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Some Statistical Approximation based on Post-Widder operators

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Abstract:

In the present paper, we study the convergence of real and modified Post-Widder operators in C, which is known as the extension of approximation of these operators from the real axis in the complex plane. In this direction, we also investigate error estimation in simultaneous approximation and a Voronovskaya-type asymptotic formula.

Key Words: Real and complex Post-Widder type operator, over convergence phenomenon, approximation estimate, Voronovskaya-type result, exact error estimation.

AMS Subject Classifications: 41A25, 41A30

Introduction

In order to achieve better approximation in the recent years, convergence properties of linear positive operators were studied by many mathematician, some of them are listed as [1], [5], [6], [11], [12] and [14] etc. Direct, inverse and saturation results on these linear positive operators can be seen in [8], [9] and [10].

In the present paper, we discuss the approximation properties of complex Post-Widder operators.

For all $n \in N,\ x > 0,$ Post-Widder operators are defined by [15] as follows

$$P_n(f, x) = \frac{1}{n!}\left(\frac{n}{x}\right)^n \int_0^\infty t^n e^{-\frac{nt}{x}} f(t)dt, \quad f \in C[0, \infty). \tag{1.1}$$

These operators preserve constant functions only.

In [13], author has given modified form of Post-Widder operators as

$$P_n(f, x) = \frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n \int_0^\infty t^{n-1} e^{-\frac{nt}{x}} f(t)dt, \quad f \in C[0, \infty). \tag{1.2}$$

Operators (1.2) preserve constant as well as linear functions.

Now, it is obvious that

$$P_{n+1}(f, x) = \frac{1}{n!}\left(\frac{n+1}{x}\right)^{n+1} \int_0^\infty t^n e^{-\frac{(n+!)t}{x}} f(t)dt, \quad f \in C[0, \infty).$$

Now by changing the form from discrete parameter *n* to a continuous parameter $k \geq 1$

$$\overline{P}_k(f, x) = \frac{1}{\Gamma(k+1)}\left(\frac{k+1}{x}\right)^{k+1}\int_0^\infty t^k e^{-\frac{(k+!)t}{x}} f(t)dt, \quad f\in C[0, \infty).$$

Changing the variable $t = xv$

$$\overline{P}_k(f, x) = \frac{1}{\Gamma(k+1)}(k + 1)^{k+1}\int_0^\infty e^{-(k+1)v} v^k f(xv)dv. \qquad (1.3)$$

Following [4] and denoting $e_n(x) = x^n,\ n = 0, 1, 2$, we have

$$\overline{P}_k\left(e_0, x\right) = 1, \overline{P}_k\left(e_1, x\right) = x,$$
$$\overline{P}_k\left(e_2, x\right) = x^2 + \frac{x^2}{k+1}, \overline{P}_k\left(\left(e_1 - x\right)^2, x\right) = \frac{x^2}{k+1}, \text{ and}$$
$$\overline{P}_k\left(e_s, x\right) = \frac{(k+1)...(k+s)}{(k+1)^s}x^s \text{ for all } k \geq 1,\ s\in N. \qquad (1.4)$$

The procedure of extension of approximation properties of the linear and positive operators from the real axis in the complex plane is known as over convergence phenomenon and it was extensively studied by some authors e.g. [2] and [7].

Present paper deals with the approximation properties of Post-Widder type complex operators studied in (1.2).

Observations in Complex Plane

The first method for constructing complex type Post-Widder operators is according to Anastassiou-Gal [3], for which, the complex convolution type operators for $|z| \leq 1$, are given as

$$\overline{M}_k(f, z) = \frac{k^{k+1}}{\Gamma(k+1)}\int_0^\infty e^{-ku}u^k f\left(ze^{\frac{iu}{k}}\right)du.$$

Now, we lead to following estimates $\left|\overline{M}_k(f, z) - f(z)\right| \leq 3\omega_1\left(f, \frac{1}{k}\right)$.

In another method for construction of these operators, we replace *x* by *z* directly in (1.3).

Now for all $|z| \leq L$, we have

$$\left|\overline{P}_k(f, z) - f(z)\right| \leq \omega_1\left(f, \frac{1}{k}\right). \left(1 + kL. \frac{(k+1)^{k+1}}{\Gamma k}\int_0^\infty e^{-(k+1)v}v^k|v - 1|dv\right)$$
$$\leq \omega_1\left(f, \frac{1}{k}\right). (1 + k. LC),$$

where $\omega_1(f, \delta)$ is the modulus of continuity of *f* on C.

By using different method and assuming *f* as an entire function, we prove quantitative estimates for $\left|\overline{P}_k(f, z) - f(z)\right|$. We also find a Voronovskaya type asymptotic formula.

Basic Results

In this section we obtain some lemmas which are useful for proofs of main theorems.

Lemma1: For every $s\in N$ and $k \geq 1$, we get

$$\prod_{r=1}^{s}\left(\frac{k+r}{k+1}\right) - 1 \leq 2\frac{s!}{k+1}.$$

Proof: Taking $s\in N$ anda fixed arbitrary $k \geq 1$, we apply mathematical induction.

Now for $s = 1$, we have$0 \leq \frac{2}{k+1}$.

Let the inequality is valid for *s*, we proceed to prove it for (*s+1*) too.

Infect we have

$$\prod_{r=1}^{s+1}\left(\frac{k+r}{k+1}\right) - 1 = \frac{k+s+1}{k+1}\left[\prod_{r=1}^{s}\left(\frac{k+r}{k+1}\right) - 1\right] + \left(\frac{k+s+1}{k+1} - 1\right)$$

$$\leq \frac{k+s+1}{k+1}.\, 2\frac{s!}{k+1} + \frac{s}{k+1}$$

$$= \left(1 + \frac{s}{k+1}\right).\, 2.\frac{s!}{k+1} + \frac{s}{k+1}.$$

And $\left(1 + \frac{s}{k+1}\right).\, 2.\frac{s!}{k+1} + \frac{s}{k+1} \leq 2\frac{(s+1)!}{k+1}$, for all $s\in N$.

Therefore above inequality leads to as $s\leq(2s + 2).\, s! - \left(1 + \frac{s}{k+1}\right).\, 2.\, s!$.

After simple computation, for all $s\in N$ and $k \geq 1$, we get

$$1 \leq \frac{2k}{k+1}.\,(s!).$$

Hence this is the proof.

Main Results

Theorem 4.1: Taking *f* as an entire function and $f(z) = \sum_{s=0}^{\infty} c_s z^s$, for all $z\in C$, such that there exists $L > 0$ and $B\in(0,\ 1)$ with the property $\left|c_s\right| \leq L\frac{B^s}{s!}$, for all $s = 0,\ 1,\ 2,\ \ldots$ (means $|f(z)| \leq Le^{B|z|}$, for all $z\in C$.). Let $1 \leq j \leq 1/B$, then for all $k \geq 1$ and$|z| \leq j$, $\overline{P}_k(f,\ z)$ is analytic.

$$\overline{P}_k(f,\ z) = \sum_{s=0}^{\infty} c_s \overline{P}_k\left(e_s,\ z\right) \text{ and } \left|\overline{P}_k(f,\ z) - f(z)\right| \leq \frac{2L}{1-Bj}.\frac{1}{k+1}.$$

Proof: According to $|f(z)| \leq Le^{B|z|}$ and $B|z| < Bj < 1 < k + 1$, we have

$$\left|\overline{P}_k(f,\ z)\right| \leq L\frac{(k+1)^{k+1}}{\Gamma(k+1)}\int_0^{\infty} e^{-(k+1)v} v^k e^{Bv|z|} dv$$

$$= L\frac{(k+1)^{k+1}}{\Gamma(k+1)}\int_0^{\infty} e^{-v((k+1)-B|z|)} dv < \infty,\ |z| \leq j,$$

which implies that $\overline{P}_k(f,\ z)$ is analytic for $|z| \le j$.

Now we can have, $\overline{P}_k(f,\ z) = \frac{(k+1)^{k+1}}{\Gamma(k+1)}\int\limits_0^\infty e^{-(k+1)v}v^k\left(\sum\limits_{s=0}^\infty c_s(vz)^s\right)dv.$

Now dealing with the infinite sum, we get

$$\overline{P}_k(f,\ z) = \sum_{s=0}^\infty c_s.\frac{(k+1)^{k+1}}{\Gamma(k+1)}\int\limits_0^\infty e^{-(k+1)v}v^k(vz)^s dv = \sum_{s=0}^\infty c_s.\overline{P}_k\left(e_s,\ z\right).$$

Applying Fubini's type sufficient condition for commutatively, we get

$$\int\limits_0^\infty e^{-(k+1)v}v^k\left(\sum_{s=0}^\infty \left|c_s\right|(|vz|)^s\right)dv \le L\int\limits_0^\infty e^{-(k+1)v}v^k\left(\sum_{s=0}^\infty \frac{B^s}{s!}|vz|^s\right)dv$$

$$\le L\int\limits_0^\infty e^{-(k+1)v}v^k e^{B|z|v}dv \le L\int\limits_0^{+\infty} e^{-v((k+1)-B|z|)}v^k dv < \infty,$$

For all $|z| \le j,\ k \ge 1$and $jB < 1$.

According to inequality (1.4), we have

$$\left|\overline{P}_k(f,\ z) - f(z)\right| \le \sum_{s=0}^\infty \left|c_s\right|\left|\overline{P}_k\left(e_s,\ z\right) - e_s(z)\right| \le 2L\sum_{s=1}^\infty \frac{B^s}{s!}|z|^s\left|\prod_{r=1}^s\left(\frac{k+r}{k+1}\right) - 1\right|$$

$$\le 2L\sum_{s=1}^\infty \frac{1}{k+1}(Bj)^s \le \frac{2L}{1-Bj}.\frac{1}{(k+1)}.$$

Hence, it is proved.

Now we derive the quantitative estimate for Voronovskaya type result for complex Post-Widder operators, for which this type of result for the operators (1.3) is given as

$$(k\ +\ 1)\left[\overline{P}_k(f,\ x) - f(x) - \frac{x^2}{2(k+1)}f''(x)\right] = 0,\ \ x\in(0,\ \ +\ \infty).$$

Theorem 4.2: Let f is an entire function means $f(z) = \sum\limits_{s=0}^\infty c_s z^s$ for all $z\in C$, such that there exists $L\ >\ 0$ and $B\in(0,\ 1)$ with the property$\left|c_s\right| \le L\frac{B^s}{s!}$, for all $s\ =\ 0,\ 1,\ 2,\ \ldots$ (means$|f(z)| \le Le^{B|z|}$, for all$z\in C$). Let $1\ \le j\ \le\ 1/B$, then for all $k\ \ge\ 1$ and$|z| \le j$, we have the following estimates

$$\left|\overline{P}_k(f,\ z) - f(z) - \frac{z^2}{2(k+1)}f''(z)\right| \le \frac{2L}{1-Bj}.\frac{1}{(k+1)^2}.$$

Proof: It is obvious that for all $k\ \ge\ 1$ and $|z| \le j$, we write

$$\left|\overline{P}_k(f,\ z) - f(z) - \frac{z^2}{2(k+1)}f''(z)\right| \le \sum_{s=2}^\infty \left|c_s\right|\left|\overline{P}_k\left(e_s,\ z\right) - e_s(z) - e_s(z)\frac{s(s-1)}{2(k+1)}\right|$$

$$\leq \sum_{s=2}^{\infty}\left|c_s\right|.j^s\left|\prod_{r=1}^{s}\left(\frac{k+r}{k+1}\right)-1-\frac{s(s-1)}{2(k+1)}\right| \leq L\sum_{s=2}^{\infty}\frac{(Bj)^s}{s!}\left|\prod_{r=1}^{s}\left(\frac{k+r}{k+1}\right)-1-\frac{s(s-1)}{2(k+1)}\right|.$$

Now taking $E_s = \prod_{r=1}^{s}\left(\frac{k+r}{k+1}\right)-1-\frac{s(s-1)}{2(k+1)}$, we get the following recurrence formula as follows

$$E_{s+1} = \left(1+\frac{s}{k+1}\right)E_s + \frac{s^2(s-1)}{2(k+1)^2}, \text{ for all } s \geq 1. \qquad (4.1)$$

We can write $E_{s+1} = \frac{k+s+1}{k+1}\left[\prod_{r=1}^{s}\left(\frac{k+r}{k+1}\right)-1\right]+\frac{k+s+1}{k+1}-1-\frac{s(s+1)}{2(k+1)}$,

$$= \frac{k+s+1}{k+1}\left[\prod_{r=1}^{s}\left(\frac{k+r}{k+1}\right)-1-\frac{s(s-1)}{2(k+1)}\right]+\frac{k+s+1}{k+1}.\frac{s(s-1)}{2(k+1)}+\frac{k+s+1}{k+1}-1-\frac{s(s+1)}{2(k+1)}$$

$$= \frac{k+s+1}{k+1}E_s + \frac{k+s+1}{k+1}.\frac{s(s-1)}{2(k+1)}+\frac{s}{k+1}-\frac{s(s+1)}{2(k+1)}$$

$$= \frac{k+s+1}{k+1}E_s + \frac{s^2(s-1)}{2(k+1)^2}.$$

In view of (4.1), we prove that $E_s \leq \frac{2(s!)}{(k+1)^2}$, for all ≥ 1. (4.2)

By mathematical induction, for $s = 1$, we get $0 < \frac{2}{k+1}$.

Consider (4.2) is valid for s and we also prove its validity for $(s+1)$.

From (4.1), we have, $E_{s+1} \leq \left(1+\frac{s}{k+1}\right).\frac{2(s!)}{(k+1)^2}+\frac{s^2(s-1)}{2(k+1)^2} \leq \left(1+\frac{s}{2}\right).\frac{2(s!)}{(k+1)^2}+\frac{s^2(s-1)}{2(k+1)^2}$.

If we inflict to have $\left(1+\frac{s}{2}\right).\frac{2(s!)}{(k+1)^2}+\frac{s^2(s-1)}{2(k+1)^2} \leq \frac{2(s+1)!}{(k+1)^2}$.

Which is equivalent to $\left(1+\frac{s}{2}\right).2(s!)+\frac{s^2(s-1)}{2} \leq 2(s+1)!$,

that is also equivalent to $\frac{s^2(s-1)}{2} \leq s(s!)$, means equivalent to $\frac{1}{2} \leq (s-2)!$ and it is valid for all $s \geq 2$.

Hence proof of (4.2) is here.

Now, $\left|P_k(f,z)-f(z)-\frac{z^2}{2(k+1)}f''(z)\right| \leq L\sum_{s=2}^{\infty}\frac{(Bj)^s}{s!}E_s \leq \frac{2L}{(k+1)^2}.\sum_{s=2}^{\infty}(Bj)^s$

$$\leq \frac{2L}{(k+1)^2}.\sum_{s=0}^{\infty}(Bj)^s = \frac{2L}{1-Bj}.\frac{1}{(k+1)^2}.$$

Hence the theorem is proved.

Theorem 4.3: If we suppose in theorem (4.2), f is not a polynomial of degree ≤ 1, then we have

$$\|\overline{P}_k f - f\|_j \sim \frac{C}{k+1}, \ \ k \geq 1, \text{ where the constant C depends on } f \text{ and } j.$$

Proof: For all $|z| \leq j$, we have the identity

$$\overline{P}_k(f, z) - f(z) = \frac{1}{(k+1)}\left[\frac{z^2}{2}f''(z) + \frac{1}{k+1}.(k+1)^2\left(\overline{P}_k(f, z) - f(z) - \frac{z^2}{2(k+1)}f''(z)\right)\right].$$

Applying the inequality, $\|P + Q\| \geq |\|P\| - \|Q\|| \geq \|P\| - \|Q\|$,

we have $\|\overline{P}_k(f) - f\|_j \geq \frac{1}{(k+1)}\left[\|\frac{e_1^2}{2}f''\|_j - \frac{1}{k+1}.(k+1)^2\|\overline{P}_k(f) - f - \frac{e_1^2}{2(k+1)}f''\|_j\right].$

Since f is not a polynomial of degree≤ 1 in any disk$\overline{D}_j$, we have $\|\frac{e_1^2}{2}f''\|_j > 0$.

In contradiction, it follows $z^2 f''(z) = 0$, for all $|z| \leq j$.

Last inequality can be written as $f''(z) = 0$ for all $z \in \overline{D}_j / \{0\}$ and it is a contradiction with taken hypothesis. Hence by theorem (4.2) and for all $k \geq 1$, we have

$$(k+1)^2\|\overline{P}_k f - f - \frac{e_1^2}{2(k+1)}f''\|_j \leq C_{j,B,L}(f), \quad \text{where } C_{j,B,L}(f) = \frac{2L}{1-Bj}.$$

Clearly there exists $k_0 > 2$ such that $k \geq k_0$, we get

$$\|\frac{e_1^2}{2}f''\|_j - \frac{1}{(k+1)}.(k+1)^2\|\overline{P}_k(f) - f - \frac{e_1^2}{2(k+1)}f''\|_j \geq \frac{1}{4}\|e_1^2 f''\|_j,$$

which shows that, $\|\overline{P}_k(f) - f\|_j \geq \frac{1}{4(k+1)}\|e_1^2 f''\|$ for all $k \geq k_0$.

For $k \in [1, k_0 - 1]$, we clearly get

$\|\overline{P}_k f - f\|_j \geq \frac{L_{j,k}(f)}{(k+1)}$ and $L_{j,k}(f) = (k+1).\|\overline{P}_k(f) - f\|_j > 0$.

As $\|\overline{P}_k f - f\|_j = 0$ is valid only for a polynomial f (degree≤ 1) for a certain k and which contradicts the hypothesis. Hence we have, $\|\overline{P}_k f - f\|_j \geq \frac{C_j(f)}{(k+1)}$ for all $k \geq 1$,

where $C_j(f) = \left\{L_{j,k}(f), \frac{1}{4}\|e_1^2 f''\|_j\right\}$,

Combining this with theorem (4.1), we get the required proof.

Conclusion: In the present paper authors have studied convergence properties of Post-Widder operators in complex plane. One can estimates these properties for other linear positive operators. We can also study convergence properties for multivariate Post-Widder operators by applying Laplace transform.

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