DOI https://doi.org/10.24297/jam.v21i.9352

On the solvability of a functional Volterra integral equation

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Abstract

In this article, we will investigate the existence of a unique bounded variation solution for a functional integral equation of Volterra type in the space $L_1(R^+)$ of Lebesgue integrable functions.

Keywords: Nemytskii operator, Volterra integral operator, Hausdorff measure of noncompactness, Functions of bounded variation, Darbo fixed point theorem.

1 Introduction

Integral equations play an important role in the theory of nonlinear analysis and its applications in mathematical physics, biology, engineering, economics, radiation transfer theory and mechanics (see [7], [8], [11], [12], [26]). For a review of various integral equations and their applications, see ([1], [3], [10], [13], [15], [19], [21], [22]).
This paper studies the existence of a unique solution of the functional Volterra integral equation

$$x(t) = g(t) + f_1(t, \int_0^t k(t,s) f_2(s, x(s))ds), \qquad t \geq 0 \tag{1}$$

in the space $L_1(R^+)$ of functions of bounded variation.

2 Preliminaries

Let R be the field of real numbers and R^+ be the interval $[0, \infty)$. Denote by $L_1 = L_1(R^+)$ the space of Lebesgue integrable functions on the interval $[0, \infty)$, with the standard norm

$$\|x\| = \int_0^\infty |x(t)|dt.$$

The most important operator in nonlinear functional analysis is the so-called Nemytskii (or superposition) operator ([2], [14]).

Definition 2.1 *If $f(t, x) = f : I \times R \to R$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in R^+$. Then to every function $x(t)$ being measurable on R^+ we may assign the function*

$$(Fx)(t) = f(t, x(t)) \qquad t \in I$$

The operator F is called the Nemytskii (or superposition) operator generated by f.

Furthermore, we propose a theorem which gives necessary and sufficient condition for the Nemytskii operator to map the space L_1 into itself continuously.

Theorem 2.1 *[2] If f satisfies Carathéodory conditions, then the Nemytskii operator F generated by the function f maps continuously the space L_1 into itself if and only if*

$$|f(t,x)| \leq a(t) + b|x|,$$

for every $t \in R^+$ and $x \in R$, where $a(t) \in L_1$ and $b \geq 0$ is a constant.

Definition 2.2 *(Volterra integral operator) [28]*
Let $k : \Delta \to R$ be a function that is measurable with respect to both variables, where $\Delta = \{(t,s) : 0 \leq s \leq t < \infty\}$. For an arbitrary function $x \in L_1(R^+)$, we define

$$(Vx)(t) = \int_0^t k(t,s)x(s)ds, \quad t \geq 0.$$

The above operator V is the well-known linear Volterra integral operator. Obviously, if $V : L_1 \to L_1$ then it is continuous [27].

Definition 2.3 *([5], [23])*

The Hausdorff measure of noncompactness $\chi(X)$ (see also [16], [17]) is defined as

$$\chi(X) = \inf\{r > 0 : \text{there exists a finite subset } Y \text{of } E \text{ such that } x \subset Y + B_r\}.$$

A more general regular measure can be defined as the space [4]:

$$c(X) = \lim_{\varepsilon \to 0}\{\sup_{x \in X}\{\sup[\int_D |x(\tau)|d\tau : \ D \subset R^+, \ \text{meas}D \leq \varepsilon]\}\} = 0 \tag{2}$$

and

$$d(X) = \lim_{T \to \infty}\{\sup[\int_T^\infty |x(\tau)|d\tau : x \in X]\}, \tag{3}$$

where meas$D$ represents the Lebesgue measure of subset D.
Put

$$\gamma(X) = c(X) + d(X). \tag{4}$$

Then we have the following theorem [18], which connects between the two measures $\chi(X)$ and $\gamma(X)$.

Theorem 2.2 *Let $X \in M_E$ and compact in measure, then*

$$\chi(X) \leq \gamma(X) \leq 2\chi(X).$$

Now, we give Darbo fixed point theorem (cf.[9], [20], [24]).

Theorem 2.3 *If Q is nonempty, bounded, closed and convex subset of E and let $A : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists a constant $k \in [0,1)$ such that*

$$\mu(AX) \leq k\mu(X),$$

for any nonempty subset X of Q. Then A has at least one fixed point in the set Q.

Definition 2.4 *(Functions of bounded variation) ([6], [23])*
Let $x : [a, b] \to R$ be a function. For each partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of the interval $[a, b]$, we define

$$Var(x) = \sup \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|,$$

where the supremum is taken over all partitions P of the interval $[a, b]$.

If $Var(x) < \infty$, we say that x has bounded variation and we write $x \in BV$. For functions $x : [a, b] \to R$ with $a < b$ we write $Var(x, [a, b])$ instead of $Var(x)$. We denote by $BV = BV[a, b]$ the space of all functions of bounded variation on $[a, b]$.

Theorem 2.4 *([4], [25]) Assume that $X \subset L_1(I)$ is of locally generalized bounded variation, then Conv X (convex hull of X) and $\bar{X}$ are of the same type.*

Corollary 2.1 *([4], [25]) Let $X \subset L_1(I)$ is of locally generalized bounded variation, then Conv X is also such.*

Next, we will have the following theorem, which we will further use (cf. [4], [25]).

Theorem 2.5 *Assume that $X \subset L_1$ is a bounded set have the following hypotheses:*

(i) there exists $t_0 \geq 0$ such that the set $x(t_0) : x \in X$ is bounded in R,

(ii) X is of locally generalized bounded variation on R^+,

(iii) for any $a > 0$ the following equality holds

$$\lim_{T \to \infty} \{\sup_{x \in X} \{meas\ \{t > T : |x(t)| \geq a\}\}\} = 0.$$

Then the set X is compact in measure.

Corollary 2.2 *[4] If $X \subset L_1$ is a bounded set satisfies the hypotheses of Theorem 2.5. Then Conv X is compact in measure.*

3 Main result

We can write (1) in operator form as

$$Gx = g + F_1 V F_2 x,$$

where F_1 and F_2 are the Nemytskii operators generated by the functions $f_1(t, x)$ and $f_2(t, x)$ respectively, as V is the Volterra operator generated by $k(t, s)$.
We will solve equation (1) under the following hypotheses listed below:

(i) $g \in L_1(R^+)$ and is of locally generalized bounded variation on R^+.

(ii) $f_1, f_2 : R^+ \times R \to R$ satisfy Carathéodory conditions and $\exists$ functions $a_1, a_2 \in L_1(R^+)$ and constants b_1, b_2 such that

$$|f_i(t,x)| \leq a_i(t) + b_i|x|, \qquad (i = 1, 2)$$

for all $t \in (0,1)$ and $x \in R$.

(iii) There exists a constant $L > 0$ such that

$$|f_i(t,x) - f_i(t,y)| \leq L[|t-s| + |x-y|], \quad i = 1, 2.$$

(iv) $k : \Delta \to R$ is measurable in both variables such that the integral operator V generated by k maps L_1 into itself ($\Delta = \{(t,s) : 0 \leq s \leq t < \infty\}$).
Moreover, $\forall\ h > 0$

$$\lim_{T\to\infty} \{\text{meas } \{t > T : |(Vx)(t)| \geq h\}\} = 0.$$

uniformly on $x \in X$, where $X \subset L_1$ is arbitrarily bounded.

(v) The generalized variation of the function $t \to k(t,s)$ is essentially bounded on $[0,T]$ $\forall\ T > 0$ and uniformly on $s \in [0,T]$. Also, the function $v(T)$ is defined as

$$v(T) = \operatorname{ess\,sup}\{\text{var}_t k(t,s), [0,T] : s \in [0,T]\},$$

then we get $v(T) < \infty\ \forall\ T \geq 0$.

(vi) $b_1 b_2 \|V\| < 1$.

Theorem 3.1 *Let the hypotheses (i)–(vi) be satisfied, then equation (1) has at least one solution $x \in L_1(R^+)$ which is a function of locally bounded variation on R^+.*

Proof. First by hypothesis (ii) and Theorem 2.1 the operators F_1, F_2 map $L_1(R^+)$ into itself and are continuous. secondly, by hypothesis (iv) the Volterra operator V maps $L_1(R^+)$ into itself and is continuous. Finally, for any $x \in L_1(R^+)$ and from a hypothesis (i) we get $Gx \in L_1(R^+)$.
Moreover, we have

$$\begin{aligned}
\|Gx\| &\leq \|g\| + \|F_1VF_2x\| \\
&\leq \|g\| + \int_0^\infty |f_1(t, \int_0^t k(t,s)f_2(s,x(s))ds)|dt \\
&\leq \|g\| + \int_0^\infty [a(t) + b_1|\int_0^t k(t,s)f_2(s,x(s))ds)|dt \\
&\leq \|g\| + \int_0^\infty [a_1(t) + b_1\|V\||f_2(s,x(s))|]dt \\
&\leq \|g\| + \int_0^\infty [a_1(t) + b_1\|V\|(a_2(t) + b_2|x(t)|)]dt \\
&\leq \|g\| + \|a_1\| + b_1\|V\|\|a_2\| + b_1b_2\|V\|\|x\| \\
&\leq \|g\| + \|a_1\| + b_1\|V\|\|a_2\| + b_1b_2\|V\|.r \\
&\leq r
\end{aligned}$$

From the above estimate, the operator $G : B_r \to B_r$, where

$$r = \frac{\|g\| + \|a_1\| + b_1\|V\|\|a_2\|}{1 - b_1b_2\|V\|} > 0.$$

In what follows, consider $x \in B_r$. In view of assumption (i), we get

$$\begin{aligned} |(Gx)(0)| &= |g(0) + f_1(0,0)| \\ &\leq |g(0)| + |f_1(0,0)| \\ &< \infty. \end{aligned} \tag{5}$$

So we get that all functions belonging to GB_r are bounded at $t = 0$.
Moreover, fix $T > 0$ and assume that the sequence t_i such that $0 = t_0 < t_1 < t_2 \ldots < t_n = T$. Then, using the above hypotheses leads us to

$$\begin{aligned} \sum_{i=1}^{n} |(Gx)(t_i) - (Gx)(t_{i-1})| &\leq \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| \\ &+ \sum_{i=1}^{n} |f_1(t_i, \int_0^{t_i} k(t_i, s) f_2(s, x(s)) ds) \\ &- f_1(t_{i-1}, \int_0^{t_{i-1}} k(t_{i-1}, s) f_2(s, x(s)) ds)| \\ V(Gx, T) &\leq V(g, T) + \sum_{i=1}^{n} |f_1(t_i, \int_0^{t_i} k(t_i, s) f_2(s, x(s)) ds) \\ &- f_1(t_i, \int_0^{t_i} k(t_{i-1}, s) f_2(s, x(s)) ds)| \\ &+ \sum_{i=1}^{n} |f_1(t_i, \int_0^{t_i} k(t_{i-1}, s) f_2(s, x(s)) ds) \\ &- f_1(t_i, \int_0^{t_i - 1} k(t_{i-1}, s) f_2(s, x(s)) ds)| \\ &+ \sum_{i=1}^{n} |f_1(t_i, \int_0^{t_{i-1}} k(t_{i-1}, s) f_2(s, x(s)) ds) \\ &- f_1(t_{i-1}, \int_0^{t_{i-1}} k(t_{i-1}, s) f_2(s, x(s)) ds)| \\ &\leq V(g, T) + L \sum_{i=1}^{n} \int_0^{t_i} |k(t_i, s) - k(t_{i-1}, s)||f_2(s, x(s))| ds \\ &+ L \sum_{i=1}^{n} | \int_0^{t_i} k(t_{i-1}, s) - \int_0^{t_{i-1}} k(t_{i-1}, s)||f_2(s, x(s))| ds + M, \end{aligned}$$

where $M = L|t_i - t_{i-1}|$

$$\begin{aligned} V(Gx, T) &\leq V(g, T) + L \int_0^T \sum_{i=1}^{n} |k(t_i, s) - k(t_{i-1}, s)|[a_2(s) + b_2|x(s)|] ds \\ &+ L \sum_{i=1}^{n} \int_{t_i - 1}^{t_i} |k(t_{i-1}, s)|[a_2(s) + b_2|x(s)|] ds + M \\ &\leq V(g, T) + L \int_0^T v(T) a_2(s) ds + L b_2 \int_0^T |x(s)| ds \\ &+ L k_0 \int_0^T a_2(s) ds + L b_2 k_0 \int_0^T |x(s)| ds + M \\ &\leq V(g, T) + L v(T) \|a_2\| + L b_2 v(T) r + L k_0 \|a_2\| + L b_2 k_0 r + M < \infty, \end{aligned} \tag{6}$$

from the previous estimate, all functions belonging to GB_r have the same constant variation over each closed subinterval of R^+.

In the following, let the set Q_r=Conv GB_r, it is clear that $Q_r \subset B_r$. We will show that Q_r is nonempty, bounded convex, closed and compact in measure.
To prove Q_r is nonempty, let $x(t) = \frac{r}{\pi}(\frac{1}{1+t^2})$, we get

$$\|x\| = \int_0^\infty |x(t)|dt = \int_0^\infty |\frac{r}{\pi}(\frac{1}{1+t^2})|dt = \frac{r}{\pi}\arctan|_0^\infty = \frac{r}{\pi}(\frac{\pi}{2}) \leq r.$$

Since, $Q_r \subset B_r$ then it is bounded.
To prove the convexity of Q_r, take $x_1, x_2 \in Q_r$ which gives $\|x_i\| \leq r, \quad i = 1, 2$. Let

$$z(t) = \lambda x_1(t) + (1-\lambda)x_2(t), \quad t \in R^+, \ \lambda \in R^+.$$

Then

$$\begin{aligned} \|z\| &\leq \lambda\|x_1\| + (1-\lambda)\|x_2\| \\ &\leq \lambda r + (1-\lambda)r = r. \end{aligned}$$

So, we get Q_r is convex.
Now, we prove that the closeness of Q_r. To do this, suppose $\{x_n\}$ is the sequence of elements in Q_r that converges to x in $L_1(R^+)$, then this sequence is convergent in measure and as a result of the Vitali convergence theorem and the characterization of convergence in measure (the Riesz theorem) this leads to the existence of $\{x_{n_k}\} \subset \{x_n\}$ that converges to x almost uniformly on R^+ that means $x \in Q_r$ and thus the set Q_r is closed.
Moreover, in view of (5),(6) and Theorem 2.5 we conclude that the set GB_r is compact in measure. By Corollary 2.2 this yields that the set Q_r is also compact in measure. Moreover, Corollary 2.1 implies that the set Q_r is of locally generalized bounded variation on R^+. Now, from assumption (i), and since $Q_r \subset B_r$, then G is a self-mapping of the set Q_r into it self and is continuous.

Finally, we prove that the operator G is a contraction with respect to the measure of noncompactness χ.
Take a subset $X \subset Q_r$ and $\varepsilon > 0$ is fixed, then $\forall x \in X$ and for a set $D \subset R^+$, meas$D \leq \varepsilon$, we get

$$\begin{aligned} \int_D |(Gx)(t)|dt &\leq \int_D g(t|dt + \int_D |f_1(t, \int_0^t f_2(s, x(s))ds)|dt \\ &\leq \int_D g(t)dt + \int_D [a_1(t) + b_1| \int_0^t k(t,s) f_2(s, x(s))ds|]dt \\ &\leq \int_D g(t)dt + \int_D a_1(t)dt + b_1\|V\| \int_D a_2(s)ds + b_1 b_2\|V\| \int_D |x(s)|ds. \end{aligned}$$

Therefore, using the fact that

$$\lim_{\varepsilon \to 0} \sup\{\int_D g(t)dt : \ D \subset R^+, \ \text{meas}D \leq \varepsilon\} = 0,$$

and

$$\lim_{\varepsilon \to 0} \sup\{\int_D a_i(t)dt, \ i = 1, 2 : \ D \subset R^+, \ \text{meas}D \leq \varepsilon\} = 0,$$

Then using (2), we get

$$c(GX) \leq b_1 b_2 \|V\| c(X). \tag{7}$$

Also, fixing $T > 0$ we have

$$\int_T^\infty |(Gx)(t)|dt \leq \int_T^\infty g(t)dt + \int_T^\infty a_1(t)dt + b_1\|V\| \int_T^\infty a_2(t)dt + b_1 b_2 \|V\| \int_T^\infty |x(t)|dt$$

As $T \to \infty$, the previous inequality yields

$$d(GX) \leq b_1 b_2 \|V\| d(X), \tag{8}$$

where $d(X)$ has been defined before in (3).
Thus from (7) and (8) we get

$$\gamma(GX) \leq b_1 b_2 \|V\| \gamma(X),$$

where γ denotes the measure of noncompactness defined in (4).
Since X is a subset of Q_r and Q_r is compact in measure, we get

$$\chi(GX) \leq b_1 b_2 \|V\| \chi(X).$$

Therefore, by using hypothesis (vi) we can apply Darbo's fixed point theorem. This completes the proof. ■

4 Uniqueness of the solution

Now, we can prove the existence of our unique solution.

Theorem 4.1 *If the hypotheses of Theorem 3.1 is satisfied but instead of assuming (vi), let $L^2\|V\| < 1$. Then, equation (1) has a unique solution on R^+.*

Proof. To prove that equation (1) has a unique solution, let $x(t), y(t)$ be any two solutions of equation (1) in B_r, we have

$$\begin{aligned} \|x - y\| &= \int_0^\infty |f_1(t, \int_0^t k(t,s) f_2(s, x(s)) ds) - f_1(t, \int_0^t k(t,s) f_2(s, y(s)) ds)| dt \\ &\leq L \int_0^\infty \int_0^t |k(t,s)| |f_2(s, x(s)) - f_2(s, y(s))| ds dt \\ &\leq L^2 \|V\| \|x - y\|. \end{aligned}$$

Therefore,

$$(1 - L^2\|V\|)\|x - y\|_{L_1} \leq 0,$$

This yields $\|x - y\| = 0, \Rightarrow x = y$, which completes the proof.

5 Example

Consider the integral equation

$$x(t) = e^{-t} + \int_0^t \frac{1}{1 + s^2 + t^2} (e^{-s} + \frac{s x(s)}{s+2}) ds, \qquad t \in R^+ \tag{9}$$

We have $g(t) = e^{-t}$, $g(t) \in L_1(R^+)$ since

$$\int_0^\infty e^{-t} dt = -e^{-t}|_0^\infty = 1 - 0 = 1,$$

so, condition (i) is satisfied.
Also, $f_1(t,x) = x$, $f_2(t,x) = e^{-t} + \frac{tx(t)}{t+2})$, so we can see that f_i, $i = 1, 2$ satisfy Carathéodory conditions i.e. it is

measurable in t and continuous in x.
Also, we get

$$\begin{aligned} |f_2(t,x)| &= e^{-t} + \frac{tx(t)}{t+2} \\ &\leq e^{-t} + \frac{1}{3}|x(t)|. \end{aligned}$$

Hence, $a_2(t) = e^{-t} \in L_1(R^+)$ and $b_2 = \frac{1}{3} > 0$. Moreover, $a_1(t) = 0$ and $b_1(t) = 1 > 0$, then condition (ii) is satisfied.
Also,

$$|f_1(t,x) - f_1(t,y)| \leq |x-y|,$$

and

$$|f_2(t,x) - f_2(t,y)| \leq \frac{1}{2}|x-y|,$$

so that condition (iii)is satisfied. Furthermore, $k(t,s) = \frac{1}{1+s^2+t^2}$ is measurable for all t, s.
Let $x \in L_1$, we will show that the Volterra operator V maps continuously the space L_1 into itself

$$\begin{aligned} \|Vx\| &\leq \int_0^\infty \int_0^t \frac{|x(s)|}{1+s^2+t^2} dsdt \\ &\leq \int_0^\infty \int_s^\infty \frac{|x(s)|}{1+s^2+t^2} dtds \\ &\leq \int_0^\infty \int_s^\infty \frac{|x(s)|}{1+t^2} dtds \\ &\leq \int_0^\infty \arctan t|_s^\infty |x(s)| ds \\ &\leq \int_0^\infty (\frac{\pi}{2} - \arctan s)|x(s)| ds \\ &\leq \frac{\pi}{2}\|x\|, \end{aligned}$$

and hence condition (iv) is satisfied.
Finally, we have $b_1 b_2 \|V\| = \frac{\pi}{6} < 1$ then condition (vi) is satisfied.
Therefore, the assumptions of our Theorem 3.1 are satisfied, so equation (9) has at least one solution $x \in BV$ on R^+.

Data Availability (excluding Review articles)

Applicable.

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Supplementary Materials

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Funding Statement

The research was self-sposored by the author.

Acknowledgments

The authors like to thank the anonymous referees for their valuable comments and suggestions, which greatly improved the presentation of this paper.