as

$$Pu = \sum_{j\in J} (u, \psi_j) \chi_j,\ u \in H, \tag{2.3}$$

where P is the projection operator from H onto the geometric eigenspace of T.

Proof: See [20, Theorem 3.2]. ⋄

Remark: From (2.2) we conclude that

$$\overline{[\chi_1, \chi_2, \chi_3, \cdots]} = T(H) = R(T).$$

where $R(T)$ means the range of T. Further, from (2.3),

$$P : H \mapsto \overline{[\chi_1, \chi_2, \chi_3, \cdots]}.$$

◇

Moreover, in [20, Theorem 3.3], we have proven the following theorem.

Theorem 2.3

Let the conditions (C1) - (C4) be fulfilled. Then, we obtain

$$u = Pu = \sum_{j\in J}(u,\psi_j)\chi_j,\ u \in H \tag{2.4}$$

and the projection operator

$$P_0 = I - P : H \mapsto N(T) = \{0\} \iff P_0 = 0. \tag{2.5}$$

◇

If condition $(C4)$ is not fulfilled, one can remedy this, e.g., by using a biorthogonalization pre-process, as the next lemma shows.

Lemma 2.4

Let the conditions (C1) - (C3) be fulfilled, and let, for instance, $\lambda_{j_1}, \lambda_{j_2}, \cdots, \lambda_{j_p}$ be eigenvalues of T with linearly independent eigenvectors $\chi_{j_1}, \chi_{j_2}, \cdots, \chi_{j_p}$; further, let $\psi_{j_1}, \psi_{j_2}, \cdots, \psi_{j_p}$ be linearly independent eigenvectors pertinent to $\overline{\lambda}_{j_1}, \overline{\lambda}_{j_2}, \cdots, \overline{\lambda}_{j_p}$ of T^*. Then, these eigenvectors can be biorthonormalized such that

$$(\chi_{j_k}, \psi_{j_l}) = \delta_{kl},\ k,l = 1,2,\cdots,p. \tag{2.6}$$

Proof: See [20, Lemma 3.4]. ◇

After appropriate application, for instance, of the biorthogonalization pre-process, condition $(C4)$ is satisfied.

(iii) *Special Case of a Selfadjoint Compact Operator $T = A$*

If $T = A$ is selfadjoint and compact and if there is a countable set of non-zero eigenvalues λ_j, $j \in J$, then it is known that the relation

$$\lim_{j\to\infty} \lambda_j = 0 \tag{2.7}$$

is fulfilled. Further, the eigenvalues are real and the pertinent eigenvectors φ_j can be chosen real so that one has

$$\varphi_j = \chi_j = \psi_j,\ j \in J \tag{2.8}$$

meaning that the biorthonormality relations (2.1) turn into the orthonormality relations

$$(\varphi_j, \varphi_k) = \delta_{jk},\ j,k \in J. \tag{2.9}$$

Thus, if $0 \neq \sigma(A)$, the relations (2.2) and (2.4) turn into the known results

$$Au = \sum_{j\in J}\lambda_j\,(u,\varphi_j)\varphi_j,\ u \in H \tag{2.10}$$

and

$$u = Pu = \sum_{j\in J}(u,\varphi_j)\varphi_j,\ u \in H. \tag{2.11}$$

For all this, see [30, Section 7].

For the next theorem, we define new subspaces of H. For every $j = 1,2,\ldots$, let

$$N_{\chi,j} := \{u \in H \,|\, u = \sum_{k=1}^{j}\alpha_k\chi_k \text{ with } \alpha_k \in \mathbb{C},\ k = 1,2,\ldots,j\} =: [\chi_1,\ldots,\chi_j], \tag{2.12}$$

$j = 1,2,\ldots$ and

$$N_{\chi,j,\mathbb{R}} := \{u \in H \,|\, u = \sum_{k=1}^{j}\beta_k\chi_k \text{ with } \beta_k \in \mathbb{R},\ k = 1,2,\ldots,j\} = [\chi_1,\ldots,\chi_j]_{\mathbb{R}}, \tag{2.13}$$

$j = 1, 2, \dots$ as well as

$$\begin{aligned} N_\chi &:= N_{\chi,\infty} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \alpha_k \chi_k \text{ exists in H with } \alpha_k \in \mathbb{C}, k = 1, 2, \dots\} \\ &= \overline{[\chi_1, \chi_2, \dots]} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} N_{\chi,\mathbb{R}} &:= N_{\chi,\infty,\mathbb{R}} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \beta_k \chi_k \text{ exists in H with } \beta_k \in \mathbb{R}, k = 1, 2, \dots\} \\ &= \overline{[\chi_1, \chi_2, \dots]}_{\mathbb{R}}. \end{aligned} \tag{2.15}$$

Likewise, we define

$$N_{\psi,j} := \{u \in H \,|\, u = \sum_{k=1}^{j} \alpha_k \psi_k \text{ with } \alpha_k \in \mathbb{C},\ k = 1, 2, \dots, j\} =: [\psi_1, \dots, \psi_j], \tag{2.16}$$

$j = 1, 2, \dots$ and

$$N_{\psi,j,\mathbb{R}} := \{u \in H \,|\, u = \sum_{k=1}^{j} \beta_k \psi_k \text{ with } \beta_k \in \mathbb{R},\ k = 1, 2, \dots, j\} = [\psi_1, \dots, \psi_j]_{\mathbb{R}}, \tag{2.17}$$

$j = 1, 2, \dots$ as well as

$$\begin{aligned} N_\psi &:= N_{\psi,\infty} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \alpha_k \psi_k \text{ exists in H with } \alpha_k \in \mathbb{C}, k = 1, 2, \dots\} \\ &= \overline{[\psi_1, \psi_2, \dots]} \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} N_{\psi,\mathbb{R}} &:= N_{\psi,\infty,\mathbb{R}} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \beta_k \psi_k \text{ exists in H with } \beta_k \in \mathbb{R}, k = 1, 2, \dots\} \\ &= \overline{[\psi_1, \psi_2, \dots]}_{\mathbb{R}}. \end{aligned} \tag{2.19}$$

After these preparations, we are able to prove the following theorem.

Theorem 2.5 *Let the conditions (C1) - (C4) be fulfilled. Then,*

$$(Tu, v) = \sum_{j \in J} \lambda_j (u, \psi_j)(\chi_j, v),\ u,\ v \in H \tag{2.20}$$

and

$$(u, v) = (Pu, v) = \sum_{j \in J} (u, \psi_j)(\chi_j, v),\ u,\ v \in H \tag{2.21}$$

where

$$(u, \psi_j),\ (\chi_j, v) \in \mathbb{R},\ u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}},\ j \in J \tag{2.22}$$

leading to

$$Re\,(Tu, v) = \sum_{j \in J} Re\,\lambda_j (u, \psi_j)(\chi_j, v),\ u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}},\ j \in J. \tag{2.23}$$

Proof: See [21, Theorem 3.4]. ⋄

In order to set up the formulas for the generalized Rayleigh quotients, we have to define the following subspaces of N_χ and N_ψ.

$$M_{\chi,1,\mathbb{R}} := N_{\chi,\mathbb{R}} = \overline{[\chi_1, \chi_2, \dots]}_{\mathbb{R}}, \tag{2.24}$$

$$\begin{aligned} M_{\chi,j,\mathbb{R}} &:= \{u \in N_{\chi,\mathbb{R}} \,|\, (u, \psi_k) = 0,\ k = 1, 2, \dots, j-1\} \\ &= [\psi_1, \dots, \psi_{j-1}]^{\perp}_{N_{\chi,\mathbb{R}}},\ j = 2, 3, \dots \end{aligned} \tag{2.25}$$

where $M_{\chi,j,\mathbb{R}}$ is called an *orthogonal complement in* $N_{\chi,\mathbb{R}}$ and

$$M_{\psi,1,\mathbb{R}} := N_{\psi,\mathbb{R}} = \overline{[\psi_1, \psi_2, \dots]}_{\mathbb{R}}, \tag{2.26}$$

$$\begin{aligned} M_{\psi,j,\mathbb{R}} &:= \{u \in N_{\psi,\mathbb{R}} \,|\, (u,\chi_k) = 0,\ k = 1,2,\ldots,j-1\} \\ &= [\chi_1,\ldots,\chi_{j-1}]^{\perp}_{N_{\psi,\mathbb{R}}},\ j = 2,3,\ldots \end{aligned} \tag{2.27}$$

where $M_{\psi,j,\mathbb{R}}$ *is called an orthogonal complement in* $N_{\psi,\mathbb{R}}$. The next lemma characterizes these spaces.

Lemma 2.6 *Let the conditions (C1) - (C4) be fulfilled as well as* $\{\chi_1,\chi_2,\ldots\}$ *and* $\{\psi_1,\psi_2,\ldots\}$ *be sets of biorthogonal eigenvectors of* T *and* T^* *respectively, i.e., such that*

$$(\chi_i,\psi_j) = \delta_{ij},\ i,j = 1,2,\ldots. \tag{2.28}$$

Then,

$$M_{\chi,j,\mathbb{R}} = \overline{[\chi_j,\chi_{j+1},\ldots]}_{\mathbb{R}}, j = 1,2,\ldots \tag{2.29}$$

and

$$M_{\psi,j,\mathbb{R}} = \overline{[\psi_j,\psi_{j+1},\ldots]}_{\mathbb{R}}, j = 1,2,\ldots \tag{2.30}$$

Proof: See proof of [21, Lemma 3.5]. $\diamond$

Now, let $u \in N_{\chi,\mathbb{R}}$ with $u = \sum_{k=1}^{\infty} \alpha_k \chi_k$ and $\alpha_k \in \mathbb{R}$ as well as $v \in N_{\psi,\mathbb{R}}$ with $v = \sum_{k=1}^{\infty} \beta_k \psi_k$ and $\beta_k \in \mathbb{R}$. Then, due to Theorem 2.1,

$$(u,v) = \sum_{k=1}^{\infty} \alpha_k \beta_k. \tag{2.31}$$

In order to facilitate the manner of speaking, we say that the *scalar product* (u,v) *of* $u \in N_{\chi,\mathbb{R}}$ *and* $v \in N_{\psi,\mathbb{R}}$ *is strongly positive iff* $\alpha_k\beta_k \geq 0,\ k = 1,2,\ldots$ *and* $\sum_{k=1}^{\infty} \alpha_k\beta_k > 0$. For short, we write

$$(u,v) \gg 0.$$

Remark: One has $\alpha_k = (u,\psi_k),\ u \in N_{\chi,\mathbb{R}}$ and $\beta_k = (\chi_k,v),\ v \in N_{\psi,\mathbb{R}}$ for $k = 1,2,\ldots$. Therefore, $(u,v) \gg 0$ means $(u,\psi_k)(\chi_k,v) \geq 0,\ k = 1,2,\ldots$ and $(u,v) = \sum_{k=1}^{\infty}(u,\psi_k)(\chi_k,v) > 0$. $\diamond$

Remark: For $(u,v) \gg 0$, one can admit linear combinations $u = \sum_{k=1}^{\infty} \alpha_k\chi_k$ and $v = \sum_{k=1}^{\infty} \beta_k\psi_k$ with $\alpha_k,\beta_k \in \mathbb{C},\ k = 1,2,\ldots$ such that $\alpha_k\overline{\beta}_k = |\alpha_k\beta_k|,\ k = 1,2,\ldots$ and $\sum_{k=1}^{\infty} |\alpha_k\beta_k| > 0$. For example, all elements $\alpha_k,\beta_k \in \mathbb{C}$ with $\alpha_k = |\alpha_k|e^{i\varphi_k}$ and $\beta_k = |\beta_k|e^{i\varphi_k}$ where φ_k is in $0 \leq \varphi_k < 2\pi,\ k = 1,2,\ldots$ are acceptable. $\diamond$

3 Generalized Rayleigh-Quotient Formulas for the Real Parts of the Eigenvalues

Again, for the readers' convenience, [21, Section 4] is recapitulated without proof, but we restrict ourselves to the formulas for the real parts of the eigenvalues.

In the sequel, we suppose that the non-zero eigenvalues are arranged according to

$$Re\lambda_1 \geq Re\lambda_2 \geq Re\lambda_3 \geq \ldots. \tag{3.1}$$

Such an arrangement is possible, for instance, if the real parts of all eigenvalues are positive. An arrangement that is always possible will be dealt with in Section 11.

One has the following generalized max-representation.

Theorem 3.1

Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T *be arranged according to (3.1). Moreover, let the vector spaces* $M_{\chi,j,\mathbb{R}}$ *resp.* $M_{\psi,j,\mathbb{R}}$ *for* $j \in J$ *be defined by (2.24), (2.25) resp. (2.26), (2.27) or (2.29) resp. (2.30). Then,*

$$Re\,\lambda_j = \max_{\substack{(u,v)\gg 0 \\ u \in M_{\chi,j,\mathbb{R}}, v \in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)},\ j \in J. \tag{3.2}$$

The maximum is attained for $u = \chi_j$, $v = \psi_j$.

Proof: See proof of [21, Theorem 4.1]. ◇

For the next theorem, we need the following *denotation of codimension.* A subspace $M \subset H$ has codimension j for $j \in J$ denoted by $\mathbf{codim\,M = j}$ if there exist linearly independent vectors $v_1, \ldots, v_j \in H$ such that

$$M = [v_1, \ldots, v_j]^{\perp} := [v_1, \ldots, v_j]^{\perp}_H = \{u \in H \,|\, (u, v_k) = 0,\ k = 1, \ldots, j\}.$$

Further, we set

$$\text{codim } M = 0$$

if $M = H$. Next, we prove a generalized min-max-representation.

Theorem 3.2

Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T *be arranged according to (3.1).*

Then, for every $j \in J$ *and every subspace* $M_\chi \subset N_{\chi,\mathbb{R}}$ *and* $M_\psi \subset N_{\psi,\mathbb{R}}$ *with* $\text{codim } M_\chi = \text{codim } M_\psi = j - 1$, *the following inequalities are valid:*

$$Re\,\lambda_j \le \max_{\substack{(u,v)\gg 0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu, v)}{(u, v)} \le Re\,\lambda_1, \tag{3.3}$$

and the following min-max-representation formulas hold:

$$Re\,\lambda_j = \min_{\substack{\text{codim } M_\chi = j-1\\ \text{codim } M_\psi = j-1}} \max_{\substack{(u,v)\gg 0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu, v)}{(u, v)},\ j \in J. \tag{3.4}$$

The minimum is attained for

$$M_\chi = M_{\chi,j,\mathbb{R}},\ M_\psi = M_{\psi,j,\mathbb{R}}.$$

Proof: See proof of [21, Theorem 4.2]. ◇

The next theorem contains a generalized min-representation.

Theorem 3.3

Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T *be arranged according to (3.1). Moreover, let the vector spaces* $N_{\chi,j,\mathbb{R}}$ *resp.* $N_{\psi,j,\mathbb{R}}$ *for* $j \in J$ *be defined by (2.13) resp. (2.17). Then,*

$$Re\,\lambda_j = \min_{\substack{(u,v)\gg 0\\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu, v)}{(u, v)},\ j \in J. \tag{3.5}$$

The minimum is attained for $u = \chi_j$, $v = \psi_j$.

Proof: See proof of [21, Theorem 4.3]. ◇

Next, we state the following generalized max-min-representation of $Re\lambda_j$.

Theorem 3.4

Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T *be arranged according to (3.1). Moreover, let the vector spaces* $N_{\chi,j,\mathbb{R}}$ *resp.* $N_{\psi,j,\mathbb{R}}$ *for* $j \in J$ *be defined by (2.13) resp. (2.17).*

Then, for every $j \in J$ *and every subspace* $N_\chi \subset N_{\chi,\mathbb{R}}$ *and* $N_\psi \subset N_{\psi,\mathbb{R}}$ *with* $\dim N_\chi = \dim N_\psi = j$, *the following inequalities are valid:*

$$\min_{\substack{(u,v)\gg 0\\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu, v)}{(u, v)} \le Re\,\lambda_j, \tag{3.6}$$

and the following max-min-representation formulas hold:

$$Re\,\lambda_j = \max_{\substack{dim\ N_\chi = j\\ dim\ N_\psi = j}} \min_{\substack{(u,v)\gg 0\\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu, v)}{(u, v)},\ j \in J. \tag{3.7}$$

The maximum is attained for

$$N_\chi = N_{\chi,j,\mathbb{R}},\ N_\psi = N_{\psi,j,\mathbb{R}}.$$

Proof: See proof of [21, Theorem 4.4]. ⋄

4 Series Expansions and Generalized Rayleigh Quotients for Compact Inverse of Differential Operator

In this section, we include the findings of [20, Sections 3 and 4]. However, we need a slight modification of the conditions. Namely, there, a differential operator $L : H_D \to H_R \subset H := L_2(0,l)$ was considered with $\overline{H}_D = \overline{H}_R = L_2(0,l)$ in the pertinent conditions $(C1_d)$ - $(C5_d)$ where d stands for densely defined. As opposed to this, in this paper, we have only $\overline{H}_R = L_2(0,l)$, and the corresponding conditions without $\overline{H}_D = L_2(0,l)$ are called $(C1_L)$ - $(C5_L)$ where L stands for the differential operator associated with the BEVP. Since in [20], we used only $\overline{H}_R = L_2(0,l)$, the results derived there are thus also valid under the new conditions $(C1_L)$ - $(C5_L)$.

This section is structured as follows. We begin with the conditions $(C1_L)$ - $(C5_L)$ on the differential operator L, its formally adjoint operator L_+ and their pertinent compact inverses G and G_+. Then, it is stated that $G_+ = G^*$ is the adjoint operator of G. Next, expansions of Gu and u for $u \in L_2(0,l)$ in series' of eigenvectors stated in [20, Section 4] resp. [21, Section 3] are given followed by generalized Rayleigh quotients for the real parts of the eigenvalues of G stated in [21, Section 4].

4.1 Series Expansions

In this subsection, in the case of simple eigenvalues, expansions in series' of eigenvectors are stated.

(i) The Conditions $(C1_L)$ - $(C5_L)$

We assume the following conditions:

$(C1_L)$ $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$

$(C2_L)$ $\{0\} \neq H_D$ and H_R are pre-Hilbert spaces with

$$H_D \subset H_R \subset H,\ \overline{H}_R = H$$

and where

$$L : D(L) := H_D \mapsto H_R$$

is a linear operator with the countably many *simple* non-zero eigenvalues $\mu_1, \mu_2, \mu_3, \cdots$ and the property $\lim_{j\to\infty} \mu_j = \infty$ as well as pertinent eigenvectors $\chi_1, \chi_2, \chi_3, \cdots \in H_D$. Further, L possesses a compact inverse

$$G := L^{-1} \in B(H)$$

$(C3_L)$ $\{0\} \neq H_{D,+}$ and H_R are pre-Hilbert spaces with

$$H_{D,+} \subset H_R \subset H,\ \overline{H}_R = H$$

and where

$$L_+ : D(L_+) := H_{D,+} \mapsto H_R$$

is a linear operator with the countably many *simple* non-zero eigenvalues $\mu_{1,+}, \mu_{2,+}, \mu_{3,+}, \cdots$ and the property $\lim_{j\to\infty} \mu_{j,+} = \infty$ as well as pertinent eigenvectors $\psi_1, \psi_2, \psi_3, \cdots \in H_{D,+}$. Further, L_+ possesses a compact inverse

$$G_+ := L_+^{-1} \in B(H)$$

$(C4_L)$ $(Lu, v) = (u, L_+ v),\ u \in H_D,\ v \in H_{D,+}$

$(C5_L)$ $\mu_j \neq \mu_k,\ j \neq k,\ j,k \in J$

Remark: We mention that due to the above conditions, $0 \notin \sigma(G)$. Further, that we also have $G \in L(H_R, H_D)$. Moreover, if $\mu_{j_0} = \mu_{k_0}$ for two indices $j_0 \neq k_0$, then $\chi_{j_0},\, \chi_{k_0}$ and $\psi_{j_0},\, \psi_{k_0}$ can be biorthonormalized such that we obtain $(\chi_j, \psi_k) = \delta_{jk}$ for $j,k \in \{j_0,\, k_0\}$. $\diamond$

(ii) Series Expansions of Gu and Pu

The first theorem reads as follows.

Theorem 4.1

Let the conditions $(C1_L)$ - $(C5_L)$ be fulfilled. Then,

$$\mu_{j,+} = \overline{\mu}_j,\ j \in J \tag{4.1}$$

and

$$G_+ = G^* \tag{4.2}$$

where $G^* \in B(H)$ is the adjoint operator of G defined by

$$(Gu, v) = (u, G^* u),\ u,\, v \in H. \tag{4.3}$$

Further, the operator G has the eigenvalues $\lambda_j = 1/\mu_j$ as well as the eigenvectors χ_j, and $G_+ = G^*$ has the eigenvalues $\lambda_{j,+} = \overline{\lambda}_j = 1/\mu_{j,+} = 1/\overline{\mu}_j$ as well as the eigenvectors ψ_j for $j \in J$. In addition, $\lim_{j\to\infty} \lambda_j = 0$.

Proof: See proof of [20, Theorem 4.1] with $(C1_d)$ - $(C5_d)$ replaced by $(C1_L)$ - $(C5_L)$. $\diamond$

From Theorem 3.1 and the results of Section 2, we obtain the following corollary.

Corollary 4.2

Let the conditions $(C1_d)$ - $(C5_d)$ be fulfilled. Then,

$$Gu = \sum_{j\in J} \lambda_j (u, \psi_j)\chi_j,\ u \in H, \tag{4.4}$$

$$u = Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H. \tag{4.5}$$

Proof: See proof of [20, Corollary 4.2] with $(C1_d)$ - $(C5_d)$ replaced by $(C1_L)$ - $(C5_L)$. $\diamond$

Remark: In the case $G = G_+ = G^*$, then - as already mentioned in Section 2 (iii) - the eigenvalues λ_j are real and $\varphi_j = \chi_j = \psi_j$, $j \in J$ as well as $\lim_{j\to\infty} \lambda_j = 0$. $\diamond$

4.2 Generalized Rayleigh Quotients

Since the conditions $(C1_L)$ - $(C5_L)$ entail the conditions $(C1)$ - $(C4)$ for $T = G \in B(H)$, we obtain the following corollaries.

One has the following generalized max-representation.

Corollary 4.3

Let the conditions $(C1_L)$ - $(C5_L)$ be fulfilled. Further, let the eigenvalues of $T = G$ be arranged according to (3.1). Moreover, let the vector spaces $M_{\chi,j,\mathbb{R}} = M_{\psi,j,\mathbb{R}} = M_{\varphi,j,\mathbb{R}}$ for $j \in J$ be defined by (2.25), (2.26) resp. (2.27), (2.28) or (2.29) resp. (2.30) with $\chi_j = \psi_j = \varphi_j$, $j \in J$. Then,

$$Re\,\lambda_j = \max_{\substack{(u,v)\gg 0 \\ u,\, v\in M_{\varphi,j,\mathbb{R}}}} \frac{Re\,(Gu, v)}{(u, v)},\ j \in J. \tag{4.6}$$

The maximum is attained for $u = v = \varphi_j$.

Proof: This follows from Therorem 3.1. ⋄

Next, we prove a generalized min-max-representation.

Corollary 4.4

Let the conditions $(C1_L)$ - $(C5_L)$ be fulfilled. Further, let the eigenvalues of $T = G$ be arranged according to (3.1).

Then, for every $j \in J$ and every subspace $M_\varphi \subset N_{\varphi,\mathbb{R}}$ with codim $M_\varphi = j-1$, the following inequalities are valid:

$$Re\,\lambda_j \leq \max_{\substack{(u,v)\gg 0\\ u,\,v\in M_\varphi}} \frac{Re\,(Gu,v)}{(u,v)} \leq Re\,\lambda_1, \tag{4.7}$$

and the following min-max-representation formulas hold:

$$Re\,\lambda_j = \min_{\text{codim } M_\varphi = j-1} \max_{\substack{(u,v)\gg 0\\ u,\,v\in M_\varphi}} \frac{Re\,(Gu,v)}{(u,v)},\ j \in J. \tag{4.8}$$

The minimum is attained for

$$M_\varphi = M_{\varphi,j,\mathbb{R}}.$$

Proof: See proof of [21, Theorem 4.2]. ⋄

The next theorem contains a generalized min-representation.

Corollary 4.5

Let the conditions $(C1_L)$ - $(C5_L)$ be fulfilled. Further, let the eigenvalues of $T = G$ be arranged according to (3.1). Moreover, let the vector spaces $N_{\varphi,j,\mathbb{R}}$ for $j \in J$ be defined by (2.13) with $\chi_j = \varphi_j$ resp. (2.17) with $\psi_j = \varphi_j$. Then,

$$Re\,\lambda_j = \min_{\substack{(u,v)\gg 0\\ u,\,v\in N_{\varphi,j,\mathbb{R}}}} \frac{Re\,(Gu,v)}{(u,v)},\ j \in J. \tag{4.9}$$

The minimum is attained for $u = v = \varphi_j$.

Proof: This follows from Theorem 3.3. ⋄

Next, we state the following generalized max-min-representation of $Re\lambda_j$.

Corollary 4.6

Let the conditions $(C1_L)$ - $(C5_L)$ be fulfilled. Further, let the eigenvalues of T be arranged according to (3.1). Moreover, let the vector space $N_{\varphi_j,\mathbb{R}}$ for $j \in J$ be defined by (2.13) with $\chi_j = \varphi_j$ resp. by (2.17) with $\psi_j = \varphi_j$.

Then, for every $j \in J$ and every subspace $N_\varphi \subset N_{\varphi,\mathbb{R}}$ with dim $N_\varphi = j$, the following inequalities are valid:

$$\min_{\substack{(u,v)\gg 0\\ u,\,v\in N_\varphi}} \frac{Re\,(Gu,v)}{(u,v)} \leq Re\,\lambda_j, \tag{4.10}$$

and the following max-min-representation formulas hold:

$$Re\,\lambda_j = \max_{dim\, N_\varphi = j} \min_{\substack{(u,v)\gg 0\\ u,\,v\in N_\varphi}} \frac{Re\,(Gu,v)}{(u,v)},\ j \in J. \tag{4.11}$$

The maximum is attained for

$$N_\varphi = N_{\varphi,j,\mathbb{R}}.$$

Proof: This follows from Theorem 3.4. ⋄

It is clear that the *Re* sign can be omitted when the eigenvalues are real.

5 A BEVP from Area of Engineering

In this section, a damped vibration model of a string is investigated whose mathematical formulation as a boundary value problem (BVP) reads $-u''(x,t) = -m\,\ddot{u}(x,t) - b\,\dot{u}(x,t),\ u(0,t) = u(l,t) = 0$. For its solution, it is transformed by the ansatz $-u''(x,t) = k\,u(x,t)$ into the boundary eigenvalue problem (BEVP) $-u''(x,t) = k\,u(x,t),\ u(0,t) = u(l,t) = 0$ with the constraint $m\ddot{u}(x,t) + b\,\dot{u}(x,t) + k\,u(x,t) = 0$. Then, with the separation of variables $u(x,t) = w(x)\,y(t)$, the constraint - after division by $w(x)$ - has the form $m\ddot{y}(t) + b\,\dot{y}(t) + k\,y(t) = 0$ whose solution is stated for various cases. The separation of variables - this time after division by $y(t)$ - transforms the BEVP $-u''(x,t) = k\,u(x,t),\ u(0,t) = u(l,t) = 0$ into the BEVP $-w''(x) = k\,w(x),\ w(0) = w(l) = 0$ whose eigenvalues $k = k_j$ and eigenfunctions $w(x) = w_j(x)$ for $j = 1, 2, \ldots$ are presented. Then, for the eigenvalues $k = k_j$, the solutions $y_{j,1}(t),\ y_{j,2}(t)$ of $m\ddot{y}(t) + b\,\dot{y}(t) + k\,y(t) = 0$ for a selected case representing small damping are found. Finally, the solutions $w_j(x)$ and $y_{j,1}(t),\ y_{j,2}(t)$ are combined to yield the eigenfunctions $\chi_{j,1}(x,t) = w_j(x)\,y_{j,1}(t),\ \chi_{j,2}(x,t) = w_j(x)\,y_{j,2}(t),\ j = 1, 2, \ldots$ of the BEVP $-u''(x,t) = k\,u(x,t),\ u(0,t) = u(l,t) = 0$ with the constraint $m\ddot{u}(x,t) + b\,\dot{u}(x,t) + k\,u(x,t) = 0$ which is equivalent to the original BVP $-u''(x,t) = -m\,\ddot{u}(x,t) - b\,\dot{u}(x,t),\ u(0,t) = u(l,t) = 0$.

5.1 Damped Vibration Model of a String

The problem is described in [6, p.42]. The author's translation from German into English reads as follows, where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$:

There are also physical problems leading to complex eigenvalues, e.g., damped vibrations of a string [with S=const: see Eqn. (2.2), K as damping constant]:

$$(4.3) \qquad S y'' = \varrho\,F\,\ddot{y} + K\,\dot{y}.$$

The same differential equation occurs also with other damped vibrations such as torsional vibrations of rods and shafts, telegraph equation, and so on.

With the ansatz

$$y(x,t) = e^{\lambda\,t}\,Y(x),$$

(4.3) turns into

$$S\,y'' = (\varrho\,F\,\lambda^2 + K\,\lambda)\,y,$$

or, with different notations (k_1, k_2 positive constants):

$$y'' = (k_1\,\lambda^2 + k_2\,\lambda)\,y.$$

For the boundary conditions (length of string $l = 1$)

$$y(0) = y(1) = 0,$$

one has the solutions

$$y = \sin n\,\pi\,x$$

with

$$k_1\lambda^2 + k_2\lambda = -n^2\pi^2.$$

The eigenvalues determined from this are, in general, complex, e.g., when the values of k_2 are not too large.

We stress that the above equation numbers (2.2) and (4.3) are those of the book [6].

5.2 Boundary Eigenvalue Problem

Using the new function u instead of y, the partial differential equation of subsection 5.1 reads

$$S\,u''(x,t) = \varrho\,F\,\ddot{u}(x,t) + K\,\dot{u}(x,t).$$

Next, we multiply this by $-1/S$ yielding

$$-u''(x,t) = -\frac{\varrho\, F}{S}\, \ddot{u}(x,t) - \frac{K}{S}\, \dot{u}(x,t).$$

Introducing the new variables

$$m := \frac{\varrho\, F}{S} > 0, \quad b := \frac{K}{S} > 0,$$

we obtain the partial differential equation (PDE)

$$\boxed{-u''(x,t) = -m\, \ddot{u}(x,t) - b\, \dot{u}(x,t)}\,. \tag{5.1}$$

In a further step, we make the solution ansatz

$$\boxed{-u''(x,t) = k\, u(x,t)}\,. \tag{5.2}$$

Equating the right-hand sides of (5.1) and (5.2) leads to

$$\boxed{m\, \ddot{u}(x,t) + b\, \dot{u}(x,t) + k\, u(x,t) = 0}\,. \tag{5.3}$$

This is interpreted as a *constraint on the solution* to the differential equation (5.2). The *boundary conditions* read

$$\boxed{u(0,t) = u(l,t) = 0}\,. \tag{5.4}$$

Thus, the original BVP (5.1), (5.4) with the new notations is equivalent to the BEVP (5.2), (5.4), where the solutions to this BEVP have to satisfy the constraint (5.3).

Equation (5.2) can also be written in the form

$$L[u] = k\, u \tag{5.5}$$

with the ordinary differential operator

$$L[u] = -u''. \tag{5.6}$$

5.3 Separation of Variables

The usual way to solve the BEVP in Subsection 5.2 is to separate the variables x and t by setting

$$u(x,t) = w(x)\, y(t). \tag{5.7}$$

Then, (5.2), (5.4) turn - after division by $y(t)$ - into

$$-\, w''(x) = k\, w(x), \quad 0 < x < l \tag{5.8}$$

$$w(0) = w(l) = 0, \tag{5.9}$$

and (5.3) becomes - after division by $w(x)$ -

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = 0, \quad 0 \le t < \infty. \tag{5.10}$$

5.4 Solution of $m\ddot{y}(t) + b\dot{y}(t) + ky(t) = 0$ for Various Cases

The differential equation

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = 0$$

is known in the Theory of Vibrations where it describes a one-mass vibratory model with displacement y, mass m, damping constant b and stiffness constant k, see, for instance, [33, Section 2.6]. In what follows we recapitulate the solution of the above differential equation whereby we use verbatim passages from the cited textbook.

For the solution, as usual, the differential equation is divided by m giving

$$\ddot{y}(t) + \frac{b}{m}\dot{y}(t) + \frac{k}{m}y(t) = 0.$$

Set

$$\delta := -\frac{b}{2m} \tag{5.11}$$

and

$$\omega := \sqrt{\frac{k}{m}}. \tag{5.12}$$

This yields

$$\ddot{y}(t) - 2\,\delta\,\dot{y}(t) + \omega^2 y(t) = 0. \tag{5.13}$$

The solution ansatz

$$y(t) = \hat{y}\,e^{\lambda\,t} \tag{5.14}$$

delivers the quadratic equation

$$\lambda^2 - 2\,\delta\,\lambda + \omega^2 = 0 \tag{5.15}$$

with the two roots

$$\lambda_{1,2} = \delta \pm \sqrt{\delta^2 - \omega^2}\,. \tag{5.16}$$

<u>Case 1:</u> <u>$\delta = 0$</u>

For $\delta = 0$, (5.16) reduces to

$$\lambda_{1,2} = \ \pm i\omega \tag{5.17}$$

so that the roots lie on the imaginary axis and correspond to the *undamped case*, ω is called (natural) *circular eigenfrequency.* The *pertinent complex basis* is given by

$$y_1(t) = \hat{y}_1 e^{i\omega t}, \quad y_2(t) = \hat{y}_2 e^{-i\omega t}\,. \tag{5.18}$$

A *corresponding real basis* of the above ordinary differential equation (ODE) is given by

$$\boxed{y_1(t) = \hat{y}_1 \sin\omega t, \quad y_2(t) = \hat{y}_2 \cos\omega t}\,. \tag{5.19}$$

<u>Case 2 a:</u> <u>$0 < \delta < \omega^2$</u>

For $0 < \delta < \omega^2$, (5.16) shows that the roots λ_1 and λ_2 are conjugate-complex:

$$\lambda_{1,2} = \delta \pm i\sqrt{\omega^2 - \delta^2} = \delta \pm i\omega_d \tag{5.20}$$

with

$$\omega_d = \sqrt{\omega^2 - \delta^2} \tag{5.21}$$

where

$$Re\,\lambda_{1,2} = -\frac{b}{2\,m} = \delta < 0 \tag{5.22}$$

and

$$Im\,\lambda_{1,2} = \pm\sqrt{\omega^2 - \delta^2} = \pm\omega_d \tag{5.23}$$

where $\omega_d > 0$. This is the so-called *underdamped case.* Here, δ is known as *logarithmic decrement* and ω_d as *damped circular eigenfrequency.* The *pertinent complex basis* is given by

$$y_1(t) = \hat{y}_1 e^{(\delta + i\omega_d)t}, \quad y_2(t) = \hat{y}_2 e^{(\delta - i\omega_d)t}\,. \tag{5.24}$$

A *corresponding real basis* reads

$$\boxed{y_1(t) = \hat{y}_1 e^{\delta t} \sin\omega_d t, \quad y_2(t) = \hat{y}_2 e^{\delta t} \cos\omega_d t}\,. \tag{5.25}$$

Remark: We avoid to call λ an eigenvalue since in our formulation of the BEVP, it is no eigenvalue. As a consequence, the functions $y_1(t)$, $y_2(t)$ are called merely basis functions and not eigenfunctions. ⋄

<u>Case 2 b:</u> $\underline{0 < \delta^2 = \omega^2}$

In this case,

$$\omega_d = 0\,. \tag{5.26}$$

Further, here we have the *double root*

$$\lambda_1 = \delta \tag{5.27}$$

with *only one pertinent real basis function*

$$\boxed{y_1(t) = \hat{y}_1 e^{\delta t}}\,. \tag{5.28}$$

This is the *critically damped case.* It is known that a *second basis function* is given by

$$\boxed{y_2(t) = \hat{y}_2\, t\, e^{\delta t}}\,. \tag{5.29}$$

Thus, (5.28) and (5.29) is a real basis of the ODE (5.10).

<u>Case 3:</u> $\underline{\delta^2 > \omega^2}$

In this case, (5.16) shows that the roots λ_1 and λ_2 are real and negative since

$$\boxed{\lambda_{1,2} = \delta \pm \sqrt{\delta^2 - \omega^2} < 0}\,. \tag{5.30}$$

This is the *overdamped case* leading to a *nonoscillatory motion.*

5.5 Solution of the BEVP $-w''(x) = k\,w(x),\ w(0) = w(l) = 0$

The solution of the BEVP

$$-w''(x) = k\,w(x),\ 0 < x < l$$
$$w(0) = w(l) = 0,$$

i.e., of (5.8), (5.9), is obtained by the ansatz

$$w(x) = \hat{w}\,\sin\sqrt{k}x\,. \tag{5.31}$$

Apparently, the function (5.31) satisfies (5.8), and the boundary condition (BC) $w(0) = 0$ is also fulfilled. Further, the BC $w(l) = 0$ leads to

$$\sin\sqrt{k}\,l = 0 \tag{5.32}$$

yielding

$$\sqrt{k}\,l = j\,\pi,\ j \in J := (1,\,2,\,3,\,\dots) \tag{5.33}$$

or

$$\boxed{k = k_j = j^2 \tfrac{\pi^2}{l^2} = j^2 k_1,\ j \in J} \tag{5.34}$$

so that

$$\boxed{w(x) = w_j(x) = \hat{w}_j \sin j\pi \tfrac{x}{l},\ j \in J} \tag{5.35}$$

are the *pertinent (real) eigenfunctions.*

5.6 Solutions of $m\ddot{y}(t) + b\dot{y}(t) + ky(t) = 0$ with $k = k_j,\ j \in J$ for Selected Case

The solutions of (5.10), i.e., of

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = 0$$

with

$$k = k_j \tag{5.36}$$

for all possible cases can be found in subsection 5.4 by the replacements

$$\omega^2 = \frac{k}{m} = \frac{1}{m}1^2\frac{\pi^2}{l^2} = \frac{k_1}{m} = \omega_1^2 = \omega^2 \quad \to \quad \omega_j^2 = \frac{k_j}{m} = \frac{1}{m}j^2\frac{\pi^2}{l^2}, \quad j \in J \tag{5.37}$$

and

$$y_1(t),\, y_2(t) \quad \to \quad y_{j,1}(t),\, y_{j,2}(t), \quad j \in J\,. \tag{5.38}$$

In the sequel, we restrict ourselves to the real basis in the case 2 a with $k = \frac{\pi^2}{l^2}$ replaced by $k_j = j^2 \frac{\pi^2}{l^2} = j^2\omega_1^2 = j^2\omega^2$. Then, case 2 a takes on the form

Case 2 a (j): $0 < \delta < \omega_j^2,\ j \in J$

Here,

$$\omega_j^2 = \frac{k_j}{m},\, j \in J \tag{5.39}$$

and

$$k_j = j^2 \frac{\pi^2}{l^2},\ j \in J\,. \tag{5.40}$$

The *pertinent real basis functions* read

$$\boxed{y_{j,1}(t) = \hat{y}_{j,1} e^{\delta t} \sin \omega_{d,j} t, \quad y_{j,2}(t) = \hat{y}_{j,2} e^{\delta t} \cos \omega_{d,j} t,\ j \in J} \tag{5.41}$$

with

$$\omega_{d,j} = \sqrt{\omega_j^2 - \delta^2},\, j \in J\,. \tag{5.42}$$

5.7 Solutions of BEVP $-u''(x,t) = k\, u(x,t),\ u(0,t) = u(l,t) = 0$ with constraint $m\ddot{u}(x,t) + b\,\dot{u}(x,t) + k\,u(x,t) = 0$

It has been shown that the original BVP $-u''(x,t) = -m\,\ddot{u}(x,t) - b\,\dot{u}(x,t),\ u(0,t) = u(l,t) = 0$ can be transformed into the BEVP $-u''(x,t) = k\,u(x,t),\ u(0,t) = u(l,t) = 0$ with constraint $m\ddot{u}(x,t) + b\,\dot{u}(x,t) + k\,u(x,t) = 0$. According to (5.34), the eigenvalues of the last one are given by

$$k = k_j = j^2 \frac{\pi^2}{l^2},\ j \in J$$

and, according to (5.35) and (5.41), the pertinent eigenfunctions by

$$\chi_{j,1}(x,t) = w_j(x)\, y_{j,1}(t) = \hat{w}_j \sin j\pi \frac{x}{l}\ \hat{y}_{j,1}\, e^{\delta t} \sin \omega_{d,j} t,\ j \in J\,, \tag{5.43}$$

$$\chi_{j,2}(x,t) = w_j(x)\, y_{j,2}(t) = \hat{w}_j \sin j\pi \frac{x}{l}\ \hat{y}_{j,2}\, e^{\delta t} \cos \omega_{d,j} t\ j \in J\,. \tag{5.44}$$

6 The Example in a Functional-Analysis Setting

In this section, we

- introduce the function spaces and operators allowing us to treat the BEVP by employing functional-analytic methods
- determine the eigenvalues and eigenfunctions of the differential operator L and of its inverse $T = G$
- give expansions of Gu and $u = Pu$ for all u in an appropriate Hilbert space H as well as of (Gu, v) and $(u, v) = (Pu, v)$ for all $u,\, v \in H$
- and present illustrative specific examples with generalized Rayleigh quotients for the positive eigenvalues of $T = G$

6.1 The Function Spaces and Operators

The operators (5.6), i.e.,

$$L[u] = Lu = -u'' \tag{6.1}$$

is here defined in the vector space H_D over the field $\mathbb{F} = \mathbb{R}$ as

$$\begin{aligned} &H_D := D(L) \\ &:= \{u \in C^2\,[0, l] \times [0, \infty) \,|\, u(0, t) = u(l, t) = 0,\ m\ddot{u}(x, t) + b\,\dot{u}(x, t) + k\,u(x, t) = 0, \\ &\textstyle\int_0^l \int_0^\infty u^2(x, t)\,dt\,dx < \infty\}. \end{aligned} \tag{6.2}$$

Further, we define the vector space H_R over the field $\mathbb{F} = \mathbb{R}$ as

$$H_R := \{u \in C\,[0, l] \times [0, \infty) \mid \int_0^l \int_0^\infty u^2(x, t)\,dt\,dx < \infty\} \tag{6.3}$$

as well as

$$H := L_2\,(0, l) \times (0, \infty) \tag{6.4}$$

where $L_2\,(0, l) \times (0, \infty)$ is the space of real measurable functions defined almost everywhere on $(0, l) \times (0, \infty)$ such that the following Lebesgue integral satisfies

$$\int_0^l \int_0^\infty u^2(x, t)\,dt\,dx < \infty.$$

We mention that the norm in H is given by

$$\|u\| = \|u\|_H = \left(\int_0^l \int_0^\infty u^2(x, t)\,dt\,dx\right)^{\frac{1}{2}},\ u \in H, \tag{6.5}$$

and the pertinent scalar product by

$$(u, v) = (u, v)_H = \int_0^l \int_0^\infty u(x, t)\,v(x, t)\,dt\,dx,\ u,\, v \in H_D \tag{6.6}$$

that is well-defined by the Cauchy-Schwarz inequality. Further, due to the BCs $u(0, t) = u(l, t) = 0$ and $v(0, t) = v(l, t) = 0$ for $u,\, v \in H$, employing partial integration, we have

$$(Lu, v) = (u, Lv),\ u,\, v \in H_D = H_{D,+} \tag{6.7}$$

so that the formally adjoint L_+ of L is equal to L in $H_D = H_{D,+} := D(L_+)$. One has

$$C_0^\infty\,(0, l) \times (0, \infty) \subset H_R \subset H = L_2\,(0, l) \times (0, \infty). \tag{6.8}$$

From [2, Section 1], we conclude that

$$\overline{C_0^\infty\,(0, l) \times (0, \infty)} = \overline{H_R} = H = L_2\,(0, l) \times (0, \infty). \tag{6.9}$$

Further,

$$H_D \subset H_R \subset \overline{H_R} = H = L_2\,(0, l) \times (0, \infty).$$

We mention that $L \in L(H_D, H_R)$ where $L(H_D, H_R)$ is the set of linear operators mapping H_D into H_R. The Green's function associated with

$$Lu = 0,\ u \in H_D$$

is given by

$$G(x, s) = \begin{cases} G_1(x, s) = \dfrac{x\,(l - s)}{l},\ 0 \le x \le s \le l, \\[2ex] G_2(x, s) = \dfrac{s\,(l - x)}{l},\ 0 \le s \le x \le l, \end{cases} \tag{6.10}$$

see [21, Section 10.5], and the pertinent inverse operator $G = L^{-1} = G_+ = L_+^{-1} \in L(H_R, H_D)$ is given by

$$(Gu)(x,t) = \int_0^l G(x,s)\, u(s,t)\, ds,\ x \in [0,l],\, t \in (0,\infty). \tag{6.11}$$

Before we continue to show that $G \in B(H,H) = B(H)$ and that G is compact, we make some remarks on the above norm $\|\cdot\|_H$ and further norms.

First, from [26, Fubini's theorem in Sections 66 and 67],

$$\|u\|_H^2 = \int_0^l \int_0^\infty u^2(x,t)\, dt\, dx = \int_0^\infty \int_0^l u^2(x,t)\, dx\, dt. \tag{6.12}$$

Next, we define

$$\|u(x,\cdot)\|^2_{L_2(0,l)} := n^2_{L_2(0,l)}(u(x,\cdot)) := \int_0^\infty u^2(x,t)\, dt,\ x \in [0,l],\ u \in C\,[0,l] \times [0,\infty) \tag{6.13}$$

and

$$\|u(\cdot,t)\|^2_{L_2(0,\infty)} := n^2_{L_2(0,\infty)}(u(\cdot,t)) := \int_0^l u^2(x,t)\, dx,\ t \in [0,\infty),\ u \in C\,[0,l] \times [0,\infty). \tag{6.14}$$

Then,

$$\boxed{\begin{aligned} \|u\|_H^2 &= \|u\|^2_{L_2\,(0,l)\times(0,\infty)} \\ &= \int_0^l \int_0^\infty u^2(x,t)\, dt\, dx = \int_0^l \|u(x,\cdot)\|^2_{L_2(0,\infty)}\, dx \\ &= \|\, \|u(\cdot,\cdot)\|^2_{L_2(0,\infty)}\, \|^2_{L_2(0,l)} = (n^2_{L_2(0,l)} \circ n^2_{L_2(0,\infty)})(u) \\ &= \int_0^\infty \int_0^l u^2(x,t)\, dx\, dt = \int_0^l \|u(\cdot,t)\|^2_{L_2(0,l)}\, dt \\ &= \|\, \|u(\cdot,\cdot)\|^2_{L_2(0,l)}\, \|^2_{L_2(0,\infty)} = (n^2_{L_2(0,\infty)} \circ n^2_{L_2(0,l)})(u) \end{aligned}}\,. \tag{6.15}$$

We note that $n_{L_2(0,l)}$ acts on the first argument of u and $n_{L_2(0,\infty)}$ on the second argument.

(i) <u>Boundedness of the Operator $T = G \in L(H)$</u>

We have

$$T = G \in L(H_R, H_D) \subset L(H_R, H_R) \subset L(H,H) = L(H)$$

with

$$Tu(x,t) = Gu(x,t) = (Gu)(x,t) = \int_0^l G(x,s)\, u(s,t)\, ds,\ x \in [0,l],\, t \in [0,\infty),\ u \in H_R, \tag{6.16}$$

and we will show that $T = G \in B(H)$ with $\|T\| = \|T\|_H \le (\int_0^l \int_0^l G^2(x,s)\, ds\, dx)^{\frac{1}{2}}$.

This is obtained as follows. From (6.16), we have

$$\begin{aligned} |Tu(x,t)| &= |\int_0^l G(x,s)\, u(s,t)\, ds| \le \int_0^l |G(x,s)|\, |u(s,t)|\, ds \\ &\le \left(\int_0^l G^2(x,s)\, ds\right)^{\frac{1}{2}} \left(\int_0^l u^2(s,t)\, ds\right)^{\frac{1}{2}},\ u \in H_R \end{aligned} \tag{6.17}$$

This entails

$$[Tu(x,t)]^2 \le \int_0^l G^2(x,s)\, ds \int_0^l u^2(s,t)\, ds,\ u \in H_R \tag{6.18}$$

yielding

$$\int_0^\infty [Tu(x,t)]^2 dt \le \int_0^l G^2(x,s)\,ds \underbrace{\int_0^\infty \int_0^l u^2(s,t)\,ds\,dt}_{=\|u\|_H^2},\ u \in H_R \tag{6.19}$$

and thus to

$$\int_0^l \int_0^\infty [Tu(x,t)]^2 dt\,dx \le \int_0^l \int_0^l G^2(x,s)\,ds\,dx\,\|u\|_H^2,\ u \in \overline{H_R} = H \tag{6.20}$$

or

$$\|Tu\|_H^2 \le \int_0^l \int_0^l G^2(x,s)\,ds\,dx\,\|u\|_H^2,\ u \in H.$$

Thus, $T \in L(H)$ is bounded, that is,

$$T \in B(H) \tag{6.21}$$

and

$$\|T\| = \|T\|_H = \sup_{0 \ne u \in H} \frac{\|Tu\|_H}{\|u\|_H} \le \left(\int_0^l \int_0^l G^2(x,s)\,ds\,dx\right)^{\frac{1}{2}} \tag{6.22}$$

(ii) Compactness of $T \in B(H)$

Starting point is the inequality

$$\int_0^\infty [Tu(x,t)]^2 dt \le \int_0^l G^2(x,s)\,ds\,\|u\|_H^2,\ u \in \overline{H_R} = H, \tag{6.23}$$

see (6.19), leading to

$$\max_{0\le x\le l} \int_0^\infty [Tu(x,t)]^2 dt \le \max_{0\le x\le l} \int_0^l G^2(x,s)\,ds\,\|u\|_H^2,\ u \in H. \tag{6.24}$$

or

$$\left(\max_{0\le x\le l} \int_0^\infty [Tu(x,t)]^2 dt\right)^{\frac{1}{2}} \le \left(\max_{0\le x\le l} \int_0^l G^2(x,s)\,ds\right)^{\frac{1}{2}} \|u\|_H,\ u \in H$$

so that also

$$\max_{0\le x\le l} \left(\int_0^\infty [Tu(x,t)]^2 dt\right)^{\frac{1}{2}} \le \max_{0\le x\le l} \left(\int_0^l G^2(x,s)\,ds\right)^{\frac{1}{2}} \|u\|_H,\ u \in H. \tag{6.25}$$

Let $f \in C[0,l]$ and

$$\|f\|_{C_\infty[0,l]} := \max_{0\le x\le l} |f(x)|. \tag{6.26}$$

Then,

$$\|\,\|Tu\|_{L_2(0,\infty)}\,\|_{C_\infty[0,l]} \le \max_{0\le x\le l} \left(\int_0^l G^2(x,s)\,ds\right)^{\frac{1}{2}} \|u\|_H,\ u \in H. \tag{6.27}$$

Further,

$$\begin{aligned} \|Tu\|_H^2 &\le \int_0^l \int_0^\infty [Tu(x,t)]^2 dt\,dx \\ &\le l \max_{0\le x\le l} \int_0^l [Tu(x,t)]^2 dt \\ &= l \max_{0\le x\le l} \|Tu(x,\cdot)\|_{L_2(0,\infty)}^2 = l\,\|\,\|Tu\|_{L_2(0,\infty)}^2\,\|_{C_\infty[0,l]}^2,\ u \in H. \end{aligned} \tag{6.28}$$

Define

$$\|Tu\|_{H_1} := \|\,\|Tu\|_{L_2(0,\infty)}\|_{C_\infty[0,l]},\ u \in H. \tag{6.29}$$

Then, due to (6.28), $T \in B(H, H_1)$ and

$$\|Tu\|_H \le \sqrt{l}\,\|Tu\|_{H_1},\ u \in H$$

entailing

$$\|Tu\|_H \leq \sqrt{l}\,\|Tu\|_{H_1} \leq \max_{0\leq x\leq l}\left(\int_0^l G^2(x,s)\,ds\right)^{\frac{1}{2}}\|u\|_H. \tag{6.30}$$

Next, we show that $T\in B(H,H_1)$ is compact. We mention that, due to the inequality (6.30), we then have also that $T\in B(H,H)=B(H)$ is compact.

For the proof of the compactness of $T\in B(H,H_1)$, we show that g defined by

$$g(x) := \|Tu(x,\cdot)\|_{L_2(0,\infty)},\ u\in H,\ x\in[0,l] \tag{6.31}$$

is equicontinuous. We have, for all $x,y\in[0,l]$,

$$\begin{aligned}|Tu(x,t)-Tu(y,t)| &\leq \int_0^l |G(x,s)-G(y,s)|\,|u(s,t)|\,ds\\ &\leq \left(\int_0^l |G(x,s)-G(y,s)|^2\,ds\right)^{\frac{1}{2}}\left(\int_0^l |u(s,t)|^2\,ds\right)^{\frac{1}{2}},\ u\in H\end{aligned} \tag{6.32}$$

implying

$$|Tu(x,t)-Tu(y,t)|^2 \leq \int_0^l |G(x,s)-G(y,s)|^2\,ds\int_0^l |u(s,t)|^2\,ds,\ u\in H$$

and thus

$$\int_0^\infty |Tu(x,t)-Tu(y,t)|^2dt \leq \int_0^l |G(x,s)-G(y,s)|^2\,ds\int_0^\infty\int_0^l |u(s,t)|^2\,ds\,dt,\ u\in H$$

or

$$\int_0^\infty |Tu(x,t)-Tu(y,t)|^2dt \leq \int_0^l |G(x,s)-G(y,s)|^2\,ds\,\|u\|_H^2,\ u\in H,\ x\in[0,l]. \tag{6.33}$$

Now,

$$G(x,s)-G(y,s) = \frac{\partial G}{\partial x}(x+\vartheta\,(y-x),s)\,(x-y)$$

with $0<\vartheta=\vartheta(x,y,s)<1$ and, due to (6.10),

$$\frac{\partial G(x,s)}{\partial x} = \begin{cases}\dfrac{(l-s)}{l}, & 0\leq x\leq s\leq l,\\[2ex] -\dfrac{s}{l}, & 0\leq s\leq x\leq l,\end{cases}$$

so that

$$\max_{0\leq x\leq l}\left|\frac{\partial G(x,s)}{\partial x}\right| = 1.$$

Thus,

$$|G(x,s)-G(y,s)| \leq |x-y|,\ 0\leq x,\ s\leq l. \tag{6.34}$$

From (6.33), (6.34), we obtain

$$\int_0^\infty |Tu(x,t)-Tu(y,t)|^2dt \leq l\,|x-y|^2\|u\|_H^2,\ u\in H$$

or

$$\|Tu(x,\cdot)-Tu(y,\cdot)\|_{L_2(0,\infty)}^2 \leq l\,|x-y|^2\|u\|_H^2,\ u\in H$$

and therefore,

$$\begin{aligned}|g(x)-g(y)| &= |\,\|Tu(x,\cdot)\|_{L_2(0,\infty)} - \|Tu(y,\cdot)\|_{L_2(0,\infty)}\,|\\ &\leq \|Tu(x,\cdot)-Tu(y,\cdot)\|_{L_2(0,\infty)} \leq \sqrt{l}\,|x-y|\|u\|_H,\ u\in H\,.\end{aligned} \tag{6.35}$$

This means that g is equicontinuous in $\|\cdot\|_{C_\infty[0,l]}$. Thereby, with the Arzelà-Ascoli theorem, it follows that $T=G\in B(H,H_1)$ is compact implying that, due to (6.30), also $T=G\in B(H,H)=B(H)$ is compact.

6.2 The Eigenvalues and Eigenfunctions of L and $T = G$

(i) *The Eigenvalues*

The eigenvalues of

$$L\,\varphi = \mu\,\varphi \tag{6.36}$$

are given by

$$\mu = \mu_j = k_j = \frac{\pi^2}{l^2} j^2,\ j \in J. \tag{6.37}$$

The inverse operator $T = G$ has the eigenvalues

$$\lambda = \lambda_j = \frac{1}{\mu_j} = \frac{1}{k_j} = \frac{l^2}{\pi^2}\frac{1}{j^2},\ j \in J \tag{6.38}$$

so that

$$\lim_{j\to\infty} \lambda_j = 0, \tag{6.39}$$

as it must be.

(ii) *The Eigenfunctions*

The eigenfunctions of L and $T = G$ are the same. According to the Subsections 5.5, 5.6, and 5.7, the eigenfunctions are given by

$$\varphi_{j,1}(x,t) = \chi_{j,1}(x,t) = \psi_{j,1}(x,t) = w_j(x)\,y_{j,1}(t) = \hat{w}_j \sin j\pi\frac{x}{l}\ \hat{y}_{j,1}\,e^{\delta t}\sin\omega_{d,j}t,\ j \in J\,, \tag{6.40}$$

$$\varphi_{j,2}(x,t) = \chi_{j,2}(x,t) = \psi_{j,2}(x,t) = w_j(x)\,y_{j,2}(t) = \hat{w}_j \sin j\pi\frac{x}{l}\ \hat{y}_{j,2}\,e^{\delta t}\cos\omega_{d,j}t\ j \in J\,. \tag{6.41}$$

Next, we determine $\hat{w}_j,\ \hat{y}_{j,1},\ \hat{y}_{j,2},\ j \in J$ such that

$$(\varphi_{j,1}, \varphi_{k,1}) = (\varphi_{j,2}, \varphi_{k,2}) = \delta_{jk},\ j,k \in J.$$

Herewith, we further calculate

$$(\varphi_{j,1}, \varphi_{j,2}),\ j \in J.$$

This is done in several steps. We mention that, for functions $\varphi(x,t) = w(x)\,y(t)$ and $\psi(x,t) = v(x)\,z(t)$ with $\varphi,\,\psi \in H_D \subset H_R \subset H$, one has

$$(\varphi, \psi) = (\varphi, \psi)_H = \int_0^l w(x)\,v(x)\,dx \int_0^\infty y(t)\,z(t)\,dt = (u,v)_{L_2(0,l)}\,(y,z)_{L_2(0,\infty)},\ u,v \in H.$$

Remark: The following points (iii) - (x) essentially state the results without derivations; the omitted details use [4, p. 327, No. 459, No. 461, and No.362]. $\diamond$

(iii) *Determination of* $\hat{w}_j,\ j \in J$

We determine $\hat{w}_j,\ j \in J$ such that $(w_j, w_j)_{L_2(0,l)} \overset{!}{=} 1,\ j \in J$ yielding

$$\boxed{\hat{w}_j = \hat{w} := \sqrt{\tfrac{2}{l}},\ j \in J}\,. \tag{6.42}$$

(iv) *Orthonormality of* $w_j,\ w_k,\ j,\ k \in J$

One obtains

$$\boxed{(w_j, w_k) = 0,\ j \neq k,\ j,\ k \in J}\,. \tag{6.43}$$

(v) *Determination of* $\hat{y}_{j,1},\ j \in J$

The condition $(y_{j,1}, y_{j,1})_{L_2(0,l)} \overset{!}{=} 1$ entails

$$\boxed{\hat{y}_{j,1} = 2\sqrt{-\delta}\,\frac{\omega_j}{\omega_{d,j}},\ j \in J} \tag{6.44}$$

resp.

$$\boxed{\hat{y}_{j,1} = 2\sqrt{-\delta}\,\frac{\omega_j}{\sqrt{\omega_j^2 - \delta^2}},\ j \in J}\,. \tag{6.45}$$

Thus, for (6.44) resp. (6.45), we have

$$\boxed{(y_{j,1}, y_{j,1})_{L_2(0,\infty)} = 1,\ j \in J}\,. \tag{6.46}$$

(vi) *Determination of* $\hat{y}_{j,2},\ j \in J$

The condition $(y_{j,1}, y_{j,1})_{L_2(0,l)} \overset{!}{=} 1$ gives

$$\boxed{\hat{y}_{j,2} = \frac{2\sqrt{-\delta}\,\omega_j\,\sqrt{\omega_{d,j}}}{\sqrt{\omega_j^2 + \delta^2}},\ j \in J}\,. \tag{6.47}$$

As a consequence, for (6.47), we have

$$\boxed{(y_{j,2}, y_{j,2})_{L_2(0,\infty)} = 1,\ j \in J}\,.$$

(vii) *Determination of* $(y_{j,1}, y_{j,2}),\ j \in J$

The condition $(y_{j,1}, y_{j,2}) \overset{!}{=} 0$ entails

$$\begin{aligned}(y_{j,1}, y_{j,2}) &= \frac{y_{j,1}, y_{j,2}}{2}\,\frac{\sqrt{\omega_j^2 - \delta^2}}{2\,\omega_j^2} = \frac{1}{2}\,\frac{2\sqrt{-\delta}\,\omega_j}{\sqrt{\omega_j^2 - \delta^2}}\,\frac{2\sqrt{-\delta}\,\omega_j\,\sqrt{\omega_{d,j}}}{\sqrt{\omega_j^2 + \delta^2}}\,\frac{\sqrt{\omega_j^2 - \delta^2}}{2\,\omega_j^2} \\ &= \frac{(-\delta)\,\sqrt{\omega_{d,j}}}{\sqrt{\omega_j^2 + \delta^2}} > 0\end{aligned}$$

so that

$$\boxed{(y_{j,1}, y_{j,2}) > 0,\ j \in J} \tag{6.48}$$

or

$$\boxed{y_{j,1} \not\perp y_{j,2},\ j \in J} \tag{6.49}$$

and thus also

$$\boxed{\varphi_{j,1} \not\perp \varphi_{j,2},\ j \in J}\,. \tag{6.50}$$

More precisely, one has

$$\boxed{\hat{y}_{j,1} = \frac{2\sqrt{-\delta}\,\omega_j}{\sqrt{\omega_j^2 - \delta^2}},\quad \hat{y}_{j,2} = \frac{2\sqrt{-\delta}\,\omega_j\,\sqrt{\omega_{d,j}}}{\sqrt{\omega_j^2 + \delta^2}},\quad (y_{j,1}, y_{j,2}) = \frac{(-\delta)\,\sqrt{\omega_{d,j}}}{\sqrt{\omega_j^2 + \delta^2}} > 0,\quad j \in J}\,. \tag{6.51}$$

In order to remedy this, one could orthogonalize the two functions $y_{j,1}$ and $y_{j,2}$ for $j \in J$, for instance, by using Schmidt's method leading to

$$\tilde{y}_{j,2} = \frac{y_{j,2} - (y_{j,1}, y_{j,2})\,y_{j,1}}{\|y_{j,2} - (y_{j,1}, y_{j,2})\,y_{j,1}\|},\ j \in J \tag{6.52}$$

with

$$\begin{aligned} y_{j,1}(t) &= \hat{y}_{j,1}\,e^{\delta\,t}\,\sin\omega_{d,j}\,t\,, \\ y_{j,2}(t) &= \hat{y}_{j,2}\,e^{\delta\,t}\,\cos\omega_{d,j}\,t\,. \end{aligned}$$

But then, the similar shape of $y_{j,1}$ and $y_{j,2}$ is lost.

An alternative method is to replace these functions by the following ones:

$$\begin{aligned} z_{j,1}(t) &= \hat{z}_{j,1}\, e^{\delta\, t}\, \sin(\omega_{d,j}\, t + \alpha_j)\,, && (6.53)\\ z_{j,2}(t) &= \hat{z}_{j,2}\, e^{\delta\, t}\, \cos(\omega_{d,j}\, t + \alpha_j)\,, && (6.54)\end{aligned}$$

$j \in J$. Due to

$$\begin{aligned} \sin(\omega_{d,j}\, t + \alpha_j) &= \sin\omega_{d,j}\, t\, \cos\alpha_j + \cos\omega_{d,j}\, t\, \sin\alpha_j\,,\\ \cos(\omega_{d,j}\, t + \alpha_j) &= \cos\omega_{d,j}\, t\, \cos\alpha_j - \sin\omega_{d,j}\, t\, \sin\alpha_j\,,\end{aligned}$$

$j \in J$, one has

$$[z_{j,1}, z_{j,2}] = [y_{j,1}, y_{j,2}],\ j \in J\,, \tag{6.55}$$

i.e., the subspace spanned by $z_{j,1}$ and $z_{j,2}$ is the same as the subspace spanned by $y_{j,1}$ and $y_{j,2}$ for $j \in J$. We shall see that α_j can be determined such that

$$(z_{j,1}, z_{j,2}) = (z_{j,1}, z_{j,2})_{L_2(0,\infty)} = 0,\ j \in J\,. \tag{6.56}$$

(viii) *Determination of* α_j *such that* $(z_{j,1}, z_{j,2}) = 0$

The condition $(z_{j,1}, z_{j,2}) \overset{!}{=} 0$ yields

$$(z_{j,1}, z_{j,2}) = \frac{\hat{z}_{j,1}\, \hat{z}_{j,2}}{2}\, \frac{e^{\frac{-2\,\delta\,\alpha_j}{\omega_{d,j}}}}{2\,\omega_{d,j}} \left[\frac{e^{\frac{\delta}{\omega_{d,j}}\, 2\,\alpha_j}}{\left(\frac{\delta}{\omega_{d,j}}\right)^2 + 1^2} \left(\frac{-\delta}{\omega_{d,j}}\, \sin 2\,\alpha_j + \cos 2\,\alpha_j \right) \right]. \tag{6.57}$$

Thus,

$$\boxed{(z_{j,1}, z_{j,2}) = 0}$$

is equivalent to

$$\frac{-\delta}{\omega_{d,j}}\, \sin 2\,\alpha_j + \cos 2\,\alpha_j = 0 \tag{6.58}$$

or

$$\tan 2\,\alpha_j = \frac{\omega_{d,j}}{\delta} < 0 \tag{6.59}$$

so that

$$\boxed{\alpha_j = \frac{1}{2}\, \arctan\left(\frac{\omega_{d,j}}{\delta}\right) < 0}\,. \tag{6.60}$$

(6.59) is equivalent to

$$\tan 2\,(-\alpha_j) = \frac{\omega_{d,j}}{-\delta} > 0 \tag{6.61}$$

(ix) *Determination of* $\hat{z}_{j,1}$ *such that* $(z_{j,1}, z_{j,1}) = 1$

The condition $(z_{j,1}, z_{j,1}) \overset{!}{=} 1$ gives

$$\begin{aligned} (z_{j,1}, z_{j,1}) &= \hat{z}_{j,1}^2\, \frac{e^{\frac{2\,(-\delta)\,\alpha_j}{\omega_{d,j}}}}{\omega_{d,j}}\, \frac{e^{\frac{2\,\delta\,\alpha_j}{\omega_{d,j}}}\, \omega_{d,j}^2}{4\,\omega_j^2} \left[2\, \sin\alpha_j \left(\frac{(-\delta)}{\omega_{d,j}}\, \sin\alpha_j + \cos\alpha_j \right) + \frac{\omega_{d,j}}{(-\delta)} \right]\\ &= \hat{z}_{j,1}^2\, f_{j,1} \overset{!}{=} 1 \end{aligned}$$

with

$$\begin{aligned} f_{j,1} &:= \frac{e^{\frac{2\,(-\delta)\,\alpha_j}{\omega_{d,j}}}}{\omega_{d,j}}\, \frac{e^{\frac{2\,\delta\,\alpha_j}{\omega_{d,j}}}\, \omega_{d,j}^2}{4\,\omega_j^2} \left[2\, \sin\alpha_j \left(\frac{(-\delta)}{\omega_{d,j}}\, \sin\alpha_j + \cos\alpha_j \right) + \frac{\omega_{d,j}}{(-\delta)} \right]\\ &= \frac{\omega_{d,j}}{4\,\omega_j^2} \left[2\, \sin\alpha_j \left(\frac{(-\delta)}{\omega_{d,j}}\, \sin\alpha_j + \cos\alpha_j \right) + \frac{\omega_{d,j}}{(-\delta)} \right] > 0\,. \end{aligned}$$

Result: With the values $\hat{z}_{j,1} = 1/\sqrt{f_{j,1}}$,

$$(z_{j,1}, z_{j,1}) = (z_{j,1}, z_{j,1})_{L_2(0,\infty)} = 1\,.$$

(x) *Determination of* $\hat{z}_{j,2}$ *such that* $(z_{j,2}, z_{j,2}) = 1$

The condition $(z_{j,2}, z_{j,2}) \overset{!}{=} 1$ leads to

$$\begin{aligned}(z_{j,2}, z_{j,2}) &= \hat{z}_{j,2}^2 \, \frac{e^{\frac{2\,(-\delta)\,\alpha_j}{\omega_{d,j}}}}{\omega_{d,j}} \, \frac{e^{\frac{2\,\delta\,\alpha_j}{\omega_{d,j}}}\,\omega_{d,j}^2}{4\,\omega_j^2} \left[2\,\cos\alpha_j \left(\frac{(-\delta)}{\omega_{d,j}} \cos\alpha_j + \sin(-\alpha_j) \right) + \frac{\omega_{d,j}}{(-\delta)} \right] \\ &= \hat{z}_{j,2}^2\, f_{j,2} \overset{!}{=} 1\end{aligned}$$

with

$$\begin{aligned}f_{j,2} &:= \frac{e^{\frac{2\,(-\delta)\,\alpha_j}{\omega_{d,j}}}}{\omega_{d,j}} \, \frac{e^{\frac{2\,\delta\,\alpha_j}{\omega_{d,j}}}\,\omega_{d,j}^2}{4\,\omega_j^2} \left[2\,\cos\alpha_j \left(\frac{(-\delta)}{\omega_{d,j}} \cos\alpha_j + \sin(-\alpha_j) \right) + \frac{\omega_{d,j}}{(-\delta)} \right] \\ &= \frac{\omega_{d,j}}{4\,\omega_j^2} \left[2\,\cos\alpha_j \left(\frac{(-\delta)}{\omega_{d,j}} \cos\alpha_j + \sin(-\alpha_j) \right) + \frac{\omega_{d,j}}{(-\delta)} \right] > 0\,.\end{aligned}$$

Result: With the values $\hat{z}_{j,2} = 1/\sqrt{f_{j,2}}$,

$$(z_{j,2}, z_{j,2}) = (z_{j,2}, z_{j,2})_{L_2(0,\infty)} = 1\,.$$

6.3 Expansions of Gu and $u = Pu$ for $u \in H$ as well as of (Gu, v) and $(u, v) = (Pu, v)$ for $u,\, v \in H$

The expansions in series of eigenvectors for Tu and $u = Pu$ with $u \in H$ as well as of (Tu, v) and $(u, v) = (Pu, v)$ with $u,\, v \in H$ can be found in Subsection 4.1. The pertinent expressions are applied to $T = G$.

For the differential operator L defined by $Lu = -u''$, the conditions $(C1_L)$ - $(C5_L)$ are fulfilled, and for the pertinent compact inverse $T = G \in B(H)$, the conditions $(C1)$ - $(C4)$ of Theorems 2.2 and 2.3 and Lemma 2.4 are satisfied. Thus, under the conditions $(C1_L)$ - $(C5_L)$, according to Theorem 2.2,

$$\boxed{Tu = Gu = \sum_{j\in J} \sum_{k=1}^{2} \lambda_{j,k} (u, \varphi_{j,k})\, \varphi_{j,k},\ u \in H}$$

and

$$\boxed{u = Pu = \sum_{j\in J} \sum_{k=1}^{2} (u, \varphi_{j,k})\, \varphi_{j,k},\ u \in H}$$

as well as, according to Theorem 2.5,

$$\boxed{(Tu, v) = (Gu, v) = \sum_{j\in J} \sum_{k=1}^{2} \lambda_{j,k} (u, \varphi_{j,k})\, (\varphi_{j,k}, v),\ u,\, v \in H}$$

and

$$\boxed{(u, v) = (Pu, v) = \sum_{j\in J} \sum_{k=1}^{2} (u, \varphi_{j,k})\, (\varphi_{j,k}, v),\ u \in H}\,.$$

6.4 Generalized Rayleigh-Quotient Formulas for $T = G$

The generalized Rayleigh quotients for the real parts of the eigenvalues of a compact operator $T \in B(H)$ are found in Subsection 4.2. For $T = G$, we only have to replace $Re\,\lambda_j$ by $Re\,\lambda_{j,k} = \lambda_{j,k}$ as well as χ_j and ψ_j by $\varphi_{j,k},\ k = 1, 2$. The details are left to the reader.

6.5 Illustrative Examples with Generalized Rayleigh Quotients for the Eigenvalues

From Theorem 3.1 for $j = 1$ and (3.2), we obtain

$$Re\,\lambda_1 = \max_{\substack{(u,v)\gg 0\\ u\in M_{\chi,1,\mathbb{R}},v\in M_{\psi,1,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} = \max_{\substack{(u,v)\gg 0\\ u\in N_{\chi,\mathbb{R}},v\in N_{\psi,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)}\,.$$

Now, we apply this to $T = G$ and take into account that the eigenvalues of $T = G$ are positive. Thus, $Re\,\lambda_1(T) = \lambda_1(G)$, and we obtain

$$\boxed{0 < \frac{(Tu,v)}{(u,v)} = \frac{(Gu,v)}{(u,v)} < \lambda_1(G),\ u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}}}\,.$$

The following examples are similar to those in [18, Subsection 8.4] in the case of a diagonalizable matrix $T = G = A$.

More precisely,

$$\lambda_j = \lambda_j(T) = \lambda_j(G) = \frac{1}{k_j} = \frac{l^2}{\pi^2}\,\frac{1}{j^2},\ j \in J,$$

and

$$\begin{aligned}\varphi_{j,1}(x,t) &= \hat{w}\,\sin j\pi\tfrac{x}{l}\cdot\hat{y}_{j,1}\,e^{\delta\,t}\,\sin\omega_{d,j}t\,,\\ \varphi_{j,2}(x,t) &= \hat{w}\,\sin j\pi\tfrac{x}{l}\cdot\hat{y}_{j,2}\,e^{\delta\,t}\,\cos\omega_{d,j}t\,,\end{aligned}$$

$x \in [0,l]$, $t \in [0,\infty)$ with

$$\omega_{d,j} = \sqrt{\omega_j^2 - \delta^2},\quad \omega_j^2 = \omega^2 j^2$$

and

$$\hat{w} = \sqrt{\frac{2}{l}},\quad \hat{y}_{j,1} = \frac{2\sqrt{-\delta}\,\omega_j}{\sqrt{\omega_j^2-\delta^2}},\quad \hat{y}_{j,2} = \frac{2\sqrt{-\delta}\,\omega_j}{\sqrt{\omega_j^2+\delta^2}},\quad j\in J\,.$$

Remark: At this point, it becomes clear that the basis functions $y_{j,1}(t) = \hat{y}_{j,1}\,e^{\delta\,t}\,\sin\omega_{d,j}t$ and $y_{j,2}(t) = \hat{y}_{j,1}\,e^{\delta\,t}\,\cos\omega_{d,j}t$ cannot be called eigenfunctions in our setting: they are just multiples depending on $T = G$ of the eigenfunctions $w_j(x) = \hat{w}\,\sin j\pi\frac{x}{l}$. $\diamond$

As to the spaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$, they both are equal to

$$N_{\varphi,\mathbb{R}} := \overline{[\varphi_{1,1},\,\varphi_{1,2};\,\varphi_{2,1},\,\varphi_{2,2};\,\varphi_{3,1};\,\varphi_{3,2};\,\ldots]}_{\mathbb{R}}\,.$$

Now, let

$$\begin{aligned}u_1 &= -5\,\varphi_{1,1} + 3\,\varphi_{3,2},\\ v_1 &= -4\,\varphi_{1,1} + 2\,\varphi_{3,2}.\end{aligned}$$

Then $u_1,\ v_1 \in N_{\varphi,\mathbb{R}}$ as well as $(u_1, v_1) \gg 0$, and one obtains

$$\begin{aligned}(Tu_1,v_1) &= (-5\,\lambda_1\,\varphi_{1,1} + 3\,\lambda_3\,\varphi_{3,2}, -4\,\varphi_{1,1} + 2\,\varphi_{3,2})\\ &= 20\,\lambda_1\,\|\varphi_{1,1}\|^2 - 10\,\lambda_1\,(\varphi_{1,1},\varphi_{3,2}) - 12\,\lambda_3\,(\varphi_{3,2},\varphi_{1,1}) + 6\,\lambda_3\,\|\varphi_{3,2}\|^2\\ &= 20\,\lambda_1 + 6\,\lambda_3\end{aligned}$$

since $\|\varphi_{1,1}\| = \|\varphi_{3,2}\| = 1$ and $(\varphi_{1,1},\varphi_{3,2}) = (\varphi_{3,2},\varphi_{1,1}) = 0$. Therefore,

$$0 < \frac{(Tu_1,v_1)}{(u_1,v_1)} = \frac{20\,\lambda_1 + 6\,\lambda_3}{20+6} \le \frac{20\,\lambda_1 + 6\,\lambda_1}{20+6} = \lambda_1 = \frac{l^2}{\pi^2}$$

so that

$$\boxed{\frac{(Tu_1, v_1)}{(u_1, v_1)} \in \left[0; \frac{l^2}{\pi^2}\right]}.$$

Let

$$\begin{aligned} u_2 &= 3\,\varphi_{2,1}, \\ v_2 &= -4\,\varphi_{4,1} + 2\,\varphi_{2,1}. \end{aligned}$$

Then u_2, $v_2 \in N_{\varphi,\mathbb{R}}$ as well as $(u_2, v_2) \gg 0$, and one obtains

$$\begin{aligned} (Tu_2, v_2) &= (3\,\lambda_2\,\varphi_{2,1}, -4\,\varphi_{4,2} + 2\,\varphi_{2,1}) \\ &= -12\,\lambda_2\,(\varphi_{2,1}, \varphi_{4,2}) + 6\,\lambda_2\,\|\varphi_{2,1}\|^2 \\ &= 6\,\lambda_2 \end{aligned}$$

since $\|\varphi_{2,1}\| = 1$ and $(\varphi_{2,1}, \varphi_{4,2}) = 0$. Therefore,

$$0 < \frac{(Tu_2, v_2)}{(u_2, v_2)} = \frac{6\,\lambda_2}{6} < \frac{6\,\lambda_1}{6} = \lambda_1 = \frac{l^2}{\pi^2}$$

so that

$$\boxed{\frac{(Tu_2, v_2)}{(u_2, v_2)} \in \left[0; \frac{l^2}{\pi^2}\right]}.$$

Let

$$\begin{aligned} u_3 &= -5\,\varphi_{1,1} + 3\,\varphi_{2,1} - 4\,\varphi_{3,2}, \\ v_3 &= -4\,\varphi_{1,1} + 2\,\varphi_{2,1} - 2\,\varphi_{3,2}. \end{aligned}$$

Then u_3, $v_3 \in N_{\varphi,\mathbb{R}}$ as well as $(u_3, v_3) \gg 0$, and one obtains

$$\begin{aligned} (Tu_3, v_3) &= (-5\,\lambda_1\,\varphi_{1,1} + 3\,\lambda_2\,\varphi_{2,1} - 4\,\lambda_3\,\varphi_{3,2}, -4\,\varphi_{1,1} + 2\,\varphi_{2,1} - 2\,\varphi_{3,2}) \\ &= 20\,\lambda_1 + 6\,\lambda_2 + 8\,\lambda_3 \end{aligned}$$

since $\|\varphi_{1,1}\| = \|\varphi_{3,2}\| = 1$ and $(\varphi_{1,1}, \varphi_{2,1}) = (\varphi_{1,1}, \varphi_{3,2}) = (\varphi_{2,1}, \varphi_{3,2}) = 0$. Therefore,

$$0 < \frac{(Tu_3, v_3)}{(u_3, v_3)} = \frac{20\,\lambda_1 + 6\,\lambda_2 + 8\lambda_3}{20 + 6 + 8} < \frac{20\,\lambda_1 + 6\,\lambda_1 + 8\,\lambda_1}{20 + 6 + 8} = \lambda_1 = \frac{l^2}{\pi^2}$$

so that

$$\boxed{\frac{(Tu_3, v_3)}{(u_3, v_3)} \in \left[0; \frac{l^2}{\pi^2}\right]}.$$

Let

$$\begin{aligned} u_4 &= -5\,\varphi_{1,1} + 3\,\varphi_{2,2}, \\ v_4 &= -2\,\varphi_{2,1}. \end{aligned}$$

Then u_4, $v_4 \in N_{\varphi,\mathbb{R}}$, but $(u_4, v_4) \not\gg 0$, and one obtains

$$\begin{aligned} (Tu_4, v_4) &= (-5\,\lambda_1\,\varphi_{1,1} + 3\,\lambda_2\,\varphi_{2,2}, -2\,\varphi_{2,2}) \\ &= -6\,\lambda_2 \end{aligned}$$

since $\|\varphi_{2,2}\| = 1$ and $(\varphi_{1,1}, \varphi_{2,2}) = 0$. Therefore,

$$0 < \frac{(Tu_4, v_4)}{(u_4, v_4)} = \frac{-6\,\lambda_2}{-6} = \lambda_2 < \lambda_1 = \frac{l^2}{\pi^2}$$

so that

$$\boxed{\frac{(Tu_4, v_4)}{(u_4, v_4)} \in \left[0; \frac{l^2}{\pi^2}\right]}$$

even though $(u_4, v_4) \not\gg 0$.

Let

$$\begin{aligned} u_5(x,t) &= w_2(x)\, z_{2,1}(t) = \hat{w}\, \sin 2\, \tfrac{\pi}{l}\, x \cdot \hat{z}_{2,1}\, e^{\delta\, t}\, \sin(\omega_{d,2}\, t + \alpha_2)\,, \\ v_5(x,t) &= w_2(x)\, z_{2,2}(t) = \hat{w}\, \sin 2\, \tfrac{\pi}{l}\, x \cdot \hat{z}_{2,2}\, e^{\delta\, t}\, \cos(\omega_{d,2}\, t + \alpha_2)\,, \end{aligned}$$

$x \in [0, l],\ t \in [0, \infty)$ with

$$\alpha_2 = \frac{1}{2}\, \arctan\left(\frac{\omega_{d,2}}{\delta}\right) < 0$$

so that, due to Subsection 6.2, (viii),

$$(z_{2,1}, z_{2,2})_{L_2(0,\infty)} = 0\,.$$

One has

$$\begin{aligned} \sin(\omega_{d,j}\, t + \alpha_j) &= \sin\omega_{d,j}\, t\, \cos\alpha_j + \cos\omega_{d,j}\, t\, \sin\alpha_j, \\ \cos(\omega_{d,j}\, t + \alpha_j) &= \cos\omega_{d,j}\, t\, \cos\alpha_j - \sin\omega_{d,j}\, t\, \sin\alpha_j; \end{aligned}$$

set $j = 2$ and

$$s_2 = \sin\alpha_2,\ c_2 = \cos\alpha_2\,.$$

Then,

$$\begin{aligned} u_5(x,t) &= w_2(x)\, z_{2,1}(t) = \hat{w}\, \sin 2\, \tfrac{\pi}{l}\, x \cdot \hat{z}_{2,1}\, e^{\delta\, t}\, [c_2\, \sin\omega_{d,2}\, t + s_2 \cos\omega_{d,2}\, t]\,, \\ &= w_2(x) \frac{\hat{z}_{2,2}\, c_2}{\hat{y}_{2,1}}\, y_{2,1}(t) + w_2(x) \frac{\hat{z}_{2,1}\, s_2}{\hat{y}_{2,2}}\, y_{2,2}(t) \\ &= \frac{\hat{z}_{2,1}\, c_2}{\hat{y}_{2,1}}\, \varphi_{2,1}(t) + \frac{\hat{z}_{2,1}\, s_2}{\hat{y}_{2,2}}\, \varphi_{2,2}(t) \\ v_5(x,t) &= w_2(x)\, z_{2,2}(t) = \hat{w}\, \sin 2\, \tfrac{\pi}{l}\, x \cdot \hat{z}_{2,2}\, e^{\delta\, t}\,, [-s_2\, \sin\omega_{d,2}\, t + c_2 \cos\omega_{d,2}\, t]\,, \\ &= w_2(x) \frac{\hat{z}_{2,2}\, (-s_2)}{\hat{y}_{2,1}}\, y_{2,1}(t) + w_2(x) \frac{\hat{z}_{2,1}\, c_2}{\hat{y}_{2,2}}\, y_{2,2}(t) \\ &= \frac{\hat{z}_{2,1}\, (-s_2)}{\hat{y}_{2,1}}\, \varphi_{2,1}(t) + \frac{\hat{z}_{2,1}\, c_2}{\hat{y}_{2,2}}\, \varphi_{2,2}(t) \end{aligned}$$

so that

$$\begin{aligned} u_5 &= \beta_{2,1}\, \varphi_{2,1} + \beta_{2,2}\, \varphi_{2,2}\,, \\ v_5 &= \gamma_{2,1}\, \varphi_{2,1} + \gamma_{2,2}\, \varphi_{2,2} \end{aligned}$$

with

$$\begin{aligned} \beta_{2,1} &= \frac{\hat{z}_{2,1}\, c_2}{\hat{y}_{2,1}} > 0, & \beta_{2,2} &= \frac{\hat{z}_{2,2}\, s_2}{\hat{y}_{2,2}} < 0, \\ \gamma_{2,1} &= \frac{\hat{z}_{2,1}\, (-s_2)}{\hat{y}_{2,1}} > 0, & \gamma_{2,2} &= \frac{\hat{z}_{2,2}\, c_2}{\hat{y}_{2,2}} > 0, \end{aligned}$$

since $s_2 = \sin\alpha_2 < 0$ and $c_2 = \cos\alpha_2 > 0$. Thus, $u_5,\ v_5 \in N_{\varphi,\mathbb{R}}$, but $(u_5, v_5) \not\gg 0$. Here,

$$(u_5, v_5) = \|w\|^2_{L_2(0,l)}\, (z_{2,1}, z_{2,2})_{L_2(0,\infty)} = 1 \cdot 0 = 0$$

so that

$$\boxed{\frac{(Tu_5, v_5)}{(u_5, v_5)} \text{ is not defined}}\,,$$

which is not surprising since $(u_5, v_5) \not\gg 0$.

7 Conclusion

In this paper, the boundary value problem (BVP) $-u''(x,t) = -m\,\ddot{u}(x,t) - b\,\dot{u}(x,t)$, $u(0,t) = u(l,t) = 0$ presented in a book by L. Collatz is taken up describing the damped vibration of a string. Whereas *there*, this problem is transformed into a quadratic eigenvalue problem (BEVP) with complex eigenvalues, *here* it is cast into a symmetric boundary eigenvalue problem with positive eigenvalues and real eigenfunctions. The main reason for this difference is that, in the book by Collatz - after separation of the variable $u(x,t) = w(x)\,y(t)$ -, the differential expression $m\,\ddot{y}(t) + b\,\dot{y}(t) + k\,y(t)$ with the time derivatives are at the center of the consideration leading with the ansatz $y(t) = \hat{y}\,e^{\lambda\,t}$ to the quadratic eigenvalue problem $(m\,\lambda^2 + b\,\lambda + k)\hat{y} = 0$, whereas in this paper, the differential expression $-u''(x,t)$ with space derivative is concentrated on leading - after separation of variables - to the BEVP $-w''(x) = k\,w(x),\ w(0) = w(l) = 0$. We have seen that the eigenvalues in Collatz[6] are complex-conjugate and the pertinent eigenfunctions are complex, but allow to construct real eigenfunctions $y_{j,1}(t),\ y_{j,2}(t)$. In our approach, however, these real functions $y_{j,1}(t)$, $y_{j,2}(t)$ are used only as multiples to the eigenfunctions $w_j(x)$ yielding the eigenfunctions $\chi_{j,1}(x,t) = y_{j,1}(t)\,w_j(x)$, $\chi_{j,2}(x,t) = y_{j,2}(t)\,w_j(x)$. Therefore, we called $y_{j,1}(t),\ y_{j,2}(t)$ merely basis functions, but not eigenfunctions because they are not eigenfunctions in our setting. A further difference is that, there, no expansion theorems occur, whereas, in our paper, the inverse $T = G$ of the considered differential operator L is determined, and a BEVP is formulated such that it can be investigated by using functional-analytic methods. This leads to expansion theorems for the inverse operator $T = G$ in series of eigenvectors as well as max-, min-max-, min-, and max-min-formulas for generalized Rayleigh quotients. Another important point is that we illustrate the results on the generalized Rayleigh quotients by specific examples that underpin the theoretical findings. Finally, we point out that the results can also be applied to other damped vibrations such as torsional vibrations of rods and shafts as well as to the telegraph equation.

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