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Solutions For Fractional Linear Systems With Hattaf Derivative

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Abstract

This paper deals with the analytical solutions of the fractional-order linear systems characterized by $DX(t) = AX(t)$ fractional differential equations. Here D, is the fractional derivative in the Hattaf sense and A is the $(2x2)$-type non-singular real coefficient matrix. We present some results related to the existence of analytical solutions of the linear equation systems of the Hattaf fractional derivative. These solutions are provided for three different fractional-order linear systems.

Keywords: linear system; Hattaf derivative; analytical solution

Introduction

The fractional derivative is a generalization of the classical derivative of integer order. Fractional derivatives are used in the literature to solve many mathematical and engineering problems. They also play an essential role in calculations modeling the dynamics of many systems present in various disciplines such as control theory, mechanics, chemistry, finance and biology [1–3].

The fractional derivative has recently attracted the attention of many researchers. In general, exact solutions of fractional differential equations are difficult to solve by some analytical methods. In 2015, Caputo and Fabrizio presented a new definition of the fractional derivative with a non-singular kernel [1]. Atangana and Baleanu introduced the new fractional derivative using the 1-parameter Mittag-Leffler function [3]. Because of importance of weighted fractional derivatives in solving various integral equation derivatives with easy methods. Al-Refai presented the weighted Atangana-Baleanu fractional derivative in 2020 [4]. After that Hattaf used a new definition of fractional derivative, which generalized the fractional derivatives with non-singular kernels for both Caputo and Riemann-Liouville derivatives [2].

Fractional systems, expressed by integro-differential equations of arbitrary degree, are one of the topics we have frequently heard about in system theory in recent years. A better understanding of the mathematical background has led to more studies on fractional-order systems [5–7].

In 2019, Wei et al. considered a non-asymptotic method of fractional derivative estimation of the pseudo-state type for fractional linear systems [8].

This paper organized as in the following way:

The second section deals with the preliminary information and properties of the Hattaf fractional derivative. The third section devotes the main results to obtaining analytical solution of the linear equation systems of Hattaf fractional derivative. The last section presents a discussion of our topic.

Materials and Methods

This section recalls the concepts and some properties of the Hattaf fractional derivative. Hattaf introducd a new definition with a non- singular kernel in th sense of Caputo and Riemann-Liouville [2].
[2] The Hattaf fractional derivative of order α of Caputo sense is given by

$$ {}^{C}D_{a,t,\omega}^{\alpha,\beta,\gamma} f(t) = \frac{N(\alpha)}{1-\alpha}\frac{1}{\omega(t)}\int_a^t E_\beta\left[-\mu_\alpha (t-x)^\gamma\right]\frac{d}{dx}(\omega f)(x)dx $$

Here, $N(\alpha)$ is a normalization function such that $\alpha \in [0,1)$ and $N(0) = N(1) = 1$, $w(t)$ is a weight function, $\omega \in C^1(a,b)$, $\omega, \omega' > 0$, $\mu_\alpha = \frac{\alpha}{1-\alpha}$ and $E_\beta(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\beta k+1)}$ is the Mittag-Leffler function with respect to the parameter β, $f \in H^1(a,b)$ and $\beta, \gamma > 0$.

If we choose $\beta = \gamma = 1$ and $\omega(t) = 1$, this fractional derivative becomes Caputo-Fabrizio fractional derivative. Moreover, if we choose $\beta = \gamma = \alpha$ and $\omega(t) = 1$, it transforms into Atangana-Baleanu fractional derivative and again choosing $\beta = \gamma = \alpha$, it becomes the weighted Atangana-Baleanu fractional derivative.

The Hattaf fractional derivative holds the following properties:

i. $D_{a,t,\omega}^{\alpha,\beta,\gamma}$ is a linear operator,

ii. $D_{a,t,1}^{\alpha,\beta,\gamma}(c) = 0$ such that $f(t) = c$,

iii. $D_{a,t,\omega}^{0,\beta,\gamma} f(t) = \frac{1}{\omega(t)}\int_a^t E_0(0)\frac{d}{dx}(\omega f)(x)dx = \frac{1}{\omega(t)}\left(\omega(t)f(t) - \omega(a)f(a)\right)$

[2] The Hattaf fractional integral is given by

$$ I_{a,t,\omega}^{\alpha,\beta,\gamma} f(t) = \frac{1-\alpha}{N(\alpha)} f(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\frac{1}{\omega(t)}\int_a^t (t-x)^{\beta-1}\omega(x)f(x)dx $$

where $\gamma = \beta$.

When $\alpha = \gamma = \beta$ and $\omega(t) = 1$, this fractional integral becomes Atangana-Baleanu fractional integral. Also, when $\alpha = \gamma = \beta$, it turns into the weighted Atangana-Baleanu fractional integral [4] [9] Assume that $f \in H^1(a,b)$, $\beta > 0$ and $\alpha \in [0,1)$. Then we have:

$$ I_{a,\omega}^{\alpha,\beta}\left(D_{a,\omega}^{\alpha,\beta} f\right)(t) = D_{a,\omega}^{\alpha,\beta}\left(I_{a,\omega}^{\alpha,\beta} f\right)(t) = f(t) - \left[\frac{\omega(a)f(a)}{\omega(t)}\right]. $$

[9] For $\omega(t) = 1$, we get

$$ I_{a,\omega}^{\alpha,\beta}\left(D_{a,\omega}^{\alpha,\beta} f\right)(t) = D_{a,\omega}^{\alpha,\beta}\left(I_{a,\omega}^{\alpha,\beta} f\right)(t) = f(t) - f(a). $$

Results

This section devotes to solving the Hattaf fractional real linear system. Then we give the following result according to the system defined by

$$ \begin{aligned} D_{a,t,\omega}^{\alpha,\beta,\gamma} x(t) &= \lambda x(t) \\ D_{a,t,\omega}^{\alpha,\beta,\gamma} y(t) &= \mu y(t). \end{aligned} \tag{1} $$

Let the initial conditions of system (1) be x_0 and y_0 . So the solutions of (1) are $x(t) = \frac{t^{\beta-1}}{t^{\beta}-K(\beta-2)}c_1$ and $y(t) = \frac{t^{\beta-1}}{t^{\beta}-L(\beta-2)}c_2$, where $x(0) = x_0$ and $y(0) = y_0$. Assume that $\omega(t) = 1, \beta = \gamma$ and $\beta \geq 0$. Then integrating both sides of (1), we have

$$I_{a,t,1}^{\alpha,\beta,\beta}\left[D_{a,t,1}^{\alpha,\beta,\beta}x(t)\right] = \lambda\left[\frac{1-\alpha}{N(\alpha)}x(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}x(s)ds\right]$$

$$I_{a,t,1}^{\alpha,\beta,\beta}\left[D_{a,t,1}^{\alpha,\beta,\beta}y(t)\right] = \mu\left[\frac{1-\alpha}{N(\alpha)}y(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}y(s)ds\right]$$

and

$$\begin{aligned} x(t)-x(a) &= \lambda\left[\frac{1-\alpha}{N(\alpha)}x(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}x(s)ds\right] \\ y(t)-y(a) &= \mu\left[\frac{1-\alpha}{N(\alpha)}y(t) + \frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}y(s)ds\right]. \end{aligned} \tag{2}$$

Taking derivation on both sides of (2) for $a = 0$, we deduce

$$\begin{aligned} x'(t) &= \lambda\left[\frac{1-\alpha}{N(\alpha)}x'(t) + \frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-2}x(s)ds\right] \\ y'(t) &= \mu\left[\frac{1-\alpha}{N(\alpha)}y'(t) + \frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-2}y(s)ds\right]. \end{aligned} \tag{3}$$

By the help of (3), it follows that

$$\begin{aligned} x'(t) &= \frac{\lambda\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\frac{N(\alpha)}{N(\alpha)-\lambda(1-\alpha)}\int_0^t (t-s)^{\beta-2}x(s)ds \\ y'(t) &= \frac{\mu\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\frac{N(\alpha)}{N(\alpha)-\mu(1-\alpha)}\int_0^t (t-s)^{\beta-2}y(s)ds. \end{aligned} \tag{4}$$

If we take the Laplace transform of (4) we obtain

$$x(t) = \frac{t^{\beta-1}}{t^{\beta}-K(\beta-2)}c_1$$

$$y(t) = \frac{t^{\beta-1}}{t^{\beta}-L(\beta-2)}c_2$$

where $K = \frac{\lambda\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\frac{N(\alpha)}{N(\alpha)-\lambda(1-\alpha)}$ and $L = \frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\frac{N(\alpha)}{N(\alpha)-(1-\alpha)}$.
Thus it completes the proof. Now, we consider the following system as follows:

$$\begin{aligned} D_{a,t,1}^{\alpha,\beta,\beta}x(t) &= \lambda x(t) \\ D_{a,t,1}^{\alpha,\beta,\beta}y(t) &= x(t) + \lambda y(t). \end{aligned} \tag{5}$$

Let the initial conditions of (5) be x_0 and y_0. Then the solutions of (5) are

$$\begin{aligned} x(t) &= \frac{t^{\beta-1}}{t^{\beta}-K(\beta-2)}c_1 \\ y(t) &= \frac{1}{tN(\alpha)+\lambda(1-\alpha)}\left[\frac{(1-\alpha)(\beta-2)Kc_1}{t^{\beta}-K(\beta-2)} + \frac{\alpha(\beta-1)(\beta-2)c_1}{\left[t^{\beta}-K(\beta-2)\right]\Gamma(\beta)} + \left(N(\alpha)-\lambda(1-\alpha)\right)c_2\right]. \end{aligned}$$

Let $\omega(t) = 1$, $\beta = \gamma$ and $\beta \geq 0$. Using the same methodology of Theorem 2, we observe that the first solution of (5) equals. So now, we shall find the $y(t)$ solution of (5). In this regard, by integrating the second equation of (5), we have

$$I_{a,t,1}^{\alpha,\beta,\beta}\left[D_{a,t,1}^{\alpha,\beta,\beta}y(t)\right]=\frac{1-\alpha}{N(\alpha)}x(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}x(s)ds$$
$$+\lambda\left[\frac{1-\alpha}{N(\alpha)}y(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}x(s)ds\right]$$

and

$$\begin{aligned}y(t)-y(a)&=\frac{1-\alpha}{N(\alpha)}x(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}x(s)ds\\&+\lambda\left[\frac{1-\alpha}{N(\alpha)}y(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}x(s)ds\right].\end{aligned}\tag{6}$$

Then taking derivation on both sides of (6) for $a=0$, we get

$$\begin{aligned}y'(t)&=\frac{1-\alpha}{N(\alpha)}x'(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}(\beta-1)\int_a^t(t-s)^{\beta-2}x(s)ds\\&+\lambda\left[\frac{1-\alpha}{N(\alpha)}y'(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}(\beta-1)\int_0^t(t-s)^{\beta-2}x(s)ds\right].\end{aligned}\tag{7}$$

Further, using the Laplace transform of (7) it yields

$$\begin{aligned}\mathcal{L}\left\{y'(t)\right\}&=\frac{1-\alpha}{N(\alpha)}\mathcal{L}\left\{x'(t)\right\}+\frac{\alpha}{N(\alpha)\Gamma(\beta)}(\beta-1)\mathcal{L}\left\{\int_0^t(t-s)^{\beta-2}x(s)ds\right\}\\&+\frac{1-\alpha}{N(\alpha)}\mathcal{L}\left\{x'(t)\right\}+\frac{\alpha}{N(\alpha)\Gamma(\beta)}(\beta-1)\mathcal{L}\left\{\int_0^t(t-s)^{\beta-2}x(s)ds\right\}.\end{aligned}\tag{8}$$

Consequently, follows from (8) we obtain

$$y(t)=\frac{1}{tN(\alpha)+\lambda(1-\alpha)}\left[\left[\frac{(1-\alpha)(\beta-2)Mc_1}{t^\beta-M(\beta-2)}+\frac{\alpha(\beta-1)(\beta-2)c_1}{\left[t^\beta-M(\beta-2)\right]\Gamma(\beta)}\right]+\left(N(\alpha)-\lambda(1-\alpha)\right)c_2\right].$$

This completes the proof. Finally, let us consider the following linear system:

$$\begin{aligned}D_{a,t,\omega}^{\alpha,\beta,\gamma}x(t)&=\eta x(t)+\gamma y(t)\\D_{a,t,\omega}^{\alpha,\beta,\gamma}y(t)&=-\gamma x(t)+\eta y(t).\end{aligned}\tag{9}$$

Let the initial conditions of systems (9) be x_0 and y_0. So the solutions of system (9) are

$$x(t)=\frac{1}{s}\frac{\left[\frac{\beta-2}{t^{\beta-1}}\right]\left[\frac{\eta\alpha(\beta-1)-\gamma\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\right]\left[\frac{N(\alpha)-(1-\alpha)(\gamma-\eta)}{N(\alpha)}\right]}{\left[\frac{N(\alpha)-\eta(1-\alpha)}{N(\alpha)}\right]^2+\left[\frac{\gamma(1-\alpha)}{N(\alpha)}\right]^2}+c_1$$

$$y(t)=\frac{1}{s}\frac{\left[\frac{\beta-2}{t^{\beta-1}}\right]\left[\frac{\eta\alpha(\beta-1)-\gamma\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\right]\left[\frac{N(\alpha)-(1-\alpha)(\gamma+\eta)}{N(\alpha)}\right]}{\left[\frac{N(\alpha)-\eta(1-\alpha)}{N(\alpha)}\right]^2+\left[\frac{\gamma(1-\alpha)}{N(\alpha)}\right]^2}+c_2.$$

We shall assume that $\omega(t)=1$, $\beta=\gamma$ and $\beta\geq 0$. By integrating (9), it follows that

$$\begin{aligned}I_{a,t,1}^{\alpha,\beta,\beta}\left[D_{a,t,1}^{\alpha,\beta,\beta}x(t)\right]&=\eta\left[\frac{1-\alpha}{N(\alpha)}x(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}x(s)ds\right]\\&+\gamma\left[\frac{1-\alpha}{N(\alpha)}y(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}y(s)ds\right]\end{aligned}$$

$$\begin{aligned}I_{a,t,1}^{\alpha,\beta,\beta}\left[D_{a,t,1}^{\alpha,\beta,\beta}x(t)\right]&=-\gamma\left[\frac{1-\alpha}{N(\alpha)}x(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}x(s)ds\right]\\&+\eta\left[\frac{1-\alpha}{N(\alpha)}y(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}y(s)ds\right].\end{aligned}$$

and

$$\begin{aligned} x(t)-x(a) &= \eta\left[\frac{1-\alpha}{N(\alpha)}x(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}x(s)ds\right] \\ &+\gamma\left[\frac{1-\alpha}{N(\alpha)}y(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}y(s)ds\right] \end{aligned}$$

$$\begin{aligned} y(t)-y(a) &= -\gamma\left[\frac{1-\alpha}{N(\alpha)}x(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}x(s)ds\right] \\ &+\eta\left[\frac{1-\alpha}{N(\alpha)}y(t)+\frac{\alpha}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}y(s)ds\right]. \end{aligned} \tag{10}$$

Taking the first derivative both sides of (10) for $a=0$, we have

$$\begin{aligned} x'(t) &= \eta\left[\frac{1-\alpha}{N(\alpha)}x'(t)+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-2}x(s)ds\right] \\ &+\gamma\left[\frac{1-\alpha}{N(\alpha)}y'(t)+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-2}y(s)ds\right] \end{aligned}$$

$$\begin{aligned} y'(t) &= -\gamma\left[\frac{1-\alpha}{N(\alpha)}x'(t)+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-2}x(s)ds\right] \\ &+\eta\left[\frac{1-\alpha}{N(\alpha)}y'(t)+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\beta-2}y(s)ds\right]. \end{aligned} \tag{11}$$

Then taking the Laplace transform of (11), we find

$$\begin{aligned} \mathcal{L}\{x'(t)\} &= \eta\left[\frac{1-\alpha}{N(\alpha)}\mathcal{L}\{x'(t)\}+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\mathcal{L}\left\{\int_0^t (t-s)^{\beta-2}x(s)ds\right\}\right] \\ &+\gamma\left[\frac{1-\alpha}{N(\alpha)}\mathcal{L}\{y'(t)\}+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\mathcal{L}\left\{\int_0^t (t-s)^{\beta-2}x(s)ds\right\}\right] \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{y'(t)\} &= -\gamma\left[\frac{1-\alpha}{N(\alpha)}\mathcal{L}\{x'(t)\}+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\mathcal{L}\left\{\int_0^t (t-s)^{\beta-2}x(s)ds\right\}\right] \\ &+\eta\left[\frac{1-\alpha}{N(\alpha)}\mathcal{L}\{y'(t)\}+\frac{\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\mathcal{L}\left\{\int_0^t (t-s)^{\beta-2}x(s)ds\right\}\right]. \end{aligned} \tag{12}$$

Hence, in view of (12), we obtain

$$x(t)=\frac{1}{s}\frac{\left[\frac{\beta-2}{t^{\beta-1}}\right]\left[\frac{\eta\alpha(\beta-1)-\gamma\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\right]\left[\frac{N(\alpha)-(1-\alpha)(\gamma-\eta)}{N(\alpha)}\right]}{\left[\frac{N(\alpha)-\eta(1-\alpha)}{N(\alpha)}\right]^2+\left[\frac{\gamma(1-\alpha)}{N(\alpha)}\right]^2}+c_1$$

$$y(t)=\frac{1}{s}\frac{\left[\frac{\beta-2}{t^{\beta-1}}\right]\left[\frac{\eta\alpha(\beta-1)-\gamma\alpha(\beta-1)}{N(\alpha)\Gamma(\beta)}\right]\left[\frac{N(\alpha)-(1-\alpha)(\gamma+\eta)}{N(\alpha)}\right]}{\left[\frac{N(\alpha)-\eta(1-\alpha)}{N(\alpha)}\right]^2+\left[\frac{\gamma(1-\alpha)}{N(\alpha)}\right]^2}+c_2$$

which ends the proof.

Conclusions

A fractional system is a dynamic system that exhibits fractional-order derivatives or integrals in its mathematical description. Unlike classical systems that involve integer-order derivatives or integrals, fractional systems involve derivatives or integrals of non-integer order.

Fractional systems have gained increasing attention in various scientific and engineering fields due to their ability to capture complex behaviors and phenomena that classical integer-order systems cannot adequately describe. They have been applied in physics, biology, control systems, signal processing, finance, etc. Modeling and analyzing fractional systems often require specialized techniques and tools, as the non-local nature of fractional calculus introduces additional challenges compared to classical systems. Numerical methods, such as fractional order integrators or discretization schemes, are often employed to approximate and simulate fractional systems.

This paper aims to study the analytical solutions of the fractional-order linear system in the sense of Hattaf. We obtained some solutions according to the different three fractional-order linear systems. Our further studies will be devoted to the concepts such as stability and the boundary value for these systems.

Data Availabity Statement

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

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