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Bifurcation analysis of dynamical systems with fractional order differential equations via the modified Riemann-Liouville derivative

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Abstract:
In this manuscript, the solutions of linear dynamical systems with fractional differential equations via the modified Riemann-Liouville derivative is derived. By using Jumarie type of derivative (JRL), we stated and proved the Existence and uniqueness theorems of the dynamical systems with fractional order equations. Also a novel stability analysis of fractional dynamical systems by Jumarie type derivative is established and some important stability conditions are determined. The achieved results have various applications in mathematics, plasma physics and almost all branches of physics that have non-conservative forces. Finally, we investigated interesting application of nonlinear space-time fractional Korteweg-de Vries (STFKdV) equation in Saturn F-ring's region. Moreover, our investigation could be basic interest to explain and interpret the effects of fractional and modification parameters on STFKdV equation. This is novel study on this model by dynamical system (DS) to describe the behavior of nonlinear waves without solve this system.
Keywords: Dynamical systems, Modified Riemann-Liouville derivative, Fractional differential equations, Mittage-Leffler function.

1 Introduction

Recently, solution of fractional differential equations has received a great deal of the attention of researchers because it has been used in various fields of sciences and engineering. It is one of the generalization of the classical calculus and it is a great tool to model and many nature dynamical system that have long memory and long-range spatial interactions. The fractional dynamical system (FDS) describes the system contain non-conservative forces in various branches of physics [1]. It is very important to point out that the fractional integrals were studied before the fractional derivatives. This is due to the fact that the derivatives of fractional order are defined by the fractional integrals. The most popular definition of fractional integral was given by Riemann-Liouville [2, 3, 4]. Regard the fractional derivatives, there are several different definitions because there is no applicable fractional derivative definition in all situations and each definition has its own advantages and disadvantages. Fractional dynamical systems have seen excitable growth because its global property, i.e. the next state of the system depends not only on present state but also on all of its historical states. Therefore, the differential equations in different fractional derivative definitions have different type of solutions. Recently, many authors used methods to solve linear and non-linear differential equations as Predictor-Corrector method and Adomian decomposition [5]. In 2006, G.Jumair developed Riemann-Liouville derivative to avoid non-zero fractional derivative of constant functions this means that, it is possible to interpret different physically phenomena [6]. Ghosh et al., developed analytical method to solve linear system of fractional differential equations with Jumarie derivative [7]. In fact and without a great loss of generality, stability of dynamical systems play a pivotal role in many applications, whether in nonlinear ordinary differential equation (ODE) or FDS. This recently technique is tremendously important in plasma physics, so many researchers investigated numerous physical models. Up to the best of our knowledge, all studies of bifurication analysis in plasma physics was carried out on Kadomtsev-Petviashvili (KP) Eq. [8], KdV [9], and recently on NSE [10, 11] but no research was done for stability of STFKdV in plasma physics, so this is new investigation on this field. Motivated by above works, this paper consists of six sections organized as following: In section 2, we have defined some most used definitions of fractional derivative that is basic Riemann-Liouville fractional derivative (R-L), the version of the Italian mathematician Caputo and the modified of Riemann-Liouville definition called Jumarie fractional derivative. In the begin of section 3, the existence and uniqueness theorems and stability analysis of DS with fractional-order via Jumarie type of derivative have been described. Also we used Jumarie type fractional derivative to find the solutions of FDS by using Mittag-Leffler functions. In section 4, application of nonlinear FDS in dusty plasma by using the bifurcation theory of planar dynamical system is investigated. Finally, some discussion of results and conclusion are given in Secs. 5 and 6.

2 Basic Definitions of Fractional Derivatives

Definition 1. *[2, 3] The Riemann-Liouville definitin (R-L) is proposed as*

$$D_a^\alpha\, f(\theta) = \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{d\theta^n}\int_a^\theta (\theta-s)^{n-\alpha-1}\, f(s)\, ds \tag{2.1}$$

where $f(\theta):\mathbb{R}\to\mathbb{R}$ is a continuous function and one time integrable and $n-1\le\alpha<n$, with n positive and $\Gamma(n-\alpha)$ is the Gamma function.

This definition is applicable for continuous functions but non-differentiable and the fractional derivative of a constant function is non zero. While the Caputo definition is given for n-times differentiable functions, it is assumed that the n^{th} derivative exists. In addition, in Caputo definition the fractional derivative of a constant function is zero that is a benefit for some physical phenomena.

Definition 2. *[2, 3] Caputo definition is introduced as*

$${}^cD_a^\alpha f(\theta) = \int_a^\theta \frac{(\theta-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}\; f^{(n)}(s)\; ds, \tag{2.2}$$

where $f(\theta)\in C^n([a,b]),\quad f^{(n)}(\theta)\in L_1[a,b]$ and $n-1<\alpha\le n$ with positive n.

To overcome the non-zero fractional derivative of a constant function by Riemann-Liouville derivative, Jumarie modified the (R-L) formula of fractional derivative. Moreover, the differentiability condition required by Caputo definition is not required by Jumarie derivative. Important remark to mention that if the functions are not continuous at the origin, then Jumarie fractional derivative does not exist.

Definition 3. *[4, 6, 12] The modified fractional derivative of (R-L) of $f(\theta)$ is proposed as*

$${}^JD_a^\alpha f(\theta) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^\theta (\theta-s)^{-\alpha-1}\, f(s)\, ds, & \alpha<0, \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_a^\theta (\theta-s)^{-\alpha}\,[f(s)-f(a)]\, ds, & 0<\alpha<1, \\ \left[f^{\alpha-n}(\theta)\right]^{(n)}, & n\le\alpha<n+1\,, n\ge 1. \end{cases}$$

such that $f(\theta):\mathbb{R}\to\mathbb{R}$ be a continuous function.

For more details on the properties of fractional derivative, refereed to,[2, 3, 4, 13]. It is important to give a brief overview for the Mittag-Leffler function which has increased the attention of researchers because it mostly appears in the solution of fractional-order integral equations or fractional-order differential equations. It is a generalization function of the exponential function and it was introduced by the Swedish mathematician Magnus Gustaf Mittag-Leffler in (1903-1904). Due to its importance, it is given the call **"the Queen function of fractional calculus"**. The one parameter Mittag-Leffler function [3] is denoted by

$$E_\alpha(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(\alpha k+1)}, \ (\alpha > 0, \theta \in \mathbb{C}). \tag{2.3}$$

The two parameters function of the Mittag-Leffler [3] is introduced as

$$E_{\alpha,\omega}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(\alpha k+\omega)}, \ (\alpha > 0, \omega > 0), \tag{2.4}$$

with the widely known properties:

$$E_{\alpha,1}(\theta) = E_\alpha(\theta).$$

$$E_{1,1}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(k+1)} = e^\theta.$$

$$E_{\alpha,\omega}(\theta) = \theta E_{\alpha,\alpha+\omega}(\theta) + \frac{1}{\Gamma(\omega)}. \tag{2.5}$$

The Mittag-Leffler function with matrix variable A is defined by

$$E_\alpha(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+1)}, \ \alpha > 0,$$

where A is an arbitrary n^{th} order matrix over complex field [14]. A relationship between the Mittag-Leffler function and fractional Sine and Cosine given by Jumarie [7, 12].

$$E_\alpha(i\theta^\alpha) = cos_\alpha(\theta^\alpha) + i\ sin(\theta^\alpha), \tag{2.6}$$

$$cos_\alpha(\theta^\alpha) = \frac{E_\alpha(i\theta^\alpha) + E_\alpha(-i\theta^\alpha)}{2} = \sum_{k=0}^{\infty}(-1)^k \frac{\theta^{2k\alpha}}{\Gamma(1+2\alpha k)}, \tag{2.7}$$

$$sin_\alpha(\theta^\alpha) = \frac{E_\alpha(i\theta^\alpha) - E_\alpha(-i\theta^\alpha)}{2} = \sum_{k=0}^{\infty}(-1)^k \frac{\theta^{(2k+1)\alpha}}{\Gamma(1+2\alpha k)}, \tag{2.8}$$

$$E_\alpha((a \pm b)\theta^\alpha) = E_\alpha(a\theta^\alpha) \pm E_\alpha(b\theta^\alpha). \tag{2.9}$$

3 Stability and Solutions of Fractional Dynamical Systems

In this section, we prove existence and uniqueness theorems of the following FDS with fractional-order α via the modified Riemann-Liouville derivative:

$$^J D_\theta^\alpha X = AX, \ \ X(0) = X_0, \tag{3.1}$$

where A $\in L(\mathbb{R}^2)$, $X(\theta) = (x_1(\theta), x_2(\theta))^T$, $0 < \alpha < 1$ and $\theta \in [0, T]$. One more objective is investigation the stability of FDS because most of the mathematical models represented by fractional dynamical systems.

3.1 Existence and Uniqueness Theorems

The existence and uniqueness of the solution of linear system with fractional order with Riemann-Liouville type of derivative has been investigated in Ref. [15]. Here, we state and prove existence and uniqueness theorem of the linear system (3.1) via Jumarie type of derivative and we solve (3.1) in several cases.

Proposition 1. *Let $A \in L(\mathbb{R}^2)$ is the Jordan form which have eigenvalues β_1, β_2. in the following forms:*

$$\begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \begin{bmatrix} \gamma & -\mu \\ \mu & \gamma \end{bmatrix}, \begin{bmatrix} \beta & 0 \\ 1 & \beta \end{bmatrix},$$

where in the first matrix the eigenvalues are distinct or repeated when β_1 equal β_2, the second matrix has complex form of eigenvalues $\beta = \gamma \pm i\mu$ and the last non diognal matrix has repeated eigenvalues.

Theorem 1. *Suppose that $A \in L(\mathbb{R}^2)$ has distinct real eigenvalues. Then, given $x_0 \in \mathbb{R}^2, \exists\, t > 0$, system (3.1) has a unique solution defined on [0,t].* ***Proof***

$$^{J}D_\theta^\alpha X(\theta) = AX(\theta),\ A = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \tag{3.2}$$

Let $\{K_1, K_2\}$ be the distinct eigenvectors corresponding to the distinct eigenvalues $\{\beta_1, \beta_2\}$ so that $A\,K_j = \beta_j K_j,\ j = 1, 2$. If all eigenvalues are real and distinct, then the eigenvectors $\{K_1, K_2\}$ forms a basis of $\mathbb{R}^2$. Then, $B = diag[\beta_1, \beta_2] = FAF^{-1}$ where $F = (K_1, K_2)^T$. Define Y=FX. Then,

$$^{J}D^\alpha Y(\theta) = F\ ^{J}D^\alpha X(\theta)$$

$$= FAX(\theta)$$

$$= FAF^{-1}Y(\theta)$$

$$= BY(\theta).$$

Here, $Y(\theta) = (y_1(\theta), y_2(\theta))^T,\ \ X(\theta) = (x_1(\theta), x_2(\theta))^T\ \ , Y(0) = (y_1^{(0)}, y_2^{(0)})^T = Y_0 = FX(0) = F(x_1^{(0)}, x_2^{(0)})^T = FX_0$. Since B is diagonal, we can write

$$^{J}D^\alpha y_i(\theta) = \beta_i y_i(\theta),\ y_i(0) = (Y_i^{(0)}),\ i = 1, 2.$$

Here, $\beta_i y_i(\theta) = g_i(\theta, y_i) : \varphi_i \to \mathbb{R}$, $\varphi_i = [0, T] \times [(y_i^{(0)}) - \ell_i, (y_i^{(0)}) + \ell_i]$ for positive ℓ_i. $g_i(\theta, y_i)$ is locally Lipschitz continuous in the second variable [13]. Hence, there is a unique solution $y_i : [0, t_i] \to \mathbb{R}$ solving $^{J}D^\alpha y_i(\theta) = \beta_i y_i(\theta),\ y_i(0) = (y_i^{(0)})$, where $t_i = min\{T, \left(\frac{\ell_i \Gamma(\alpha+1)}{\|g_i\|_\infty}\right)^{1/\alpha}\},\ i = 1, 2$. Let $t = min\{t_1, t_2\}$, then $X(\theta) = F^{-1}Y(\theta)$ uniquely solves (3.1) with $\theta \in [0, t]$, for having $\left(\frac{\ell_i \Gamma(\alpha+1)}{\|g_i\|_\infty}\right)^{1/\alpha}$.

Theorem 2. *Consider the system*

$$^{J}D_\theta^\alpha X(\theta) = AX(\theta),\ \ X(0) = X_0, \tag{3.3}$$

where $\alpha \in (0, 1), \theta \in [0, T]$, $X(0) = (x_1(0), x_2(0))^T, X_0 = (x_1^{(0)}, x_2^{(0)})^T$ and

$$A = \begin{bmatrix} \beta & 0 \\ 1 & \beta \end{bmatrix}$$

is the Jordan matrix. Then, (3.1) has a unique solution defined on [0,t]. ***Proof:***
Substituting $X(\theta) = (x_1(\theta), x_2(\theta))^T$ into (3.3) and using the matrix A we obtain firstly,

$$^{J}D_\theta^\alpha x_1(\theta) = \beta x_1(\theta),$$

$$x_1(0) = x_1^{(0)}.$$

Here, $g_1(\theta, x_1) = \beta x_1$ is defined on $\varphi_1 = [0,T] \times [x_1(0) - \ell_1, x_1(0) + \ell_1]$, for ℓ_1 positive , g_1 is continuous and Lipschitz in the second variable. Hence, it has a unique solution $x_1(\theta), \theta \in [0, t_1]$ where $t_1 = min\{T, \left(\frac{\ell_1 \Gamma(\alpha+1)}{\|g_1\|_\infty}\right)^{1/\alpha}\}$.
Secondly,

$${}^J D_\theta^\alpha x_2(\theta) = x_1(\theta) + \beta x_2(\theta),$$

now $x_1(\theta)$ is a known function.
Here, $g_2(\theta, x_2) = x_1(\theta) + \beta x_2$ is defined on $\varphi_2 = [0,T] \times [x_2(0) - \ell_2, x_2(0) + \ell_2]$ for $\ell_2 > 0$, g_2 is continuous and Lipschitz in the second variable. Hence, it has a unique solution $x_2(\theta)$, $\theta \in [0, t_2]$, $t_2 = min\{T, \left(\frac{\ell_2 \Gamma(\alpha+1)}{\|g_2\|_\infty}\right)^{1/\alpha}\}$. Hence, $x_1(\theta)$ and $x_2(\theta)$ are known functions. Therefore, the system(3.3) has a unique solution on [0,t] where $t = min\{t_1, t_2\}$.

Theorem 3. ***(i)*** *Consider the system*

$${}^J D^\alpha X(\theta) = \begin{bmatrix} \gamma & -\mu \\ \mu & \gamma \end{bmatrix} X(\theta), \tag{3.4}$$

where $\gamma, \mu \in \mathbb{R}, X(0) = X_0, \theta \in [0,T]$ and $0 < \alpha < 1$.
Define $z(\theta) = x_1(\theta) + ix_2(\theta)$. Then,

$${}^J D^\alpha z(\theta) = \beta z, \quad z(0) = z_0 = x_1(0) + ix_2(0), \beta = \gamma + i\mu, \tag{3.5}$$

is equivalent to (3.3), it can be shown that (3.4) has a unique solution.
(ii) *Consider the system ${}^J D^\alpha X(\theta) = AX(\theta); X(0) = X_0$ and $\alpha \in (0,1)$, $A \in L(\mathbb{R}^2)$ has eigenvalues $\beta = \gamma \pm i\mu$, where γ and $\mu \in \mathbb{R}$. Then, there is a matrix F such that $A = FAF^{-1}$. i.e*

$$A = F \begin{bmatrix} \gamma & -\mu \\ \mu & \gamma \end{bmatrix} F^{-1}.$$

Define $Y(\theta) = F^{-1}X(\theta)$, then

$${}^J D^\alpha Y(\theta) = \begin{bmatrix} \gamma & -\mu \\ \mu & \gamma \end{bmatrix} Y(\theta), \quad Y(0) = Y_0. \tag{3.6}$$

Hence, from (i) equation (3.5) has a unique solution.

Theorem 4. *If the matrix A in the system (3.1) has any type of previous eigenvalues. Then, there is a unique solution to (3.1) defined on [0,t].*
Proof:
Since there is a basis of $\mathbb{R}^2$ as mentiend in Theorem 1, in which the system of fractional differential equations becomes

$${}^J D^\alpha Y(\theta) = BY(\theta), \quad Y(0) = Y_0\,.$$

Where B is composed of diagonal blocks as defined in Theorem 1, in this basis, the system decouples into simpler subsystems. Then, from Theorem 1 and 2 $\exists\, t > 0$ and a unique solution to (3.1) defined on [0,t] which can be obtained by simple formula $X(\theta) = F^{-1}Y(\theta)$, where F is defined in Theorem 1.

3.2 Stability of Dynamical System with Fractional-Order via Jumarie Type of Derivative

The first discussion for the stability of the FDS using Caputo and Riemann-Liouville form of derivative has been established in [16] and developed in [17, 18]. In the framework, the dynamical system (3.1) will be established and some stability conditions will be determined by the use of Jumarie type of derivative.

Theorem 5. *the unique solution of the system*

$$ {}^{J}D^{\alpha}_{\theta} X(\theta) = AX(\theta), \quad X(0) = X_0, \tag{3.7} $$

where A is an n^{th} order matrix over the complex field, X is a specified vector, $0 < \alpha < 1$ and $\theta \in [0, T]$ is $E_\alpha(\theta^\alpha A)\ X_0$.

Proof: we use Laplace transform of Jumarie type of derivative [12]

$$ L\{{}^{J}D^{\alpha} f(\theta)\} = s^{\alpha}\ F(s) - s^{\alpha-1} f(0), \quad (0 < \alpha < 1) \tag{3.8} $$

for a given initial condition $X(\theta_0) = X_0$. Let

$$ L\{X(\theta)\} = X(s)\ , L\{X(0)\} = X_0. $$

Then,

$$ s^{\alpha} X(s) - s^{\alpha-1} X_0 = AX(s), $$

$$ (s^{\alpha} I - A)X(s) = s^{\alpha-1} X_0 \tag{3.9} $$

Since,

$$ |s^{\alpha} I - A| = \prod_{i=1}^{n} (s^{\alpha} - \beta_i), $$

where β_i are the eigenvalues of A, $(s^{\alpha} I - A)$ is an invertable matrix. Thus, we obtain

$$ X(s) = s^{\alpha-1} (s^{\alpha} I - A)^{-1} X_0. \tag{3.10} $$

Inserting the formula

$$ (s^{\alpha} I - A)^{-1} = \sum_{k=0}^{\infty} s^{-\alpha k - \alpha} A^k, $$

into (3.10) and taking the inverse Laplace transform term by term (3.10) becomes

$$ \widetilde{X}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^{k\alpha} A^k}{\Gamma(\alpha k + 1)} X_0 $$

$$ = E_{\alpha}(\theta^{\alpha} A)\ X_0. \tag{3.11} $$

Definition 1. *The autonomous system (3.7) is said to be*
(i) *stable if and only if $\forall\ x_0, \exists\ A$ such that $\|X(\theta)\| \leq A\ , \forall\ \theta \geq 0$.*
(ii) *asymptotically stable if and only if $\lim_{\theta \to \infty}\ \|X(\theta)\| = 0$.*

Definition 2. *The point $X_{eq} = (x_{1,eq}, x_{2,eq}, ...)$ is an equilibrium point of a fractional differential system (3.7), if and only if ${}^{J}D^{\alpha} X_{eq} = 0$. Note, for the autonomous linear planar system the equilibrium point is the origin, that is $X_{eq} = (0, 0)^T$.*

The analysis of stability of the FDS is more complicated than that of classical differential equations. This is because that the fractional derivatives are non-local and have weakly singular kernels. Additionally, the behavior of the Mittag-Leffler function plays an important role in the study of the stability of the FDS. The following propositions explain the behavior of Mittag-Leffler function in different cases.

Proposition 2. *[14, 19] Let $0 < \alpha < 2$ and ω be an arbitrary complex number, then for an arbitrary integer number $r \geq 1$ the following hold*

$$E_{\alpha,\omega}(\theta) = \frac{1}{\alpha}\theta^{(1-\omega)/\alpha} exp(\theta^{1/\alpha}) - \sum_{k=1}^{r} \frac{1}{\Gamma(\omega - \alpha k)} \frac{1}{\theta^k} + \xi\left(|\theta|^{-(r+1)}\right),$$

with $|\theta| \to \infty, |arg(\theta)| \leq \frac{\alpha\,\pi}{2}$, and

$$E_{\alpha,\omega}(\theta) = -\sum_{k=1}^{r} \frac{1}{\Gamma(\omega - \alpha k)} \frac{1}{\theta^k} + \xi\left(|\theta|^{-(r+1)}\right),$$

with $|\theta| \to \infty, |arg(\theta)| > \frac{\alpha\,\pi}{2}$.

Proposition 3. *[16] The asymptotic for $E_\alpha(\beta\theta^\alpha)$ as θ reaches infinity, is introduced as follows:*
(i) *for $|arg(\beta)| \leq \alpha\pi/2, E_\alpha(\beta\theta) \sim \frac{1}{\alpha}e^{\beta^{1/\alpha}\theta}$,*
(ii) *for $|arg(\beta)| > \alpha\pi/2, E_\alpha(\beta\theta) \sim \frac{1}{-\beta,\Gamma(1-\alpha)}\theta^{-\alpha}$, which decays slowly to 0.*

Now, the characteristic equation of system (3.1) will be established and some stability conditions will be determined by using Jumarie type of derivative. Let $\Delta s = (Is^\alpha - A)$, then equation (3.9) can be written as

$$\Delta s X(s) = s^{\alpha-1}X_0. \tag{3.12}$$

The character equation of (3.12) is given by

$$det\,(\Delta s) = det(Is^\alpha - A) = 0. \tag{3.13}$$

Since (3.13) contains fractional order, it is difficult to be solved. However, we can determine the stability of system (3.7) by the eigenvalues of Δs when all the roots of the transcdental equation (3.13) lie in the open left half complex plane, i.e $Re(s) < 0$. Let $\beta = s^\alpha$, then

$$det(\beta I - A) = 0, \tag{3.14}$$

it follows that, $s = \beta^{1/\alpha}$ and

$$|arg(s)| = |arg(\beta^{1/\alpha})| \geq \frac{\pi}{2}.$$

Thus,

$$|arg(\beta)| \geq \frac{\alpha\pi}{2}. \tag{3.15}$$

All the characteristic roots of system (3.7) have a negative real part, which described by the stable and unstable areas in the Fig. 1 and means that the eigenvalues of this system lie in the complex plane except the sector bordered by the angles $\pm\frac{\alpha\pi}{2}$. The unstable area becomes larger as long as α increases and symmetric about the positive x-axis.

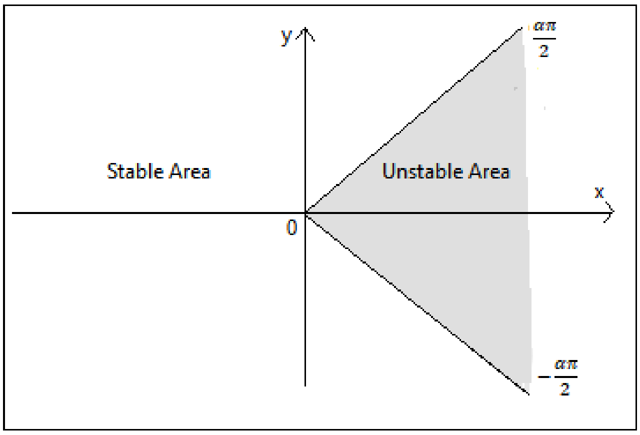

Figure 1: Stable and unstable regions of system (3.7).

Theorem 6. *If all the eigenvalues of the characteristic polynomials of Δs have a negative real part, then the equilibrium solution of system (3.7) is asymptotically stable.*

Theorem 7. *The autonomous system (3.7) is said to be*
(i) asymptotically stable *if and only if*

$$|arg(\sigma(A))| > \frac{\alpha\pi}{2}, \tag{3.16}$$

where $arg(\sigma(A))$ is the argument of all eigenvalues of A, giving in the formula $arg(\gamma \pm i\mu) = tan^{-1}(\frac{\mu}{\gamma})$. In this case, the stability is also called $\theta^{-\alpha}$ stability where the components of the state decay towards 0 like θ^{-1}.
(ii) stable *if and only if either it is asymptotically stable or the critical roots which satisfy $|arg(\sigma(A))| = \frac{\alpha\pi}{2}$, have geometric multiplicity one.*

Proof: The proof is straightforward by the proposition (2) and (3).

Special cases:

(i) for the case of pure imaginary $\beta_{1,2} = \pm i\mu$, we have, $|arg(\beta_{1,2})| = \pm\frac{\pi}{2} > \frac{\alpha\pi}{2}$, and equilibrium point of (3.7) is asymptotically stable if $\alpha \in (0,1)$ while it is stable if $\alpha \in (0,1]$.
(ii) the case of zero eigenvalues of A in (3.7) is not covered in theorems since the argument of zero in the complex plane can be arbitrary.

Now, we apply the obtained results with the characteristic method which has received much attention in recent years (see, for example, [7]). This method is employed to solve the fractional-order system (3.1). The analytical solutions are expressed in terms of Mittag-Leffler function and the generalized cosine and sine functions.

3.3 Characteristic Method of Solutions of FDS

Consider the system of linear fractional differential equations

$$\begin{cases} {}^{J}D^{\alpha}[x] = ax + by, \\ \\ {}^{J}D^{\alpha}[y] = cx + dy, \end{cases} \tag{3.17}$$

where $\alpha \in (0,1)$, a, b, c and d are constants, x and y are functions of θ.
Since ${}^{J}D^{\alpha}[x(\theta)] = \beta x(\theta)$ has a solution in the form $x(\theta) = AE_{\alpha}(\beta\theta^{\alpha})$, where A is arbitrary constant, we put

$x = AE_\alpha(\beta\theta^\alpha)$ and $y = BE_\alpha(\beta\theta^\alpha)$ in (3.17) and using
${}^J D^\alpha E_\alpha(\beta\theta^\alpha) = \beta E_\alpha(\beta\theta^\alpha)$, we have

$$\begin{cases} A(\beta - a) - bB = 0, \\ \\ -cA + (\beta - d)B = 0. \end{cases} \tag{3.18}$$

Eliminating A and B from (3.18) yields

$$\begin{vmatrix} a-\beta & b \\ c & d-\beta \end{vmatrix} = 0,$$

which gives the characteristic equation

$$\beta^2 - (a+d)\beta + (ad - bc) = 0, \tag{3.19}$$

with roots β_1 and β_2, we consider three main cases:

Case I : Suppose that β_1 *and* β_2 are real and distinct roots for the characteristic equation (3.19). Then, the solution of (3.17) is given by

$$x = x_1 + x_2 = A_1 E_\alpha(\beta_1\theta^\alpha) + A_2 E_\alpha(\beta_2\theta^\alpha), \;\; y = y_1 + y_2 = B_1 E_\alpha(\beta_1\theta^\alpha) + B_2 E_\alpha(\beta_2\theta^\alpha).$$

And the general solution of system (3.17) can be represented by

$$x = c_1 A_1 E_\alpha(\beta_1\theta^\alpha) + c_2 A_2 E_\alpha(\beta_2\theta^\alpha), \;\; y = c_1 B_1 E_\alpha(\beta_1\theta^\alpha) + c_2 B_2 E_\alpha(\beta_2\theta^\alpha),$$

where $c_1,\ c_2,\ A_1,\ A_2,\ B_1$ and B_2 are arbitrary constants.
This case arises in the following example.

Example 1. *Use the method of linear algebra to find the solutions of the following system:*

$$\begin{cases} {}^J D^\alpha x = -x - 2y, \\ \\ {}^J D^\alpha y = x - 4y, \end{cases} \tag{3.20}$$

subject to

$$0 < \alpha \leq 1,\ x(0) = 2 \text{ and } y(0) = 0.$$

Solution: *Let*

$$x = AE_\alpha(\beta\theta^\alpha) \;\text{ and }\; y = BE_\alpha(\beta\theta^\alpha),$$

be the solutions of system (3.20). Then, substituting x and y into the considered system gives

$$\begin{cases} (-1-\beta)A - 2B = 0, \\ \\ A + (-4-\beta)B = 0. \end{cases} \tag{3.21}$$

The corresponding characteristic equation is $\beta^2 + 5\beta + 6 = 0$, *from which we have* $\beta = -2, -3$. *Substituting* $\beta = -2$ *into (3.21), we get* $A = 1$*and* $B = \frac{1}{2}$. *Substituting* $\beta = -3$ *into (3.21) yields* $A = B = 1$. *Therefore, the solutions are*

$$x = E_\alpha(-2\theta^\alpha) + 0.5E_\alpha(-3\theta^\alpha)$$

$$y = E_\alpha(-2\theta^\alpha) + E_\alpha(-3\theta^\alpha).$$

Using the initial conditions leads to $c_1 = 1, c_2 = 2$. *Therefore, the required solutions are*

$$x = E_\alpha(-\theta^\alpha) + E_\alpha(-3\theta^\alpha),$$

$$y = E_\alpha(-\theta^\alpha) + 2E_\alpha(-3\theta^\alpha).$$

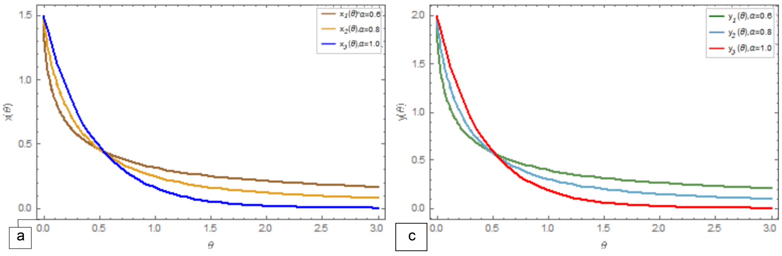

Figure 2: The solutions $x(\theta)$ and $y(\theta)$ for negative eigenvalues of the system (3.20) for different values of α

.

In Fig. 2, it is found that the curves of the solutions $x(\theta)$ and $y(\theta)$, within $\theta = [0, 3]$ at different values of $\alpha \in (0, 1]$, decline and tend to equilibrium point as θ increases. It is worth to mention that as long as α increases the declines go faster to equilibrium point. Therefore, the equilibrium point is asymptotically stable.

Case II : If the roots of the characteristic equation (3.19) are complex $\beta_{1,2} = \gamma \pm i\mu$. Then, the solutions can be written as following:

$$x = [E_\alpha(\gamma\theta^\alpha)] \left[(A_1 cos_\alpha(\mu\theta^\alpha) - A_2 sin_\alpha(\mu\theta^\alpha)) + i\left(A_2 cos_\alpha(\mu\theta^\alpha) + A_1 sin_\alpha(\mu\theta^\alpha)\right)\right].$$

$$y = [E_\alpha(\gamma\theta^\alpha)] \left[(B_1 cos_\alpha(\mu\theta^\alpha) - B_2 sin_\alpha(\mu\theta^\alpha)) + i\left(B_2 cos_\alpha(\mu\theta^\alpha) + B_1 sin_\alpha(\mu\theta^\alpha)\right)\right].$$

We can write x as linear combination of x_1 $(Re[x])$ and x_2 $(Im[x])$, Similar is done for y. Therefore, the general solution of (3.17) is represented as following:

$$x = [E_\alpha(\gamma\theta^\alpha)] \left[M((A_1 cos_\alpha(\mu\theta^\alpha) - A_2 sin_\alpha(\mu\theta^\alpha)) + N((A_2 cos_\alpha(\mu\theta^\alpha) + A_1 sin_\alpha(\mu\theta^\alpha))\right],$$

$$y = [E_\alpha(\gamma\theta^\alpha)] \left[M(B_1 cos_\alpha(\mu\theta^\alpha) - B_2 sin_\alpha(\mu\theta^\alpha)) + N((B_2 cos_\alpha(\mu\theta^\alpha) + B_1 sin_\alpha(\mu\theta^\alpha))\right].$$

Where M and N can be determined from the initial conditions.

Example 2. *Use the characteristic method to find the solutions of the following system:*

$$\begin{cases} {}^J D^\alpha x = y, \\ {}^J D^\alpha y = -4x, \end{cases} \tag{3.22}$$

subject to

$$0 < \alpha < 1,\ x(0) = 1 \text{ and } y(0) = 2.$$

Solution: *Let*

$$x = AE_\alpha(\beta\theta^\alpha) \text{ and } y = BE_\alpha(\beta\theta^\alpha),$$

be the solutions of system (3.22). Then substituting x, y into the given system leads to

$$\begin{cases} \beta\, A - B = 0, \\ 4A + \beta\, B = 0. \end{cases} \tag{3.23}$$

From the corresponding characteristic equation, we have $\beta = \pm 2i$. Substituting $\beta = 2i$ into (3.23) to obtain $A = 1$, $B = 2i$. Then, the solutions are

$$x = cos_\alpha(2\theta^\alpha) + isin_\alpha(2\theta^\alpha), \quad y = -2sin_\alpha(2\theta^\alpha) + 2icos_\alpha(2\theta^\alpha).$$

Therefore, the solutions can be written in the form

$$x = M cos_{\alpha}(2\theta^{\alpha}) + N sin_{\alpha}(2\theta^{\alpha}), \quad y = -2M sin_{\alpha}(2\theta^{\alpha}) + 2N\, cos_{\alpha}(2\theta^{\alpha}).$$

The initial conditions give us that $M = 1$ *and* $N = 1$, *so we get*

$$x = cos_{\alpha}(2\theta^{\alpha}) + sin_{\alpha}(2\theta^{\alpha}), \;\; y = -2sin_{\alpha}(2\theta^{\alpha}) + 2\, cos_{\alpha}(2\theta^{\alpha}).$$

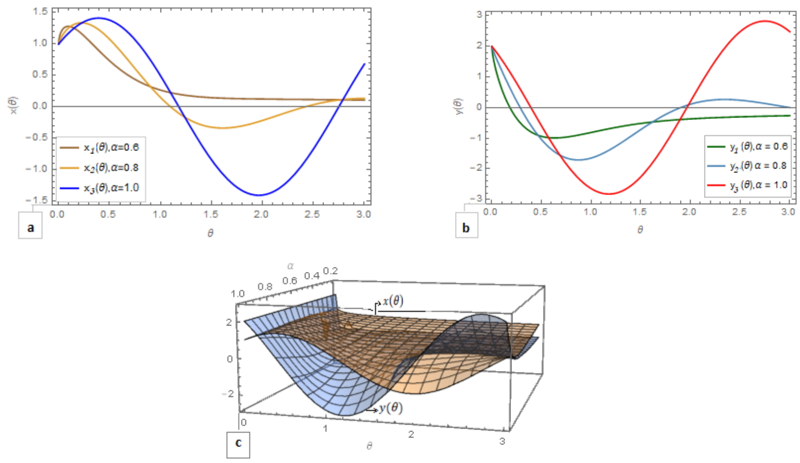

Figure 3: The particular solutions $x(\theta)$ and $y(\theta)$ of system (3.22) for pure imaginary eigenvalues with $\theta = [0, 3]$ for different values of α, in 2-dimensional (a, b) and 3-dimensional (c).

From Fig. 3, it is clear that the wave is oscillating as a periodic type and its amplitude increases as α increases. graphs (a,b) exhibit that as α increases up to one, the wave tends away from equilibrium point (0,0). This result successfully agrees with the stability classification of the equilibrium point of special cases.

Case III : If the roots of the characteristic equation (3.19) are real and repeated $\beta_1 = \beta_2 = \beta$. Then, the solutions of system (3.17) are given by

$$x = x_1 + x_2 = AE_{\alpha}(\beta\theta^{\alpha}) + (A_1\theta^{\alpha} + A_2)E_{\alpha}(\beta\theta^{\alpha}), \quad y = y_1 + y_2 = BE_{\alpha}(\beta\theta^{\alpha}) + (B_1\theta^{\alpha} + B_2)E_{\alpha}(\beta\theta^{\alpha}).$$

The general solutions are,

$$x = c_1 AE_{\alpha}(\beta\theta^{\alpha}) + c_2(A_1\theta^{\alpha} + A_2)E_{\alpha}(\beta\theta^{\alpha}), \;\; y = c_1 BE_{\alpha}(\beta\theta^{\alpha}) + c_2(B_1\theta^{\alpha} + B_2)E_{\alpha}(\beta\theta^{\alpha}),$$

where c_1 and c_2 can be determined from the initial conditions and $A,\ A_1,\ A_2,\ B,\ B_1$ and B_2 are arbitrary constants.

Example 3. *Use the characteristic method to find the solutions of the following system:*

$$\begin{cases} {}^{J}D^{\alpha}x = 2x - y, \\ \\ {}^{J}D^{\alpha}y = x + 4y, \end{cases} \tag{3.24}$$

subject to

$$0 < \alpha \leq 1, \; x(0) = 2 \; and \; y(0) = 1.$$

Solution: *Let* $x = AE_{\alpha}(\beta\theta^{\alpha})$ *and* $y = BE_{\alpha}(\beta\theta^{\alpha})$, *be the solutions of (3.24), then substituting* x *and* y *into (3.24) gives*

$$\begin{cases} (\beta - 2)A + B = 0, \\ \\ -A + (\beta - 4)\, B = 0. \end{cases} \tag{3.25}$$

From the corresponding characteristic equation, we have $\beta = 3, 3$. *Substituting* $\beta = 3$ *into (3.25) yields* $A = -B$. *Taking* $A = -1$ *leads to* $B = 1$ *so, the first solution is in the form*

$$x_1 = -E_\alpha(3\theta^\alpha)\ ,\ y_1 = E_\alpha(3\theta^\alpha).$$

The second solution will be

$$x_2 = (A_1\theta^\alpha + A_2)\left[E_\alpha(3\theta^\alpha)\right],\ \ y_2 = (B_1\theta^\alpha + B_2)\left[E_\alpha(3\theta^\alpha)\right].$$

Differentiating x_2 *and* y_2 α^{th} *order and applying the product rule for Jumarie type of derivative yield*

$${}^J D^\alpha x = 3(A_1\theta^\alpha + A_2)E_\alpha(3\theta^\alpha) + \Gamma(1+\alpha)A_1E_\alpha(3\theta^\alpha).$$

$${}^J D^\alpha y = 3(B_1\theta^\alpha + B_2)E_\alpha(3\theta^\alpha) + \Gamma(1+\alpha)B_1E_\alpha(3\theta^\alpha).$$

Substituting ${}^J D^\alpha x$ *and* ${}^J D^\alpha y$ *into (3.24) and Comparing the coefficients of* θ, *simplifying gives*

$$A_1 = -B_1,$$

$$A_2 + A_1\Gamma(1+\alpha) = -B_2.$$

For simple non-zero values, we choose $A_1 = -1, B_1 = 1$ *and* $B_2 = 0$ *then,* $A_2 = \Gamma(1+\alpha)$. *Using the initial conditions, we find* $c_1 = 1$ *and* $c_2 = \frac{3}{\Gamma(1+\alpha)}$. *Hence, the solutions are*

$$x = \left[2 - \frac{3\theta^\alpha}{\Gamma(1+\alpha)}\right]E_\alpha(3\theta^\alpha),\ \ y = \left[1 + \frac{3\theta^\alpha}{\Gamma(1+\alpha)}\right]E_\alpha(3\theta^\alpha).$$

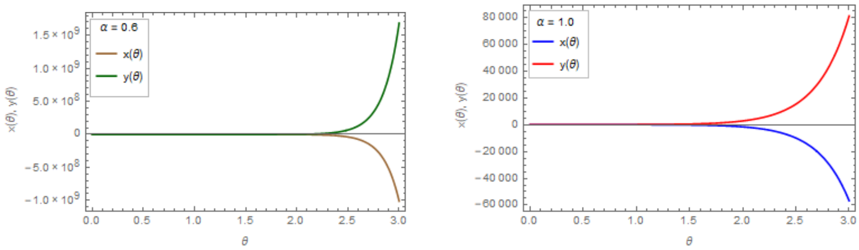

Figure 4: The solutions of system (3.24) for $\alpha = 0.8$ and 1.0 .

In the Fig. 4, we noticed that as α increases, the amplitude of $x(\theta)$ tends to $-\infty$ and the amplitude of $y(\theta)$ tends to ∞. Then, the equilibrium point $X_{eq} = (0, 0)$ is unstable.

For the special cases we have the following example:

Example 4. *Use the characteristic method to find the solutions of the following system:*

$$\begin{cases} {}^J D^\alpha x = -3y, \\ \\ {}^J D^\alpha y = 3y, \end{cases} \tag{3.26}$$

subject to

$$0 < \alpha \leq 1, x(0) = 1 \text{ and } y(0) = -1.$$

Solution: *The solutions of the given system are*

$$x(\theta) = E_\alpha(3\theta^\alpha),\ \ y(\theta) = -E_\alpha(3\theta^\alpha).$$

Here $\beta = 0, 3$. *From the Fig. 5, we can conclude that the equilibrium point is unstable.*

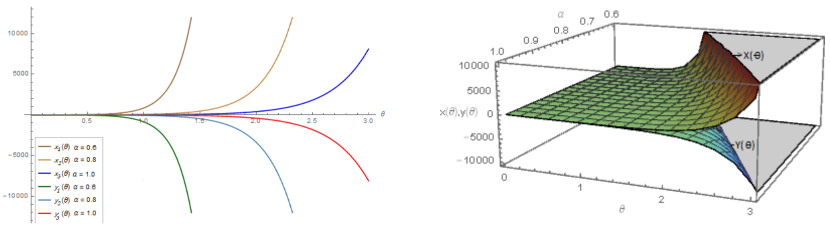

Figure 5: The solutions of system (3.26) for various values of α.

4 Applications of Nonlinear Fractional Dynamical System in Dusty Plasma

Consider an unmagnetized dusty plasma consists of isothermal electrons and ions, hot dust, cold dust grain. In equilibruim, the condition of charge neutrality is $N_{i0} = N_{c0}Z_c + N_{h0}Z_h + N_{e0}$, where N_{e0}, N_{h0}, N_{c0} and N_{i0} are the unperturbated number density of species. The charge number for negatively charged hot (cold) dust is Z_h (Z_c) . The nonlinear fluid model is described by the following one dimensional system of continuity, motion for cold (hot) dust and Poisson [20].

$$\frac{\partial N_c}{\partial t} + \frac{\partial (N_c V_c)}{\partial x} = 0 \tag{4.1}$$

$$\frac{\partial V_c}{\partial t} + V_c \frac{\partial V_c}{\partial x} = \gamma_c \frac{\partial \varphi}{\partial x}, \tag{4.2}$$

when $(T_e, T_i \gg T_h)$ for adiabatic hot grains, similar equations are

$$\frac{\partial N_h}{\partial t} + \frac{\partial (N_h V_h)}{\partial x} = 0, \tag{4.3}$$

$$\frac{\partial V_h}{\partial t} + V_h \frac{\partial V_h}{\partial x} + \frac{1}{M_h N_h}\frac{\partial P_h}{\partial x} = \gamma_h \frac{\partial \varphi}{\partial x}, \tag{4.4}$$

where

$$P_h = P_{ho}(\frac{N_h}{N_{h0}})^{\rho}, \tag{4.5}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e (Z_c N_c + Z_h N_h + N_e - N_i), \tag{4.6}$$

where V_c (V_h) represents the velocity of cold (hot) dust grain, M_h (M_c) is the mass of hot (cold) grain, φ is the electrostatic potential of plasma medium, $P_{h0} = N_{h0}T_h$ and $\gamma_c = \frac{eZ_c}{M_c}$, $\gamma_h = \frac{eZ_h}{M_h}$ and $\rho = 3$ in one dimensional fluid. The temperature of electron (ions) is T_e (T_i), T_h (T_c) is the temperature of hot (cold)grains. The density N_e and N_i of electrons and ions obey the Maxwell Boltzmann distribution as,

$$N_e = N_{e0} \exp(\frac{e\varphi}{T_e}), \tag{4.7}$$

$$N_i = N_{i0} \exp(\frac{-e\varphi}{T_i}). \tag{4.8}$$

For formulation of KdV equation, the reductive perturbation method consider,

$$\xi = \varepsilon^{\frac{1}{2}} \left(x - V_{ph} t \right), \tau = \varepsilon^{\frac{3}{2}} t, \tag{4.9}$$

where the strength of nonlinearity is ε and V_{ph} for wave speed propagation. Eqs.(4.1)-(4.6) are expanded as:

$$\begin{pmatrix} N_c \\ V_c \\ N_h \\ V_h \\ \varphi \end{pmatrix} = \begin{pmatrix} N_{c0} \\ 0 \\ N_{h0} \\ 0 \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} N_{c1} \\ V_{c1} \\ N_{h1} \\ V_{h1} \\ \varphi_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} N_{c2} \\ V_{c2} \\ N_{h2} \\ V_{h2} \\ \varphi_2 \end{pmatrix} + \cdots \tag{4.10}$$

The boundary conditions for this system are $|\xi| \to \infty, N_c = N_{c0}, N_h = N_{h0}, V_c = N_h = \varphi = 0$. Put Eqs.(4.7)-(4.10) in Eqs.(4.1)-(4.6) and equating the coefficients of like power of ε .
The KdV equation is obtained as Ref. [20].

$$\frac{\partial \varphi_1}{\partial \tau} + C\,\varphi_1 \frac{\partial \varphi_1}{\partial \xi} + D\,\frac{\partial^3 \phi_1}{\partial \xi^3} = 0, \tag{4.11}$$

where the dispersion equation is,

$$3N_{c0}P_{h0}T_eT_iZ_c\gamma_c + V_{ph}^4 eM_hN_{h0}\left[(N_{e0}T_e + N_{e0}T_i + N_{c0}T_eZ_c) + N_{h0}T_eZ_h\right] +$$
$$V_{ph}^2\left[-3eP_{h0}(N_{e0}(T_e+T_i) + T_e(N_{c0}Z_c + N_{h0}Z_h)) - M_hN_{h0}T_eT_i(N_{c0}Z_c\gamma_c + N_{h0}Z_h\gamma_h)\right] = 0,$$

and the coefficients of terms in KdV equation are,

$$C = \frac{1}{2T_e^2T_i^2V_{ph}\gamma_1(N_{c0}Z_c\gamma_1^2\gamma_c + M_h^2N_{h0}^3V_{ph}^4Z_h\gamma_h)}$$
$$\Big[-3M_h^2N_{h0}^3Z_h\gamma_h^2T_e^2T_i^2V_{ph}^4(P_{h0} + M_hN_{h0}V_{ph}^2) - 3\gamma_1^3N_{c0}T_e^2T_i^2Z_c\gamma_c^2 +$$
$$\gamma_1^3\left(e^2V_{ph}^4\left(N_{e0}T_e^2 - N_{e0}T_i^2 + N_{c0}T_e^2Z_c + N_{h0}T_e^2Z_h\right)\right)\Big]$$

$$D = \frac{V_{ph}^3\gamma_1^2}{8e\pi\left(N_{c0}Z_c\gamma_1^2\gamma_c + M_h^2N_{h0}^3V_{ph}^4Z_h\gamma_h\right)}, \quad \gamma_1 = M_hN_{h0}V_{ph}^2 - 3P_{h0}$$

4.1 Planar Dynamical System for Space-Time Fractional KdV Equation

The KdV Eq.(4.11) is converted to the space-time fractional KdV equation (STFKdV) by using the Agrawal technique[21]-[23]

$$^J D_\tau^\alpha \varphi_1(\xi,\tau) + C\,\varphi_1(\xi,\tau)\,^J D_\xi^\lambda \varphi_1(\xi,\tau) + D\ ^J D_\xi^{\lambda\lambda\lambda}\varphi_1(\xi,\tau) = 0. \tag{4.12}$$

The STFKdV equation is transformed to ODE by introduce a new variable[20]

$$\eta = \frac{\xi^\lambda}{\Gamma(\lambda+1)} - \frac{(\tau\delta)^\alpha}{\Gamma(\alpha+1)}, \tag{4.13}$$

where δ defined as a modification parameter in the transformation law, and we use some properties of JRL fractional derivative.

$$^J D_y^\alpha y^\delta = \frac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)} y^{(\delta-\alpha)},\ ^J D_y^\alpha f(u(y)) = \ ^J D_y^\alpha u(y)(\frac{df}{du}).$$

Let $\varphi_1(\xi,\tau) = \varphi(\eta)$, so we get the ODE

$$-\delta^\alpha \frac{d\varphi(\eta)}{d\eta} + C\,\varphi(\eta)\,\frac{d\varphi(\eta)}{d\eta} + D\,\frac{d^3\varphi(\eta)}{d\eta^3} = 0. \tag{4.14}$$

By integrate Eq.(4.14) with zero constant due to satisfy the boundary conditions, we get

$$-\delta^\alpha \varphi(\eta) + C\,\varphi^2(\eta) + D\,\frac{d^2\varphi(\eta)}{d\eta^2} = 0. \tag{4.15}$$

The Eq.(4.15) is equal to autonomous planar dynamical system (DS):

$$\begin{cases} \frac{d\varphi}{d\eta} = Z \\ \frac{dZ}{d\eta} = \frac{\delta\,\varphi}{D} - \frac{C}{D}\,\varphi^2 \end{cases}. \tag{4.16}$$

We investigate the bifurcations of phase portraits of system (4.16) in the (φ, Z) with α and δ are changed. Bifurcation analysis of dynamic system having important role in our research. This importance is the fact that we can essentially draw the phase portrait which describe the behavior of different solutions without solving the system.

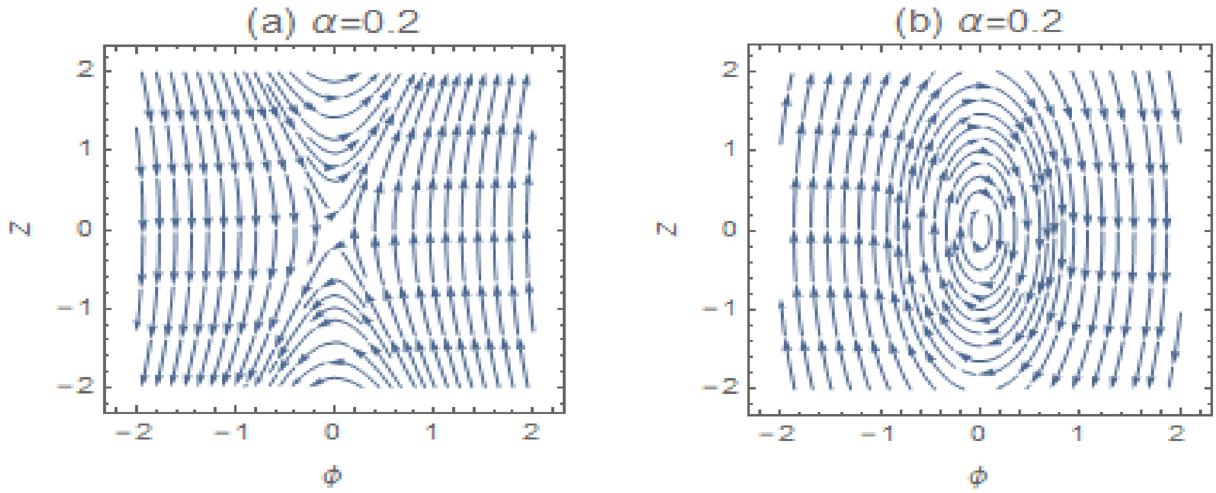

Figure: 6 Phase portrait for dynamical system (4.16) with different values of α at $\delta = 0.6, \beta = 1$; .

4.2 Bifurcations

We had the Jacobian matrix to linearizated the system (4.16) as

$$m = \begin{pmatrix} 0 & 1 \\ \frac{\delta^\alpha}{D} - \frac{2C\varphi}{D} & 0 \end{pmatrix}.$$

The system (4.16) has two equilibrium points at $q_1(0,0)$ and $q_2(\frac{\delta^\alpha}{C}, 0)$. The eigenvalues of Jacobian matric m at $q_1(0,0)$ is $\lambda_1 = \pm\sqrt{\frac{\delta^\alpha}{D}}$ are distinct real, where $\frac{\delta^\alpha}{D}$ be insure positive, so we classify the equilibrium point q_1 as unstable saddle point, see Fig.(6-a). Eigenvalues of m at $q_2(\frac{\delta^\alpha}{C}, 0)$ is $\lambda_2 = \pm\, i\,\sqrt{\frac{\delta^\alpha}{D}}$, so we classify the equilibrium point q_2 as stable center, so that $\varphi(\eta)$ becomes periodic as exhibited in Fig. (6-b). Trajectories are closed curves that are known as homoclinic orbits as shown in Figs.(7). in Fig.(7-a,b), there are two points in the phase portrait correspond to, a saddle point at $q_1(0,0)$ and other to a center point at $q_2(\frac{\delta^\alpha}{C}, 0)$. We concluded that, we have two types of solutions for system (4.16), periodic and solitary solutions due to the phase portrait of a dynamical system. A solitary wave solution analogous to the homoclinic orbit at an equilibrium point, also the phase portrait has a family of periodic orbits about an equilibrium point of the system, then the system has a family of periodic wave solutions corresponding to the family of periodic orbits about that point [24].

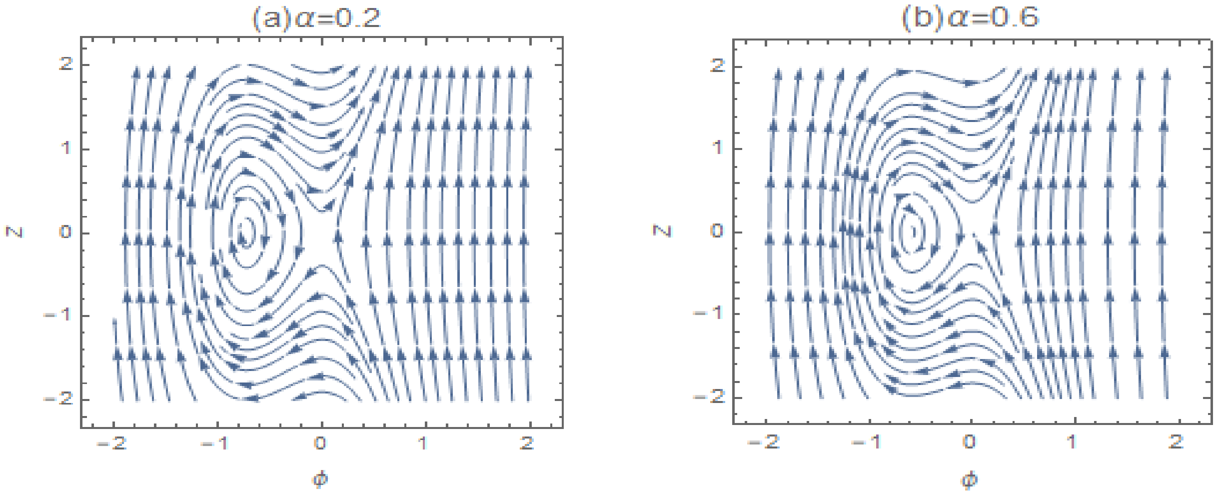

Figure: 7 Phase portrait for homoclinic orbits with different values of α at $\delta = 0.6, \beta = 1$; .

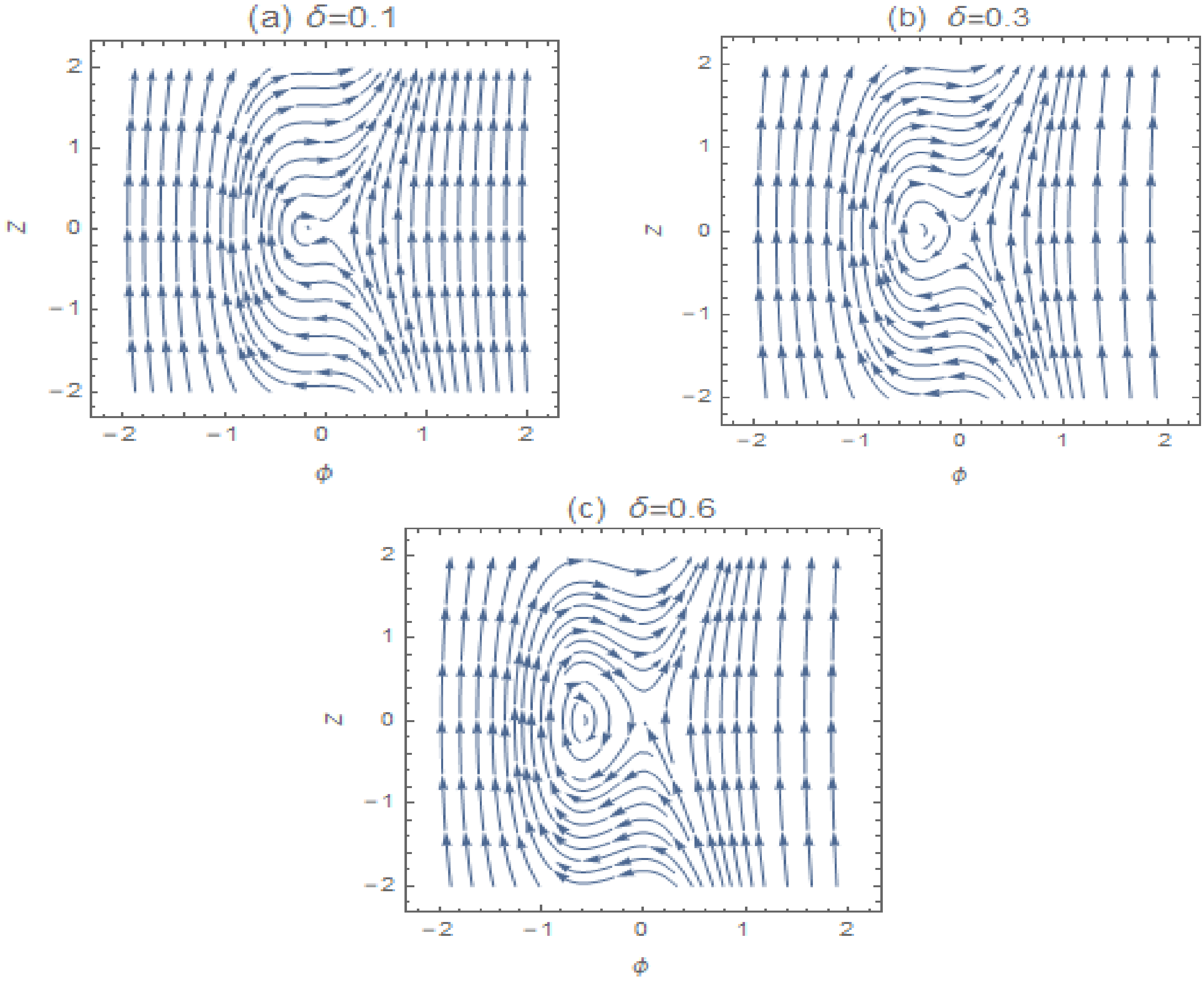

Figure: 8 Phase portrait for homoclinic orbits with different values of δ at $\alpha = 0.6, \beta = 1$; .

5 Results and Discussion

The nonlinear STFKdV equation is converted to nonlinear D.S. in plasma physics that containing electrons, ions, cold and hot dust grains have been examined. We investigated the influence of two parameters; space-time fractional α and the modification parameter δ on the behavior of solutions. Our numerical studies and all phase portraits of our system have been carried using the parameters of Saturn F-ring's i.e., the mass of dust grains are $M_c = M_h = 10^{12} M_i$, dust charges are $Z_h = 1000, Z_c = 10$, the equilibrium densities are $N_{c0} = 8\, cm^{-3}, N_{h0} = 6\, cm^{-3}, N_{e0} = 10\, cm^{-3}$, and the temperature of hot grains, electrons and ions are $T_h = 0.05\ eV$, $T_e = T_i = 1\ eV$, respectively, as given in (Salim et al., 2015 [25]; Akhtar et al., 2007 [26]; El-Shewy et al., 2011 [27]). We achieved a solitary and periodic waves by the bifurcation analysis. The parameters $\delta = 0.6$, $\alpha = 0.2$ give the saddle phase portrait in Fig. (6-a) and center in Fig.(6-b). Furthermore there is one homoclinic orbit to q_1 enclosing one center q_2 at $\alpha = 0.2$ as Fig.(7-a) and at $\alpha = 0.6$ as Fig.(7-b). It is observed that from these graphs that as α increases the amplitude of solitary wave decrease. In Fig.(7-a,b) as the parameter of fractional α increases, the number of periodic solutions around second equilibrium point decrease in same interval. So that the waves become rarefaction and this observation is in a good agrement with results in Ref.[20] where the amplitude of soliton wave decreases and width increases as α increases . In Figs.(8) at $\alpha = 0.6$, it is observed that from these figures as δ increases, the envelope solitary wave reverses its behavior from rarefactive to compressive wave. This effect of modification parameter is congruent with the observation in Ref.[20] where the width of wave decreases and the amplitude increases as δ increases .

6 Conclusions

The stability, the existence and uniqueness of fractional linear systems are an active research area nowadays. Here we presented the conditions under which the solutions of the proposed systems exist and unique. The solutions of some FDS are obtained by linear algebra method. The general solutions are obtained in terms of Mittag-Leffler function which helps us to understand the long behaviour of the solutions. The behaviour of solutions of the given systems are studied in the neighbourhood of the equilibrium point (original) JRL. Moreover it seems to be the best among others because this operator can be used to model some natural phenomena and it does not require differentiable functions. Finally we introduced a very important application of the effect of fractional parameters in astrophysical plasmas. The results from this work are expected to contribute to in-depth understanding of nonlinear waves that may appear in the interstellar regions of Saturn.

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