DOI: https://doi.org/10.24297/jam.v22i.9558

A Novel Approach for Solving Nonlinear Time Fractional Fisher Partial Differential Equations

Rana T. Shwayyea

Department of Physics, College of Education / University of AL-Qadisiyah

rana.alrubai@qu.edu.iq

Abstract:

This study sheds light upon solving non-linear time fractional Fisher partial differential equations by adapting analytical series solutions. What increase the accuracy of the result is that many authors in their formulas adapt the caputo fractional derivation. Precise analytical and numerical solution for these equations is obtained as influential tool which is known as LRPS. This is a novel method which is introduces by those authors. Precision, effectiveness, and practical application are highly considered by LRPS, and this what makes it applicable and suitable for different fields such as engineering, physics, and finance. Because of the approach's accuracy, effectiveness and application, it is noted that if there is a pattern in the parts of the series, accurate solution will be achieved, while approximate estimates are provided otherwise. Consequently, for solving non-linear time fractional Fisher partial differential equations the LRPS method is made and produced as one of significant and technique.

This ensures the accuracy of the solutions obtained and allows for further modifications and improvements to address this type of problem effectively. The obtained results demonstrate the suitability and efficiency of the proposed LRPS method for solving non-linear time fractional Fisher partial differential equations.

Key word: fractional Fisher equations, LRPS method, Inverse of Laplace transform.

1. Introduction:

Economics, biochemistry, operational research and other scientific areas are the main reasons why they can effectively get used to indicators and derivatives [1-5], that is due to the fact that this is due to the fact that moderate modeling in the acceptable real world depends on the current time and the date of the previous age, which can be completed with indicators [6-9].

As a result, many scientific and engineering scientists focus on the evaluation of differential equation (FDE), while creating procedures for linear and non -linear problems and talking about dynamic systems. In addition, FDE solutions have been studied using approximation and numerical methods [10-13]. Many academics are interested in the topic of Fractional Initial Value Problems (FIVPs), which are extensions of standard initial value problems that can capture certain real-life features more realistically than standard DE. A lot of attention has been paid to the concepts of existence and existence of FIVP structural solutions [2, 14-16]. Indeed, analytical and numerical methods have been developed to examine different types of FPDE responses. For example, pseudo spectral strategy, transform homotopy evaluation method, wavelet transform variance method, iterative expansion method, sharpened Adomian method and homotopy analysis method [17-22]. RPS extensions have been observed in several PDEs, especially partial PDEs. The fractional time diffraction PDE, the KdV-Burger formula and the fractional Boussinesq formula are some examples [1,23,24]. More recently, the LRPS approach has been adopted. It was first proposed and demonstrated by [25]. The LRPS method combines the Laplace transform method and the RPS method by transforming the central problem into the Laplace domain and generating a new algebraic solution formula to provide accurate results in the form of a faster power set (FPS) approximation. The values obtained by the Laplacian inversion can then be used to solve the identification challenge. The unknown parameters of the proposed Laplace expansion can be expressed using the concept of limits, in contrast to the concept of FRPS, which, according to the derivative, requires the calculation of many fractional derivatives that are consumed step by step to find the solution.

In this paper, we present a new unbiased method, namely LRPS, which is a powerful tool for the accurate analysis and numerical solution of these problems. By leading by example, we emphasize accuracy, efficiency and style of application. We can find exact answers if there is a pattern between the parts of the series, or we can only provide approximate data.

2. Fundamentals of fractional arithmetic concepts

There are several ways to define partial integrals of sequences, and they are not all interchangeable. Riemann defined the Liouville concept and the Caputo concept as the two most commonly used types.

Definition 2.1 [26] The Mittag-Leffler formula defined bellow:

$$E_{\beta}(t) := \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\beta k+1)},\ \beta > 0 \tag{1}$$

Definition 2.2 [26] The integral of the Riemann-Liouville time-fractional operator with a positive α, applied to the multivariable function $\Phi(x,\ t)$ with $t > 0$, is formulated as follows:

$$J_{t}^{\alpha}\Phi(x,t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t} (t - \kappa)^{\alpha-1}\Phi(x,t)d\kappa, \tag{2}$$

Definition 2.3 [25] The derivative of the Caputo time-fractional operator with an α value between $\eta - 1 < \alpha < \eta$, applied to the multivariable function $\Phi(x,\ t)$, is

$$D^{\alpha}\Phi(x,t) = \{J_{t}^{\eta-\alpha}\frac{\partial^{\eta}}{\partial t^{\eta}}\Phi(x,t),\ \eta - 1 < \alpha < \eta,\ \frac{\partial^{\eta}}{\partial t^{\eta}}\Phi(x,t),\ \alpha = \eta. \tag{3}$$

Definition 2.4 [25] For the given improper integral:

$$\Omega(x,s) = \int_{0}^{\infty} e^{-st}\Phi(x,t)dt, \tag{4}$$

If the improper integral exists for all s, then it represents the Laplace transform of the function $\Phi(x,t)$. The Laplace transform of $\Phi(x,\ t)$ is denoted as $L[\Phi(x,t)](x,s)$.

Moreover, by utilizing the provided inverse Laplace transform, the original function $\Phi(x,t)$ can be retrieved from its Laplace transform $\Omega(x,s)$.

$$\Phi(x,t) = \int_{-\infty}^{\infty} e^{st}\Omega(x,s)ds, \tag{5}$$

Theorem 2.5 [25] Given that $\Omega(x,s)$ represents the Laplace transform of $\Phi(x,t)$, and $\Omega(x,s)$ represents the Laplace transform of $\Phi(x,t)$, let us consider the following characteristics, where ξ, ν, μ, and ς are constants:

1. $L[\xi\Phi(x,t) + \nu\varphi(x,t)] = \xi L[\Phi(x,t)] + \nu L[\varphi(x,t)] = \xi\Omega(x,s) + \nu P(x,s)$ for ξ *and* $\nu \in R$.
2. $L^{-1}[\xi\Omega(x,s) + \nu P(x,s)] = \xi L^{-1}[\Omega(x,s)] + \nu L^{-1}[P(x,s)] = \xi\Phi(x,t) + \nu\varphi(x,t),\ for\ \xi\ and\ \nu \in R$.
3. $L\left[e^{\varsigma t}\Phi(x,t)\right] = \Omega(s - \lambda)$.
4. $L[\psi(\mu x,\mu t)] = \frac{1}{\mu}\Psi\left(\frac{x}{\mu},\frac{s}{\mu}\right), \mu > 0$.
5. $lim_{s\to\infty} s\Omega(x,s) = \Phi(x,0)$.
6. $L\left[\partial_{t}^{n}\Phi(x,t)\right] = s^{n}\Omega(x,s) - \sum_{j}^{n-1} s^{n-j-1}\partial_{t}^{j}\Phi(x,0)$.

Definition 2.6 [27] A fractional series defined as:

$$\sum_{n=0}^{\infty} h_{n}(x)\left(t - t_{0}\right)^{n\alpha} = \zeta_{0} + \zeta_{1}\left(t - t_{0}\right)^{\alpha} + \zeta_{2}\left(t - t_{0}\right)^{2\alpha} + \ldots, \tag{6}$$

The condition $t \geq t_0$ establishes fractional power series (FPS) with respect to t_0, where t_0 serves as a reference point. Within this series, the coefficients $h_n(x)$ are functions dependent on the variable x.

Theorem 2.7 [25] The Laplace transform $\Omega(x,s)$ of the function $\Phi(x,t)$ can be expressed as the following manner:

$$\Omega(x,s) = \sum_{n=0}^{\infty} \frac{h_{n}(x)}{s^{n\alpha+1}},\ 0 < \alpha\leq 1, s > 0. \tag{7}$$

Thus, $h_{n}(x) = \left(D_{t}^{n\alpha}\Phi\right)(0)$.

Lemma 2.8 [25] The inverse Laplace transform as described in Theorem 2.7, exhibits a specific form or type in subsequent calculations.

$$\Phi(x,\ t)\ =\ \sum_{n=0}^{\infty}\frac{D_t^{n\alpha}\Phi(x,0)}{\Gamma(n\alpha+1)}t^{n\alpha},0\ <\ \alpha\leq 1,t\geq 0. \tag{8}$$

3. LRPS Technique to Construct Series Solution to Time Fractional Fisher Partial Differential Equations:

In this section, we will outline the procedure for solving non-linear time fractional Fisher partial differential equations using the LRPS (Laplace-RPS) technique. The main objective of the LRPS technique is to analytically solve non-linear equations by initially applying the Laplace transform to the time fractional Fisher partial differential equations and subsequently utilizing the RPS technique. The findings presented in reference [30] demonstrate that the obtained results are then converted back to the original domain.

To proceed, let us rephrase formula of Time Fractional Fisher as follows:

$$D_t^{\alpha}u(x,t) = \rho^{\alpha}u_{xx}(x,t) + \ 6u(x,t) -\ 6u^2(x,t),\quad t\ >\ 0,\rho\ >\ 0,0\ <\ \alpha\leq 1, \tag{9}$$

where:

$$u(x,0)\ =\ \varsigma(x), \tag{10}$$

To begin, we utilize the Laplace transform (LT) on Eq. (9) to obtain:

$$L\left[D_t^{\alpha}u(x,t)\right] = \rho^{\alpha}L\left[u_{xx}(x,t)\right] +\ 6L[u(x,t)] -\ 6L\left[\left(L^{-1}[U(x,s)]\right)^2\right], \tag{11}$$
$$t\ >\ 0,s\ >\ 0,\ \rho\ >\ 0,\ 0\ <\ \alpha\leq 1,$$

By employing Lemma 2.8, we can establish Eq. (11) in the following manner:

$$\begin{aligned} s^{\alpha}U(x,s) - s^{\alpha-1}\,u(x,0) & \\ &= \rho^{\alpha}U_{xx}(x,s) + 6U(x,s) - 6\mathcal{L}[(\mathcal{L}^{-1}[U(x,s)])^2],\ \ s>0. \end{aligned} \tag{12}$$

vhere $U(x,s)=\mathcal{L}[u(x,t)]$ and $U_{xx}(x,s)=\mathcal{L}[u_{xx}(x,t)]$.

By dividing Eq. (12) by s^{α} and incorporating the initial condition from Eq. (10), we obtain the following form:

$$U(x,s) = \frac{\varsigma(x)}{s} + \frac{\rho^{\alpha}}{s^{\alpha}}U_{xx}(x,s) + \frac{6}{s^{\alpha}}U(x,s) - \frac{6}{s^{\alpha}}L\left[\left(L^{-1}[U(x,s)]\right)^2\right],\qquad s\ >\ 0. \tag{13}$$

Now let's examine the subsequent outcome derived from Eq. (13):

$$U(x,s)\ =\ \sum_{j=0}^{\infty}\frac{\varsigma_j(x)}{s^{1+\alpha j}}\ ,s\ >\ 0. \tag{14}$$

The series that is truncated at the k-th term, as described in (14), can be expressed as follows:

$$U_k(x,s)\ = \frac{\varsigma(x)}{s} +\ \sum_{j=1}^{k}\frac{\varsigma_j(x)}{s^{1+\alpha j}}\ ,s\ >\ 0. \tag{15}$$

To determine the unknown value of the parameter $\varsigma_j(x)$ for), $j\ =\ 1, 2,...$, the major techniques of LRPS, such as the LRF presented in (13), can be defined as follows:

$$\begin{aligned} \mathrm{LRes}(x,s) = \mathrm{U}(x,s) - \frac{\varsigma(x)}{s} - \frac{\rho^{\alpha}}{s^{\alpha}}U_{xx}(x,s) - \frac{6}{s^{\alpha}}U(x,s) & \\ + \frac{6}{s^{\alpha}}\mathcal{L}[(\mathcal{L}^{-1}[U(x,s)])^2], s>0. & \end{aligned} \tag{16}$$

The k-th of LRF can be defined as follows:

$$\mathrm{LRes}_k(\mathrm{x},\mathrm{s}) = \mathrm{U}_k(x,s) - \frac{\varsigma(x)}{\mathrm{s}} - \frac{\rho^\alpha}{s^\alpha}(U_k)_{xx}(x,s) - \frac{6}{s^\alpha}U_k(x,s) + \frac{6}{s^\alpha}\mathcal{L}[(\mathcal{L}^{-1}[U_k(x,s)])^2], s > 0. \quad (17)$$

Assuming that $U_1(x,s) = \frac{\varsigma(x)}{s} + \frac{\varsigma_1(x)}{s^{1+\alpha}}$, Eq. (17) yields the results that follow:

$$\mathrm{LRes}_1(\mathrm{x},\mathrm{s}) = \frac{\varsigma_1(x)}{s^{1+\alpha}} - \frac{\rho^\alpha}{s^\alpha}\left(\frac{\varsigma''(x)}{s} + \frac{\varsigma_1''(x)}{s^{1+\alpha}}\right) - \frac{6}{s^\alpha}\left(\frac{\varsigma(x)}{s} + \frac{\varsigma_1(x)}{s^{1+\alpha}}\right) + \frac{6}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\left(\frac{\varsigma(x)}{s} + \frac{\varsigma_1(x)}{s^{1+\alpha}}\right)\right]\right)^2\right], s > 0. \quad (18)$$

By performing the operations in (18), we can derive the following simplified expression:

$$\begin{aligned}\mathrm{LRes}_1(\mathrm{x},\mathrm{s}) &= -\rho^\alpha s^{-\alpha-1}\,\varsigma''(x) - \rho^\alpha s^{-2\alpha-1}\,\varsigma''_1(x) + 6s^{-\alpha-1}\,\varsigma^2(x) \\ &+ 6s^{-\alpha-1}\,\varsigma(x) + s^{-\alpha-1}\,\varsigma_1(x) + 12s^{-2\alpha-1}\,\varsigma(x)\,\varsigma_1(x) \\ &- 6s^{-2\alpha-1}\,\varsigma_1(x) + \frac{6\Gamma(2\alpha+1)s^{-3\alpha-1}(\varsigma_1(x))^2}{\Gamma(\alpha+1)^2}, s > 0.\end{aligned} \quad (19)$$

Then, multiplying by $s^{1+\alpha}$ both sides of Eq. (19), we obtain:

$$\begin{aligned}s^{1+\alpha}\mathrm{LRes}_1(\mathrm{x},\mathrm{s}) &= -\rho^\alpha s^{-\alpha}\,\varsigma''_1(x) + 12s^{-\alpha}\,\varsigma(x)\,\varsigma_1(x) - 6s^{-\alpha}\,\varsigma_1(x) \\ &+ \frac{6\,\Gamma(2\alpha+1)\,s^{-2\alpha}(\varsigma(x))^2}{\Gamma(\alpha+1)^2} - \rho^\alpha\,\varsigma''(x) + 6(\varsigma(x))^2 + 6\,\varsigma(x) \\ &+ \varsigma_1(x), \qquad s > 0.\end{aligned} \quad (20)$$

It is obvious that for $s > 0$ and $k = 0, 1, 2, 3, \ldots$. $Lim_{k\to\infty} LRes_k(x,s) = LRes(x,s)$, $LRes(x,s) = 0$. As a result, $Lim_{s\to\infty}\left(s^k LRes(x,s)\right) = 0$. Additionally, it was established [25] and

$$Lim_{s\to\infty}\left(s^{k+1}LRes(x,s)\right) = Lim_{s\to\infty}\left(s^{k+1}LRes_k(x,s)\right) = 0, k = 1, 2, 3, \ldots$$

By solving the provided formula for $\varsigma_1(x)$, we can quickly determine its value.

$$0 = -\rho^\alpha\varsigma''(x) + 6(\varsigma(x))^2 + 6\,\varsigma(x) + \varsigma_1(x). \quad (21)$$

It is simple to get the following by calculating $\varsigma_1(x)$ in the ensuing algebraic formula (21).

$$\varsigma_1(x) = \rho^\alpha\varsigma''(x) - 6(\varsigma(x))^2 - 6\,\varsigma(x). \quad (22)$$

The 2nd-truncated series of Eq. (17), $U_2(x,s) = \frac{\varsigma(x)}{s} + \frac{\varsigma_1(x)}{s^{1+\alpha}} + \frac{\varsigma_2(x)}{s^{1+2\alpha}}$,

The obtained value of $\varsigma_2(x)$ is then substituted into the 2nd-LRF to calculate the value of the next unknown parameter $\varsigma_2(x)$ using the following procedure:

$$\begin{aligned}\mathrm{LRes}_2(\mathrm{x},\mathrm{s}) = {} & \frac{\varsigma_2(x)}{s^{1+2\alpha}} - \frac{\rho^\alpha}{s^\alpha}\left(\frac{\varsigma''(x)}{s} + \frac{\varsigma_1''(x)}{s^{1+\alpha}} + \frac{\varsigma_2''(x)}{s^{1+2\alpha}}\right) \\ & - \frac{6}{s^\alpha}\left(\frac{\varsigma(x)}{\mathrm{s}} + \frac{\varsigma_1(x)}{s^{1+\alpha}} + \frac{\varsigma_2(x)}{s^{1+2\alpha}}\right) \\ & + \frac{6}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\left(\frac{\varsigma(x)}{\mathrm{s}} + \frac{\varsigma_1(x)}{s^{1+\alpha}} + \frac{\varsigma_2(x)}{s^{1+2\alpha}}\right)\right]\right)^2\right], s > 0.\end{aligned} \tag{23}$$

By manipulating the operators described in (23), we can simplify the expression and arrive at the following more concise form:

$$\begin{aligned}\mathrm{LRes}_2(\mathrm{x},\mathrm{s}) = {} & -\rho^\alpha s^{-\alpha-1}\,\varsigma''(x) - \rho^\alpha s^{-2\alpha-1}\,\varsigma''_1(x) - \rho^\alpha s^{-3\alpha-1}\,\varsigma''_2(x) \\ & + 6s^{-\alpha-1}(\varsigma(x))^2 - 6s^{-\alpha-1}\,\varsigma(x) + 12s^{-2\alpha-1}\,\varsigma(x)\,\varsigma_1(x) \\ & - 6s^{-2\alpha-1}\,\varsigma_1(x) + s^{-2\alpha-1}\,\varsigma_2(x) - 6s^{-3\alpha-1}\,\varsigma_1(x) \\ & + 12s^{-3\alpha-1}\,\varsigma(x)\,\varsigma_2(x) + \frac{6\,\Gamma(2\alpha+1)\,s^{-3\alpha-1}(\varsigma_1(x))^2}{\Gamma(\alpha+1)^2} \\ & + \frac{12\,\Gamma(3\alpha+1)\,s^{-4\alpha-1}\,\varsigma_1(x)\,\varsigma_2(x)}{\Gamma(\alpha+1)\,\Gamma(2\alpha+1)} \\ & + \frac{6\,\Gamma(4\alpha+1)\,s^{-5\alpha-1}(\varsigma_2(x))^2}{\Gamma(2\alpha+1)^2}, \quad s > 0.\end{aligned} \tag{24}$$

The outcome obtained by performing the multiplication operation with $s^{1+2\alpha}$ both sides of (24), is:

$$\begin{aligned} & s^{1+2\alpha}\mathrm{LRes}_2(\mathrm{x},\mathrm{s}) \\ & = -\rho^\alpha s^{-\alpha}\,\varsigma''_2(x) - 6s^{-\alpha}\,\varsigma_1(x) + 12s^{-\alpha}\,\varsigma(x)\,\varsigma_2(x) \\ & + \frac{6\,\Gamma(4\alpha+1)\,s^{-3\alpha}(\varsigma_2(x))^2}{\Gamma(2\alpha+1)^2} + \frac{12\,\Gamma(3\alpha+1)\,s^{-2\alpha}\,\varsigma_1(x)\,\varsigma_2(x)}{\Gamma(\alpha+1)\,\Gamma(2\alpha+1)} \\ & + \frac{6\,\Gamma(2\alpha+1)\,s^{-\alpha}(\varsigma_1(x))^2}{\Gamma(\alpha+1)^2} - \rho^\alpha s^\alpha\,\varsigma''(x) + 6s^\alpha(\varsigma(x))^2 - 6s^\alpha\,\varsigma(x) \\ & - \rho^\alpha\,\varsigma''_1(x) + 12\,\varsigma(x)\,\varsigma_1(x) - 6\,\varsigma_1(x) + \varsigma_2(x), s > 0.\end{aligned} \tag{25}$$

It is obvious that for $s > 0$ and $k = 0, 1, 2, 3, ...$ $Lim_{k\to\infty} LRes_k(x, s) = LRes(x, s)$, $LRes(x, s) = 0$. As a result, $Lim_{s\to\infty}\left(s^k LRes(x, s)\right) = 0$. Additionally, it was established [25] and

$$Lim_{s\to\infty}\left(s^{k+1}LRes(x, s)\right) = Lim_{s\to\infty}\left(s^{k+1}LRes_k(x, s)\right) = 0, k = 1, 2, 3,..$$

To derive the subsequent formula, we can evaluate the limit as $s\to\infty$ on both sides of (25), and then proceed with the following steps:

$$0 = -\rho^\alpha\,\varsigma''_1(x) + 12\,\varsigma(x)\,\varsigma_1(x) - 6\,\varsigma_1(x) + \varsigma_2(x).\ \varsigma \tag{26}$$

When we solve the algebraic equation that follows from $\varsigma_2(x)$, we get:

$$\varsigma_2(x) = \rho^\alpha\,\varsigma''_1(x) + 6\,\varsigma_1(x) - 12\,\varsigma(x)\,\varsigma_1(x). \tag{27}$$

Following the previous steps, we substitute the 3rd-truncated series from (17), $U_3(x, s) = \frac{\varsigma(x)}{s} + \frac{\varsigma_1(x)}{s^{1+\alpha}} + \frac{\varsigma_2(x)}{s^{1+2\alpha}} + \frac{\varsigma_3(x)}{s^{1+3\alpha}}$ into the 3rd-LRF to calculate the value of the $\varsigma_3(x)$, using the following procedure:

$$LRes_3(x,s) = \frac{\varsigma_3(x)}{s^{1+3\alpha}} - \frac{\rho^\alpha}{s^\alpha}\left(\frac{\varsigma''(x)}{s} + \frac{\varsigma_1''(x)}{s^{1+\alpha}} + \frac{\varsigma_2''(x)}{s^{1+2\alpha}} + \frac{\varsigma_3''(x)}{s^{1+2\alpha}}\right) - \frac{6}{s^\alpha}\left(\frac{\varsigma(x)}{s} + \frac{\varsigma_1(x)}{s^{1+\alpha}} + \frac{\varsigma_2(x)}{s^{1+2\alpha}} + \frac{\varsigma_3(x)}{s^{1+3\alpha}}\right) + \frac{6}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\left(\frac{\varsigma(x)}{s} + \frac{\varsigma_1(x)}{s^{1+\alpha}} + \frac{\varsigma_2(x)}{s^{1+2\alpha}} + \frac{\varsigma_3(x)}{s^{1+3\alpha}}\right)\right]\right)^2\right], \quad s > 0. \tag{28}$$

By performing the operations specified in (28), we can simplify the expression and obtain the following simplified form:

$$\begin{aligned} LRes_3(x,s) = &-\rho^\alpha s^{-\alpha-1}\,\varsigma''(x) - \rho^\alpha s^{-2\alpha-1}\,\varsigma''_1(x) - \rho^\alpha s^{-3\alpha-1}\,\varsigma''_2(x) \\ &- \rho^\alpha s^{-4\alpha-1}\,\varsigma''_3(x) + 6s^{-\alpha-1}(\varsigma(x))^2 - 6s^{-\alpha-1}\,\varsigma(x) \\ &+ 12s^{-2\alpha-1}\,\varsigma(x)\,\varsigma_1(x) - 6s^{-2\alpha-1}\,\varsigma_1(x) + 12s^{-3\alpha-1}\,\varsigma(x)\,\varsigma_1(x) \\ &- 6s^{-3\alpha-1}\,\varsigma_1(x) + s^{-3\alpha-1}\,\varsigma_3(x) + 12s^{-4\alpha-1}\,\varsigma(x)\,\varsigma_3(x) \\ &- 6s^{-4\alpha-1}\,\varsigma_3(x) + \frac{6\Gamma(2\alpha+1)s^{-3\alpha-1}(\varsigma_1(x))^2}{\Gamma(\alpha+1)^2} \\ &+ \frac{12\Gamma(3\alpha+1)s^{-4\alpha-1}(\varsigma_1(x))^2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{6\Gamma(4\alpha+1)s^{-5\alpha-1}(\varsigma_1(x))^2}{\Gamma(2\alpha+1)^2} \\ &+ \frac{12\Gamma(4\alpha+1)s^{-5\alpha-1}\,\varsigma_1(x)\,\varsigma_3(x)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)} + \frac{12\Gamma(5\alpha+1)s^{-6\alpha-1}\,\varsigma_1(x)\,\varsigma_3(x)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} \\ &+ \frac{6\Gamma(6\alpha+1)s^{-7\alpha-1}(\varsigma_3(x))^2}{\Gamma(3\alpha+1)^2}, s > 0. \end{aligned} \tag{29}$$

Multiplying by $s^{1+3\alpha}$ both sides of the Eq. (29), we get:

$$\begin{aligned} &s^{1+3\alpha}LRes_3(x,s) \\ &= -\rho^\alpha s^{-\alpha}\,\varsigma''_3(x) - \rho^\alpha s^{2\alpha}\,\varsigma''(x) + 12s^{-\alpha}\,\varsigma(x)\,\varsigma_3(x) - 6s^{-\alpha}\,\varsigma_3(x) \\ &+ 6s^{2\alpha}(\varsigma(x))^2 - 6s^{2\alpha}\,\varsigma(x) + \frac{6\,\Gamma(6\alpha+1)\,s^{-4\alpha}\left(\varsigma_3(x)\right)^2}{\Gamma(3\alpha+1)^2} \\ &+ \frac{12\,\Gamma(5\alpha+1)\,s^{-3\alpha}\,\varsigma_1(x)\,\varsigma_3(x)}{\Gamma(2\alpha+1)\,\Gamma(3\alpha+1)} + \frac{6\,\Gamma(4\alpha+1)\,s^{-2\alpha}\left(\varsigma_1(x)\right)^2}{\Gamma(2\alpha+1)^2} \\ &+ \frac{12\,\Gamma(4\alpha+1)\,s^{-2\alpha}\,\varsigma_1(x)\,\varsigma_3(x)}{\Gamma(\alpha+1)\,\Gamma(3\alpha+1)} + \frac{12\,\Gamma(3\alpha+1)\,s^{-\alpha}\left(\varsigma_1(x)\right)^2}{\Gamma(\alpha+1)\,\Gamma(2\alpha+1)} - \rho^\alpha s^\alpha\,\varsigma''_1(x) \\ &+ 12s^\alpha\,\varsigma(x)\,\varsigma_1(x) - 6s^\alpha\,\varsigma_1(x) - \rho^\alpha\,\varsigma''_2(x) + \frac{6\,\Gamma(2\alpha+1)\left(\varsigma_1(x)\right)^2}{\Gamma(\alpha+1)^2} \\ &+ 12\,\varsigma(x)\,\varsigma_1(x) - 6\,\varsigma_1(x) + \varsigma_3(x), \qquad s > 0. \end{aligned} \tag{30}$$

It is obvious that for $s > 0$ and $k = 0, 1, 2, 3, \ldots$. $Lim_{k\to\infty} LRes_k(x,s) = LRes(x,s)$, $LRes(x,s) = 0$. As a result, $Lim_{s\to\infty}\left(s^k LRes(x,s)\right) = 0$. Additionally, it was established [25] and

$$Lim_{s\to\infty}\left(s^{k+1}LRes(x,s)\right) = Lim_{s\to\infty}\left(s^{k+1}LRes_k(x,s)\right) = 0, k = 1, 2, 3, \ldots$$

Using the above fact and take $Lim_{s\to\infty}$ for each side of Eq. (30), we obtain:

$$0 =- \rho^{\alpha} \varsigma''_{2}(x) + \frac{6\Gamma(2\alpha+1)\left(\varsigma_1(x)\right)^2}{\Gamma(\alpha+1)^2} + 12\,\varsigma(x)\,\varsigma_1(x) - 6\,\varsigma_1(x) + \varsigma_3(x). \tag{31}$$

Solving formula (31) for $\varsigma_3(x)$, results in

$$\varsigma_3(x) = \rho^{\alpha} \varsigma''_{2}(x) + 6\,\varsigma_1(x) - \frac{6\Gamma(2\alpha+1)\left(\varsigma_1(x)^2\right)}{\Gamma(\alpha+1)^2} - 12\,\varsigma(x)\,\varsigma_1(x). \tag{32}$$

Once again, we can represent the results of Eq. (14) as an infinite series, which can be expressed as follows:

$$\begin{aligned} U(x,s) &= \frac{\varsigma(x)}{s} + \frac{(\rho^{\alpha}\varsigma''(x) - 6(\varsigma(x))^2 - 6\,\varsigma(x))}{s^{1+\alpha}} + \frac{(\rho^{\alpha}\varsigma''_{1}(x) + 6\,\varsigma_1(x) - 12\,\varsigma(x)\,\varsigma_1(x))}{s^{1+2\alpha}} \\ &+ \frac{\left(\rho^{\alpha}\varsigma''_{2}(x) + 6\,\varsigma_1(x) - \frac{6\Gamma(2\alpha+1)(\varsigma_1(x)^2)}{\Gamma(\alpha+1)^2} - 12\,\varsigma(x)\,\varsigma_1(x)\right)}{s^{1+3\alpha}} \\ &+ \frac{\left(\rho^{\alpha}\varsigma''_{3}(x) + 6\,\varsigma_3(x) - \frac{12\Gamma(3\alpha+1)\,\varsigma_1(x)^2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} - 12\,\varsigma(x)\,\varsigma_3(x)\right)}{s^{1+4\alpha}} \\ &+ \frac{\left(\rho^{\alpha}\varsigma''_{4}(x) + 6\,\varsigma_4(x) - \frac{6\,\Gamma(4\alpha+1)\,\varsigma_1(x)^2}{\Gamma(2\alpha+1)^2} - \frac{12\Gamma(4\alpha+1)\,\varsigma_1(x)\,\varsigma_3(x)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)} - 12\,\varsigma(\right.}{s^{1+5\alpha}} \end{aligned} \tag{33}$$

Hence, the result of the LRPS for Eqs. (9) and (10) is obtained by employing the inverse LT of (33), which can be

$$\begin{aligned} u(x,t) &= \varsigma(x) + \frac{(\rho^{\alpha}\varsigma''(x) - 6(\varsigma(x))^2 - 6\,\varsigma(x))}{\Gamma(1+\alpha)} t^{\alpha} + \frac{\left((\rho^{\alpha}\varsigma''_{1}(x) + 6\,\varsigma_1(x) - 12\,\varsigma(x)\,\varsigma_1(x))\right)}{\Gamma(1+2\alpha)} t^{2\alpha} \\ &+ \frac{\left(\rho^{\alpha}\varsigma''_{2}(x) + 6\,\varsigma_1(x) - \frac{6\,\Gamma(2\alpha+1)\,(\varsigma_1(x)^2)}{\Gamma(\alpha+1)^2} - 12\,\varsigma(x)\,\varsigma_1(x)\right)}{\Gamma(1+3\alpha)} t^{3\alpha} \\ &+ \frac{\left(\rho^{\alpha}\varsigma''_{3}(x) + 6\,\varsigma_3(x) - \frac{12\,\Gamma(3\alpha+1)\,\varsigma_1(x)^2}{\Gamma(\alpha+1)\,\Gamma(2\alpha+1)} - 12\,\varsigma(x)\,\varsigma_3(x)\right)}{\Gamma(1+4\alpha)} t^{4\alpha} \\ &+ \frac{\left(\rho^{\alpha}\varsigma''_{4}(x) + 6\,\varsigma_4(x) - \frac{6\,\Gamma(4\alpha+1)\,\varsigma_1(x)^2}{\Gamma(2\alpha+1)^2} - \frac{12\,\Gamma(4\alpha+1)\,\varsigma_1(x)\,\varsigma_3(x)}{\Gamma(\alpha+1)\,\Gamma(3\alpha+1)} - 12\,\varsigma(x)\,\varsigma_4(x)\right)}{\Gamma(1+5\alpha)} t^{5\alpha} + \ldots \end{aligned} \tag{34}$$

represented in the following fundamental form:**4- Application:** **Problem 4.1.** Given the non-linear Time Fractional Fisher shown below:

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t) + 6u(x,t)(1 - u(x,t)), \quad x \in R, \quad t > 0, 0 < \alpha \leq 1, \tag{35}$$

with initial value:

$$u(x,0) = \frac{1}{\left(1+e^{x}\right)^2} \tag{36}$$

The initial step in implementing the LT on (35) is to transfer it to the Laplace domain according to the following transformation:

When Eqs. (35) and (36) are liken to Eqs. (9) and (10), We find out that $\rho^{\alpha} = 1$ and $\varsigma(x) = \frac{1}{\left(1+e^{x}\right)^2}$.

To shift Eq. (35) to Laplace space, an initial stage is to apply the Laplace transform, as indicated that follow:

$$L[D_t^{\alpha}u(x,t)] = L\left[u_{xx}(x,t) + 6u(x,t)(1 - u(x,t))\right], \tag{37}$$

Lemma (2.8) (Part 5) states that $U(x,s) = L\{u(x,t)\}$ and applying the requirements of Eq. (37), Eq. (35) may be stated in the following way:

$$s^{\alpha}U(x,s) - s^{\alpha-1}\frac{1}{\left(1+e^{x}\right)^{2}} = U_{xx}(x,s) + 6U(x,s) - 6L\left\{\left(L^{-1}\{U(x,s)\}\right)^{2}\right\}, \ s > 0. \tag{38}$$

By dividing s^{α} both sides of Eq. (38), we obtain a new form, which is represented as follows:

$$U(x,s) = \frac{\frac{1}{\left(1+e^{x}\right)^{2}}}{s} + \frac{1}{s^{\alpha}}U_{xx}(x,s) + \frac{6}{s^{\alpha}}U(x,s) - \frac{6}{s^{\alpha}}L\left\{\left(L^{-1}\{U(x,s)\}\right)^{2}\right\}, \tag{39}$$

$$0 < \alpha \leq 1, x \in I, \ s > 0.$$

In the subsequent stage, we assume that the solution of $U(x,s)$ in (39) can be represented by FLS expansion, given by the following expressions:

$$U(x,s) = \sum_{j=0}^{\infty}\frac{\varsigma_j(x)}{s^{1+j\alpha}}, \ 0 < \alpha \leq 1, x \in I, \ s > 0, \tag{40}$$

Utilizing Lemma (2.9), we can rephrase the first coefficient of (40) as follows:

$$U(x,s) = \frac{\frac{1}{\left(1+e^{x}\right)^{2}}}{s} + \sum_{j=1}^{\infty}\frac{\varsigma_j(x)}{s^{1+j\alpha}}, \ 0 < \alpha \leq 1, x \in I, \ s > 0, \tag{41}$$

The subsequent definition of the mathematical formula LRF (41) is the third step in the construction of the LRPS appro ach:

$$LRes(x,s) = U(x,s) - \frac{\frac{1}{\left(1+e^{x}\right)^{2}}}{s} - \frac{1}{s^{\alpha}}U_{xx}(x,s) - \frac{6}{s^{\alpha}}U(x,s) + \frac{6}{s^{\alpha}}L\left\{\left(L^{-1}\{U(x,s)\}\right)^{2}\right\}, \tag{42}$$

Currently, if we utilize the kth-truncated series from Eq (41), we obtain:

$$U_k(x,s) = \frac{\frac{1}{\left(1+e^{x}\right)^{2}}}{s} + \sum_{j=1}^{k}\frac{\varsigma_j(x)}{s^{1+j\alpha}}, \ 0 < \alpha \leq 1, x \in I, \ s > 0. \tag{43}$$

Consequently, the kth LRFs are

$$LRes_k(x,s) = U_k(x,s) - \frac{\frac{1}{\left(1+e^{x}\right)^{2}}}{s} - \frac{1}{s^{\alpha}}U_{(k)xx}(x,s) - \frac{6}{s^{\alpha}}U_k(x,s) \qquad + \frac{6}{s^{\alpha}}L\left\{\left(L^{-1}\{U(x,s)\}\right)^{2}\right\}, \ 0 \tag{44}$$

Thus, the following is a series approach to Eq. (40):

$$\begin{aligned} &U(x,s) \\ &= \frac{\frac{1}{(1+e^x)^2}}{s} + \frac{\left(\frac{-6+4e^x}{(1+e^x)^3}\right)}{s^{1+\alpha}} + \frac{\left(\frac{72+2e^x(-49+8e^x)}{(1+e^x)^4}\right)}{s^{1+2\alpha}} \\ &+ \frac{\left(\frac{2(e^x(e^x(e^x(32e^x-537)+1002)+395)-432)\Gamma(\alpha+1)^2-24(3-2e^x)^2\Gamma(2\alpha+1)}{(e^x+1)^6\Gamma(\alpha+1)^2}\right)}{s^{1+3\alpha}} \\ &+ \left(\frac{1}{(e^x+1)^8}2\left(\frac{1}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)}24e^{2x}(\Gamma(2\alpha+1)^2(233\sinh(x)-86\sinh(2x)\right.\right. \\ &+155\cosh(x)+22\cosh(2x)-242)+\Gamma(\alpha+1)\Gamma(3\alpha+1)(-5\cosh(x)+92\cosh(2x) \\ &+31\sinh(x)(7-8\cosh(x))-97))+e^{3x}(e^x(e^x(128e^x-5537)+29048)-10558) \\ &\left.-1753e^x+5184)\right)/s^{1+4\alpha}, \end{aligned} \tag{45}$$

We conclude the 4th approximate LRPS finding with Eqs. (35), (36) in a series type by employing the opposite of LT of Eq. (45). This allows for us to derive:We conclude the 4th approximate LRPS finding with Eqs. (35), (36) in a series typ by employing the opposite of LT of Eq. (45). This allows for us to derive:

$$\begin{aligned} u(x,t) = {} & \frac{1}{(e^x+1)^2} + \frac{2e^x(8e^x-49)t^{2\alpha}}{(e^x+1)^4\Gamma(2\alpha+1)} + \frac{72t^{2\alpha}}{(e^x+1)^4\Gamma(2\alpha+1)} \\ & + \frac{2e^x(e^x(e^x(32e^x-537)+1002)+395)t^{3\alpha}}{(e^x+1)^6\Gamma(3\alpha+1)} \\ & - \frac{864t^{3\alpha}}{(e^x+1)^6\Gamma(3\alpha+1)} + \frac{288e^x t^{3\alpha}\Gamma(2\alpha+1)}{(e^x+1)^6\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} \\ & - \frac{96e^{2x}t^{3\alpha}\Gamma(2\alpha+1)}{(e^x+1)^6\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} \\ & - \frac{216t^{3\alpha}\Gamma(2\alpha+1)}{(e^x+1)^6\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{3506e^x t^{4\alpha}}{(e^x+1)^8\Gamma(4\alpha+1)} \\ & + \frac{2e^{3x}(e^x(e^x(128e^x-5537)+29048)-10558)t^{4\alpha}}{(e^x+1)^8\Gamma(4\alpha+1)} \\ & + \frac{10368t^{4\alpha}}{(e^x+1)^8\Gamma(4\alpha+1)} + \frac{4e^x t^\alpha}{(e^x+1)^3\Gamma(\alpha+1)} \\ & - \frac{6t^\alpha}{(e^x+1)^3\Gamma(\alpha+1)} + (48t^{4\alpha}e^{2x}(\Gamma(2\alpha+1)^2(233sinh(x) \\ & - 86sinh(2x) + 155cosh(x) + 22cosh(2x) - 242) + \Gamma(\alpha \\ & + 1)\Gamma(3\alpha+1)(-5cosh(x) + 92cosh(2x) + 31sinh(x)(7 \\ & - 8cosh(x)) - 97)))/((e^x+1)^8\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(4\alpha \\ & + 1)) \end{aligned} \tag{46}$$

As a result, the precise answer found in Eq. (46) looks like this:

$$u(x,t) = \frac{1}{\left(1+e^{x-5t}\right)^2} \tag{47}$$

To evaluate the accuracy of the approximate solution obtained in Eq. (47) and assess the correctness of the technique, we will examine the numerical results and utilize two error measures as follows:

$$Abs.\ Err.(x,t) = \left|\omega(x,t) - \omega_k(x,t)\right|, \tag{48}$$

and

$$Re.\ Err.(x,t) = \left|\frac{\omega(x,t)-\omega_k(x,t)}{\omega(x,t)}\right|, \tag{49}$$

where $\omega(x,t)$ is accurate solution and $\omega_k(x,t)$ is an approximation solution of order k.

Tables 4.1, 4.2, and 4.3 contain a table with the numerical findings for $u_j(x,t)$ for $j = 4$.
It is contrasted between estimations of solutions that have known results.

The homotopy perturbation technique and the fractions variational iteration technique [30] were used to arrive at these findings. [29] and RPSM [28].

The correspondence among the precise outcome of α=1 and the outcome of the fourthapproximation is depicted i n Figure 4.1, and every graph helps illustrate the related actions of 4th results for various values of alpha.

The region of convergent to Eqs. (35) & (36) if $\alpha = 1$ is as indicated in the table
that follows, and the surface plots illustrating the 4th LRPS findings and precise solutions, $u_4(x,t)$
and the precise outcome, are displayed in Fig. 4.1.

The graphs also illustrate that the outcomes operate whenever $\alpha = 1, \propto = 0.75$ $\alpha = 0.50$ and $\alpha = 0.25$. Therefore, As α →1, the function exhibits convergence. Conversely, as α → 0, the function diverges further.

Displayed in Figure 4.1 represents the LRPS approximating answer $u_4(x,t)$ for $\propto = [0.65, 1]$ which approaches the true outcome as α increases.

This graph unequivocally shows that there is a rising precise inaccuracies and a diminishing convergence for the approximate remedies to the accurate ones when the size of the result increases.

Table 4.1. Numerical comparisons between the exact value of $u(x,t)$ and the 4th-approximation of $u_4(x,t)$ at $\alpha = 1$

x	t	$u_4(x,t)$	$u(x,t)$	Re. Err.	Abs. Err.
0.5	0.01	0.151602	0.151602	2.28746×10^{-7}	3.46783×10^{-8}
	0.05	0.191713	0.191689	1.2251×10^{-4}	2.34838×10^{-5}
	0.1	0.250483	0.25	1.93108×10^{-3}	4.82771×10^{-4}
	0.15	0.318677	0.316042	8.33757×10^{-3}	2.63503×10^{-3}
	0.2	0.395971	0.387456	2.19785×10^{-2}	8.51568×10^{-3}

Table 4.2. Comparison among approximate solutions among u_{LRPS}, u_{RPS},u_{FVIM} and u_{HPM} at $x = 0.5$ and $\alpha = 0.75$

x	t	u_{LRPS}	u_{RPS} [29]	u_{FVIM}[30]	u_{HPM}[31]
0.5	0.01	0.175961	0.175966	0.0242651	0.146808
	0.05	0.276696	0.277218	0.0818379	0.157343
	0.1	0.402725	0.406898	0.139343	0.168312
	0.15	0.532175	0.54626	0.186505	0.178529
	0.2	0.662269	0.695654	0.224693	0.188419

Employing the previously indicated recurring directions, we are able to illustrate certain graphics implications of Eqs. (35) & (36) in Figure 4.1 to give the following succinct overview of the subject in our job:

Table 4.3. Comparison among approximate solutions among u_{LRPS}, u_{RPS},u_{FVIM} and u_{HPM} at $x = 0.5$ and $\alpha = 0.5$

x	t	u_{LRPS}	u_{RPS} [8]	u_{FVIM}[64]	u_{HPM}[51]
0.5	0.01	0.280272	0.281524	0.125003	0.157224
	0.05	0.531906	0.563205	0.316523	0.177994
	0.1	0.7312	0.856396	0.450853	0.195594
	0.15	0.848702	1.13039	0.528601	0.210357
	0.2	0.894364	1.39515	0.567751	0.22369

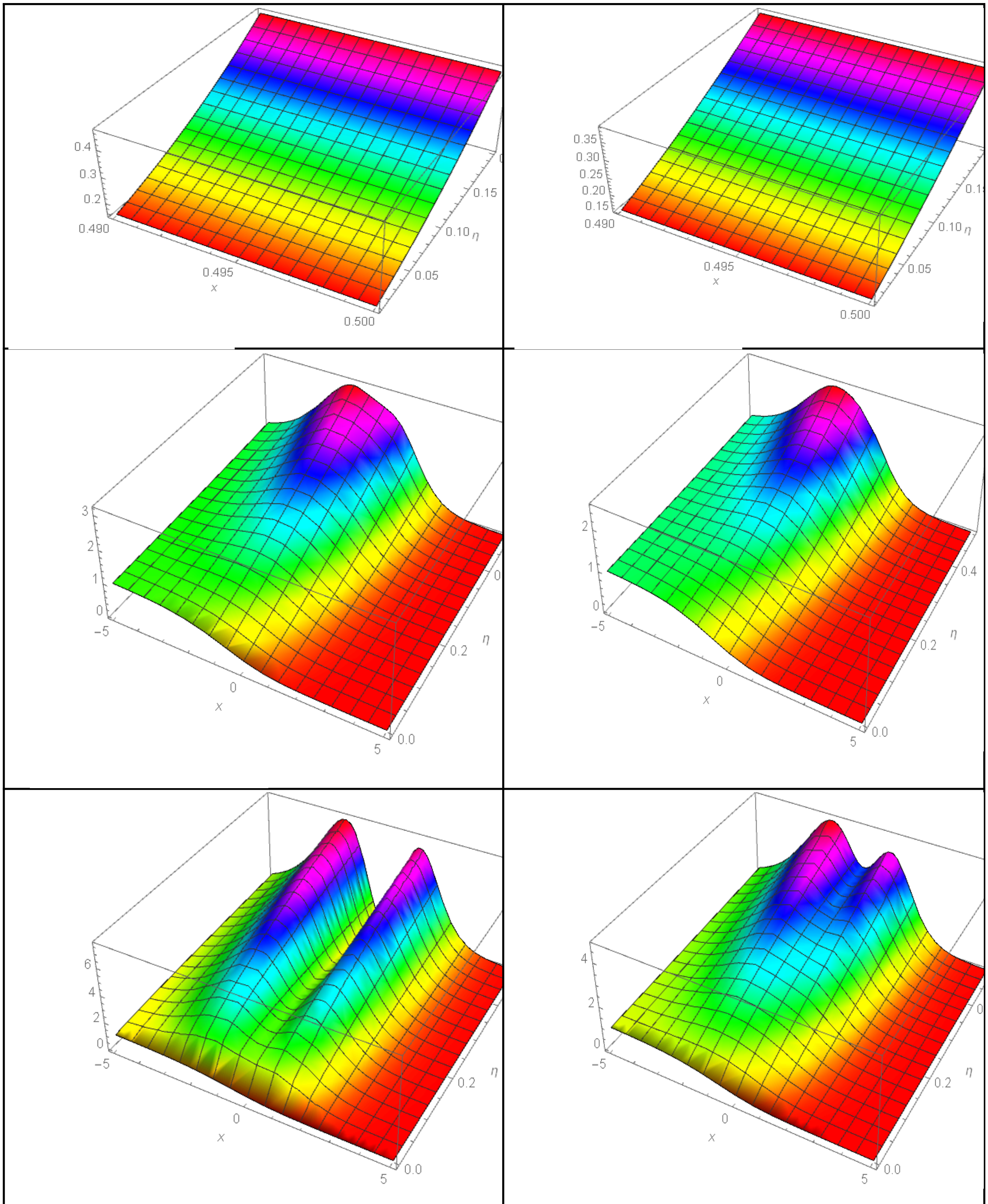

Figure 4.1. The 3D graphic for the exact solution $u(x,t)$ and the $u_4(x,t)$ approximate solution of the time-fractional Fisher equation: (a) $u_4(x,t)$, when $\alpha = 1$ (b) $u(x,t)$

is exact solution, (c) $u_4(x,t)$, when $\alpha = 0.75$, (d) $u_4(x,t)$,when $\alpha = 0.90$.,

(e) , $u_4(x,t)$, when $\alpha = 0.50$, (f) $u_4(x,t)$ when $\alpha = 0.50$.

5- Conclusions

This research introduces LRPS, a novel analytical-numerical method that proves to be a valuable tool for solving non-linear time fractional Fisher partial differential equations. By employing this approach, the non-linear time fractional Fisher partial differential equations are numerically solved, leading to accurate results. Our proposed approach successfully identifies approximate methods that exhibit fast series convergence, facilitated by easily calculable components. The results demonstrate the high effectiveness of our proposed approach, which yields estimated solutions that closely match the exact solutions with acceptable error rates.

References

[1]. Abu Arqub, O., Application of residual power series method for the solution of time-fractional Schrödinger equations in one-dimensional space. Fundamenta Informaticae, 166(2), 87-110,2019. DOI:10.3233/FI-2019-1795.

[2]. Akdemir, A. O., Dutta, H., & Atangana, A. (Eds.)., Fractional order analysis: theory, methods and applications. John Wiley & Sons,2020, ISBN: 9781119654162, 1119654165

[3]. Alabedalhadi, M., Al-Smadi, M., Al-Omari, S., Baleanu, D., & Momani, S., Structure of optical soliton solution for nonlinear resonant space-time Schrödinger equation in conformable sense with full nonlinearity term. Physica Scripta, 95(10), 105215,2020,DOI:10.1088/1402-4896/abb739.

[4]. Al-Smadi, M., & Arqub, O. A., Computational algorithm for solving Fredholm time-fractional partial integrodifferential equations of dirichlet functions type with error estimates. Applied Mathematics and Computation, 342, 280-294,2019.

[5]. Bira, B., Raja Sekhar, T., & Zeidan, D., Exact solutions for some time- fractional evolution equations using Lie group theory. Mathematical Methods in the Applied Sciences, 41(16), 6717-6725,2018.

[6]. AL-Smadi, M. H., & Gumah, G. N., On the homotopy analysis method for fractional SEIR epidemic model. Research Journal of Applied Sciences, Engineering and Technology, 7(18), 3809-3820,2014, DOI: 10.19026/rjaset.7.738

[7]. Al-Smadi, M., Freihat, A., Arqub, O. A., & Shawagfeh, N., A Novel Multistep Generalized Differential Transform Method for Solving Fractional-order Lü Chaotic and Hyperchaotic Systems. Journal of Computational Analysis & Applications, 19(1),2015.

[8]. Baleanu, D., Golmankhaneh, A. K., Golmankhaneh, A. K., & Baleanu, M. C., Fractional electromagnetic equations using fractional forms. International Journal of theoretical Physics, 48(11), 3114-3123,2009, Doi.org/10.1007/s10773-009-0109-8

[9]. Kumar, S., A new fractional modeling arising in engineering sciences and its analytical approximate solution. Alexandria Engineering Journal, 52(4), 813-819,2013, Doi.org/10.1016/j.aej.2013.09.005

[10]. Al-Smadi, M., Freihat, A., Khalil, H., Momani, S., & Ali Khan, R., Numerical multistep approach for solving fractional partial differential equations. International Journal of Computational Methods, 14(03), 1750029,2017. Doi.org/10.1142/S0219876217500293

[11]. Al-Smadi, M., Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation. Ain Shams Engineering Journal, 9(4), 2517-2525,2018, Doi.org/10.1016/j.asej.2017.04.006 .

[12]. Jleli, M., Kumar, S., Kumar, R., & Samet, B., Analytical approach for time fractional wave equations in the sense of Yang-Abdel-Aty-Cattani via the homotopy perturbation transform method. Alexandria Engineering Journal, 59(5), 2859-2863,2020, Doi.org/10.1016/j.aej.2019.12.022

[13]. Momani, S., Freihat, A., & Al-Smadi, M., Analytical study of fractional- order multiple chaotic FitzHugh-Nagumo neurons model using multistep generalized differential transform method. In Abstract and Applied Analysis (Vol. 2014). Hindawi, 2014, January, Doi.org/10.1155/2014/276279

[14]. Al-Smadi, M., Abu Arqub, O., & Gaith, M., Numerical simulation of telegraph and Cattaneo fractional-type models using adaptive reproducing kernel framework.Mathematical Methods in the Applied Sciences, 44(10), 8472-8489,2021, Doi.org/10.1002/mma.6998.

[15]. Hasan, S., Al-Smadi, M., El-Ajou, A., Momani, S., Hadid, S., & Al-Zhour, Z., Numerical approach in the Hilbert space to solve a fuzzy Atangana-Baleanu fractional hybrid system. Chaos, Solitons & Fractals, 143, 110506,2021, Doi.org/10.1016/j.chaos.2020.110506

[16]. Talafha, A. G., Alqaraleh, S. M., Al-Smadi, M., Hadid, S., & Momani, S., Analytic solutions for a modified fractional three wave interaction equations with conformable derivative by unified method. Alexandria Engineering Journal, 59(5),3731-3739,2020, Doi.org/10.1016/j.aej.2020.06.027

[17]. Al-Smadi, M., Freihat, A., Hammad, M. M. A., Momani, S., & Arqub, O. A., Analytical approximations of partial differential equations of fractional order with multistep approach. Journal of Computational and Theoretical Nanoscience, 13(11),7793-7801,2016, Doi.org/10.1166/jctn.2016.5780

[18]. Gambo, Y. Y., Ameen, R., Jarad, F., & Abdeljawad, T., Existence and uniqueness of solutions to fractional differential equations in the frame of generalized Caputo fractional derivatives. Advances in Difference Equations, 2018(1), 1-13.

[19]. Hasan, S., El-Ajou, A., Hadid, S., Al-Smadi, M., & Momani, S., Atangana- Baleanu fractional framework of reproducing kernel technique in solving fractional population dynamics system. Chaos, Solitons & Fractals, 133, 109624,2020, Doi.org/10.1016/j.chaos.2020.109624 .

[20]. He, J. H., Elagan, S. K., & Li, Z. B., Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. Physics letters A,376(4), 257-259,2012, Doi.org/10.1016/j.physleta.2011.11.030

[21]. Khader, M. M., Saad, K. M., Hammouch, Z., & Baleanu, D., A spectral collocation method for solving fractional KdV and KdV-Burgers equations with non-singular kernel derivatives. Applied Numerical Mathematics, 161, 137-146,2021, Doi.org/10.1016/j.apnum.2020.10.024

[22]. Momani, S., Arqub, O. A., Freihat, A., & Al-Smadi, M., Analytical approximations for Fokker-Planck equations of fractional order in multistep schemes. Applied and computational mathematics, 15(3), 319-330,2016.

[23]. Arqub, O. A., El-Ajou, A., & Momani, S., Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations.Journal of Computational Physics, 293, 385-399,2015, Doi.org/10.1016/j.jcp.2014.09.034

[24]. Xu, F., Gao, Y., Yang, X., & Zhang, S., Constructing of fractional power series solutions to fractional Boussinesq equations using residual power series method.Mathematical Problems in Engineering, 2016.

[25]. Eriqat, T., El-Ajou, A., Moa'ath, N. O., Al-Zhour, Z., & Momani, S., A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations. Chaos, Solitons & Fractals, 138, 109957,2020, Doi.org/10.1016/j.chaos.2020.109957

[26]. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J., Theory and applications of fractional differential equations (Vol. 204). Elsevier,2006.

[27]. El-Ajou, A., Arqub, O. A., & Momani, S., Approximate analytical solution of the nonlinear fractional KdV–Burgers equation: a new iterative algorithm. Journal of Computational Physics, 293, 81-95,2015, Doi.org/10.1016/j.jcp.2014.08.004

[28]. Al Qurashi, M. M., Korpinar, Z., Baleanu, D., & Inc, M. (2017). A new iterative algorithm on the time-fractional Fisher equation: Residual power series method. Advances in Mechanical Engineering, 9(9), Doi.org/10.1177/1687814017716009

[29]. Khan, N. A., Ayaz, M., Jin, L., & Yildirim, A. (2011). On approximate solutions for the time-fractional reaction-diffusion equation of Fisher type. Int. J. Phys. Sci, 6(10), 2483-2496, DOI: 10.5897/IJPS11.181.

[30]. Merdan, M. (2012). Solutions of time-fractional reaction-diffusion equation with modified Riemann-Liouville derivative. Int. J. Phys. Sci, 7(15), 2317-2326, DOI: 10.5897/IJPS12.027.