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**Use of Mixed Operator Method to a fractional Hadamard Dirichlet boundary value problem**

Lakhdar Ragoub

Mathematics Department, University of Prince Mugrin,
P.O. Box 41040, Al Madinah 42241,
Saudi Arabia
Email : radhkla@hotmail.com

**Abstract:**

**The purpose of this paper is to deal with the following nonlinear Hadamard fractional boundary value problem**

$$\begin{aligned} &{}^{\mathbb{H}}D^{\alpha}_{1^+}u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, \\ &\quad 1 < t < e,\ 1 < \alpha \leq 2, \\ &u(1) = u(e) = 0, \end{aligned}$$

**where ${}^{\mathbb{H}}D^{\alpha}_{1^+}$ is the Hadamard fractional derivative operator. Using the mixed monotone operator method, we prove an existence and uniqueness result for this mixed fractional Hadamard boundary value problem. As an application of this result, we give one example to establish an existence and uniqueness of a positive solution.**

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**Key words: Fractional differential equation; Dirichlet boundary condition; positive solution, Hadamard fractional derivative operator, Mixed monotone operator.**

## 1 Introduction

**A large class of fractional ordinary differential equations as well as partial differential equations is of essential importance and rapidely developed.Illustrated by several applications, it attracted a lot researchers and therefore many results derived in different fields of sciences. The novelty that we prove here is based on the motivation due to the rapid development of such fractional theory and several applications for a class of ordinary as well as partial differential equations. One may observe that such applications spread in a variety of sciences such as engineering, physics, biology, medicine and related fields. For additional lectures and details, we refer the reader to** [2, 8, 13, 14, 15] **and the references therein. Several research papers and monographs on fractional calculus are essentially established to the existence of solutions of fractional ordinary differential equations, and partial differential equations on terms of fixed points of some special operators. Some of them are recently devoted with the existence of solutions of nonlinear initial (or singular and nonsingular boundary) value problems using nonlinear analysis methods focusing on fixed point theorems, Leray-Schauder theory, etc.), see** [1, 3, 4, 5, 6, 7, 9, 10, 11, 12, 16, 17] **and the references therein.**

**The purpose of this paper is to deal with existence of non trivial solutions to the Dirichlet boundary value problems of Hadamard fractional order differential equations**

$$^{\mathbb{H}}D^{\alpha}_{1^+}u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, \quad 1 < t < e,\ 1 < \alpha \leq 2, \tag{1}$$

$$u(1) = u(e) = 0, \tag{2}$$

**where** $^{\mathbb{H}}D^{\alpha}_{1^+}$ **is the Hadamard fractional derivative operator,** $f : [1, e] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ **and** $g : [1, e] \times [0, +\infty) \to [0, +\infty)$ **are given continuous functions. We show that under appropriate conditions on the nonlinear terms** $f$ **and** $g$**, the fractional boundary value problem** $(1) - (2)$ **has a unique positive solution. Mathematical tools mainly used here are fixed point theory for operators acting on cones in a Banach space and Green's function associated to these operators. Such constructions of Green's function and cone are the backbone of this paper. We illustrate this result by providing an example .**

## 2 Notations, Definitions and Lemmas

**For completeness sake, let us dwell with some references, definitions, lemmas and basic results needed in the proof of the main result of this paper.**

**Definition 2.1** ***The Riemann-Liouville fractional derivative of order*** $\alpha > 0$ ***of a continuous function*** $\varphi : (0, +\infty) \to \mathbb{R}$ ***is given by***

$$D^{\alpha}_{0^+}\varphi(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{\varphi(s)}{(t-s)^{\alpha-n+1}} \frac{ds}{s}.$$

***Here*** $n = [\alpha] + 1$ ***stands for integer part such that the right side is defined on*** $(0, 1)$ ***point by point.***

$[\alpha]$ ***denotes the integer part of number*** $\alpha$***, provided that the right side on*** $(0, +\infty)$ ***is defined point-wisely, and*** $\Gamma$ ***is the Euler gamma function.***

**Definition 2.2** ***If*** $g \in C([a, b], \mathbb{R})$ ***and*** $\alpha > 0$***, then the Hadamard fractional integral is defined by***

$$I^{\alpha}_{a^+}\, g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(\log t - \log s)^{1-\alpha}} \frac{ds}{s}.$$

**Definition 2.3** ***Let*** $\alpha \geq 0$***, and*** $n = [\alpha] + 1$. ***If*** $f \in AC^n([a, b])$ ***then the Hadamard fractional derivative of order*** $\alpha$ ***of*** $f$ ***defined by***

$$^{\mathbb{H}}D^{\alpha}_{a^+}\, f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(\log t - \log s)^{\alpha-n+1}} \frac{ds}{s},$$

***exists almost everywhere on*** $[a, b]$ ($[\alpha]$ ***is the entire part of*** $\alpha$).

$$^{\mathbb{H}}I^{\alpha}_{a^+}\, f(t) = \frac{1}{\Gamma\alpha} \int_a^t \frac{f(s)}{(\log t - \log s)^{1-\alpha}} \frac{ds}{s}.$$

**The following theorem gives the relation between the solution** $u$ **and the Green's function** $G$ **defined in a general interval** $[a, b]$.

**Theorem 2.1** ***Let*** $1 < \alpha \le 2$ ***and*** $q \in C([a,b],\mathbb{R})$***. Then the unique solution of the problem***

$$\begin{aligned} (L) \quad {}^{\mathbb{H}}D^{\alpha}_{1+}u(t) + q(t)u(t) &= 0, \quad a < t < b, \ 1 < \alpha \le 2, \\ u(a) &= u(b) = 0, \end{aligned}$$

***is given by***

$$u(t) = \int_a^t G(t,s)u(s)q(s)\frac{ds}{s} + \int_t^b G(t,s)u(s)q(s)\frac{ds}{s},$$

***where the Green function*** $G(t,s)$ ***is defined by***

$$\Gamma(\alpha)G(t,s) = \begin{cases} (\log t - \log a)^{\alpha-1}\left(\frac{\log b - \log s}{\log b - \log a}\right)^{\alpha-1} - (\log t - \log s)^{\alpha-1}, \\ a \le s \le t, \\ (\log t - \log a)^{\alpha-1}\left(\frac{\log b - \log s}{\log b - \log a}\right)^{\alpha-1}, \\ t \le s \le b. \end{cases} \tag{3}$$

**Proof:**
**As argued in** $[8,14]$**, the solution of Hadamard differential equation in** $(L)$ **can be written in the following integral equation**

$$u(t) = c_1(\log t - \log a)^{\alpha-1} + c_2(\log t - \log a)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_a^t (\log t - \log s)^{\alpha-1} q(s)\frac{ds}{s}.$$

**Using the boundary conditions** $(2)$**, we found**

$$c_2 = 0, \text{ and } \quad c_1 = \frac{1}{\Gamma(\alpha)}\int_a^b \left(\frac{\log t - \log s}{\log b - \log a}\right)^{\alpha-1} q(s)\frac{ds}{s},$$

**and therefore**

$$u(t) = (\log t - \log a)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_a^b \left(\frac{\log b - \log s}{\log b - \log a}\right)^{\alpha-1} q(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_a^t (\log t - \log s)^{\alpha-1} q(s)\frac{ds}{s}.$$

**The next lemma represents an important auxiliary result which is essential in proving the main result of this paper.**

**Lemma 2.1** ***Let*** $1 < \alpha \le 2$***. Then the Green function*** $G$ ***defined in*** $(3)$ ***is positive, continuous, and satisfies***

$$\frac{1}{\Gamma(\alpha)}h(t)[(1-\log s)^{\alpha-1} - 1] \le G(t,s) \le \frac{1}{\Gamma(\alpha)}h(t)(1-\log s)^{\alpha-1}, \ \textit{for all } t,s \in [1,e],$$

***where*** $h(t) = (\log t)^{\alpha-1}$***,*** $t \in [1,e]$***.***

**For the proof of Lemma** $2.2$**, to obtain estimates for** $G$**, we proceed as follows. It is easy to see for the right hand-side inequality that for** $(t,s) \in [1,e]\times[1,e]$**,**

$$\Gamma(\alpha)G(t,s) \le (\log t)^{\alpha-1}(1-\log s)^{\alpha-1}.$$

**It remains to prove that the left hand-side inequality is satisfied. To do this, we first consider two cases** $t \geq s$, **and** $t \geq s$. **Before starting, one may we remark that for** $t \leq e$,

$$(\log t - \log s) \leq (\log t - \log t \log s) = \log t(1 - \log s).$$

**Hence, we conclude that for** $t \geq s$, **we have**

$$\begin{aligned} \Gamma(\alpha)G(t,s) &\geq (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} - (\log t)^{\alpha-1} \\ &= (\log t)^{\alpha-1}[(1-\log s)^{\alpha-1} - 1]. \end{aligned} \tag{4}$$

**This due to the fact that**

$$\begin{aligned} (\log t - \log s) &\leq (\log t - \log t \log s) \\ &= \log t(1 - \log s) \\ &\leq \log t, \end{aligned} \tag{5}$$

**valid for** $(t, s) \in [1, e] \times [1, e]$.

**It remains now to prove the case corresponding to** $s \geq t$. **Indeed,**

$$\begin{aligned} \Gamma(\alpha)G(t,s) &\geq (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} \\ &\geq (\log t)^{\alpha-1}[(1-\log s)^{\alpha-1} - 1] \\ &\geq ((\log t)^{\alpha-1} - (\log t)^{\alpha} + (\log t)^{\alpha})(1-\log s)^{\alpha-1}. \end{aligned} \tag{6}$$

**In what follows, we introduce some basic concepts in ordered Banach spaces and a fixed point theorem needed to accomplish the main finding.**

**Suppose that** $(E, \|\cdot\|)$ **is a real Banach space which is partially ordered by a cone** $P \subset E$**, i.e.,**

$$x, y \in E, \quad x \preceq y \Longleftrightarrow y - x \in P.$$

**If** $x \preceq y$ **and** $x \neq y$, **then we denote** $x \prec y$ **or** $y \succ x$. **By** $\theta_E$ **we denote the zero element of** $E$. **Recall that a nonempty closed convex set** $P \subset E$ **is a cone if it satisfies two conditions**

**(a).** $x \in P$, $\lambda \geq 0 \Longrightarrow \lambda x \in P$**;**

**(b).** $-x$, $x \in P \Longrightarrow x = \theta_E$**.**

**An other issue that we look for here is to investigate the cone as well as its properties to construct some subset of it. We denote the interior of the cone by Int**$(P)$, **which is defined by Int**$(P) := \{x \in P | x \text{ is an interior point of} P\}$. **If** $P$ **is not empty, it is called a solid cone. The cone** $P$ **is called normal if there exists a constant** $N$ **positive such that for all** $x, y \in E$, $\theta_E \preceq x \preceq y$ **implies** $\|x\| \leq N\|y\|$**. This**

inequality is sharp in the sense that $N$ the normality constant of $P$ is the best one. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that

$$\lambda y \preceq x \preceq \mu y.$$

Clearly, $\sim$ is an equivalence relation. Given $h \succ \theta_E$, we denote by $P_h$ the set

$$P_h = \{x \in E \,|\, x \sim h\}.$$

It is easy to see that $P_h \subset P$.

**Definition 2.4** *An operator $A : E \to E$ is said to be increasing (resp. decreasing) if for all $x, y \in E$, $x \preceq y$ implies $Ax \preceq Ay$ (resp. $Ax \succeq Ay$).*

**Definition 2.5** *An operator $A : P \times P \to P$ is said to be a mixed monotone operator if it satisfies*

$$(x, y), (u, v) \in P \times P, \quad x \preceq u,\, y \succeq v \Longrightarrow A(x, y) \preceq A(u, v).$$

*An element $x^* \in P$ is called a fixed point of $A$ if $A(x^*, x^*) = x^*$.*

**Definition 2.6** *If an operator $A : P \to P$ satisfies*

$$A(tx) \succeq tAx, \quad \forall t \in (1, e),\, x \in P,$$

*it is called sub-homogeneous operator.*

It is worth to mention that C. Zhai and M. Hao [16] established the following fixed point result.

**Lemma 2.2 (16)** *Let $\beta \in (0, 1)$. Let $A : P \times P \to P$ be a mixed monotone operator that satisfies*

$$A(tx, t^{-1}y) \succeq t^{\beta} A(x, y), \quad t \in (0, 1),\; x, y \in P. \tag{7}$$

*Let $B : P \to P$ be an increasing sub-homogeneous operator. Assume that*

(i) *there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*

(ii) *there exists a constant $\delta_0 > 0$ such that $A(x, y) \succeq \delta_0 Bx$, for all $x, y \in P$.*

*Then*

(I) $A : P_h \times P_h \to P_h$, $B : P_h \to P_h$;

(II) *there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that*

$$rv_0 \preceq u_0 \prec v_0,\; u_0 \preceq A(u_0, v_0) + Bu_0 \preceq A(v_0, u_0) + Bv_0 \preceq v_0;$$

(III) *there exists a unique $x^* \in P_h$ such that $x^* = A(x^*, x^*) + Bx^*$;*

(IV) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1},\; y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots$$

*we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.*

## 3 Main result

**This section is devoted to establish the main result of this paper.To do this, we define $E = C([1,e])$ the Banach space of continuous functions on $[1,e]$ with the norm $\|y\| = \max\{|y(t)| : t \in [1,e]\}$, $P = \{y \in C([1,e]) \,|\, y(t) \geq 0,\, t \in [1,e]\}$.**

**Now, we are ready to prove the result of this paper formulated in the following theorem**

**Theorem 3.1** ***Let*** $1 < \alpha \leq 2$ ***and assume that***

(H1) ***the functions*** $f : [1,e] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ ***and*** $g : [1,e] \times [0,+\infty) \to [0,+\infty)$ ***are continuous with*** $M(\{t \in [1,e] \,|\, g(t,0) \neq 0\}) > 0$***, where*** $M$ ***denotes the Lebesgue measure;***

(H2) $f(t,x,y)$ ***is increasing in*** $x \in [0,+\infty)$ ***for fixed*** $t \in [1,e]$ ***and*** $y \in [0,+\infty)$***, decreasing in*** $y \in [0,+\infty)$ ***for fixed*** $t \in 10,e]$ ***and*** $x \in [0,+\infty)$***, and*** $g(t,x)$ ***is increasing in*** $x \in [0,+\infty)$ ***for fixed*** $t \in [1,e]$***;***

(H3) $g(t,\lambda x) \geq \lambda g(t,x)$ ***for all*** $\lambda \in (0,1)$***,*** $t \in [1,e]$***,*** $x \in [0,+\infty)$***, and there exists a constant*** $\beta \in (0,1)$ ***such that*** $f(t,\lambda x,\lambda^{-1}y) \geq \lambda^{\beta} f(t,x,y)$ ***for all*** $\lambda \in (1,e)$***,*** $t \in [1,e]$***,*** $x,y \in [0,+\infty)$***;***

(H4) ***there exists a constant*** $\delta_0 > 0$ ***such that*** $f(t,x,y) \geq \delta_0 g(t,x)$ ***for all*** $t \in [1,e]$***,*** $x,y \in [0,+\infty)$***.***

***Then***

(1) ***there exist*** $u_0, v_0 \in P_h$ ***and*** $r \in (0,1)$ ***such that*** $rv_0 \preceq u_0 \prec v_0$ ***and***

$$\begin{aligned} u_0(t) &\leq \int_1^e G(t,s)f(s,u_0(s),v_0(s))\,\frac{ds}{s} + \int_1^e G(t,s)g(s,u_0(s))\,\frac{ds}{s}, && t \in [1,e],\\ v_0(t) &\geq \int_1^e G(t,s)f(s,v_0(s),u_0(s))\,\frac{ds}{s} + \int_1^e G(t,s)g(s,v_0(s))\,\frac{ds}{s}, && t \in [1,e], \end{aligned}$$

***where*** $h(t) = (\log t)^{\alpha-1}(1-\log t)$***,*** $t \in [1,e]$***;***

(2) ***Problem*** $(1)-(2)$ ***has a unique positive solution*** $x^* \in P_h$***;***

(3) ***For any*** $x_0, y_0 \in P_h$***, constructing successively the sequences***

$$\begin{aligned} x_n(t) &= \int_1^e G(t,s)f(s,x_{n-1}(s),y_{n-1}(s))\,\frac{ds}{s} + \int_1^e G(t,s)g(s,x_{n-1}(s))\,\frac{ds}{s}, && n = 1,2,\ldots,\\ y_n(t) &= \int_1^e G(t,s)f(s,y_{n-1}(s),x_{n-1}(s))\,\frac{ds}{s} + \int_1^e G(t,s)g(s,y_{n-1}(s))\,\frac{ds}{s}, && n = 1,2,\ldots, \end{aligned}$$

***we have*** $\|x_n - x^*\| \to 0$ ***and*** $\|y_n - x^*\| \to 0$ ***as*** $n \to \infty$***.***

**Proof. From Theorem** 2.1**, Problem** $(1)-(2)$ **has the following integral formulation**

$$u(t) = \int_1^e G(t,s)f(s,u(s),u(s))\,\frac{ds}{s} + \int_1^e G(t,s)g(s,u(s))\,\frac{ds}{s}.$$

**Let us introduce two operators** $A : P \times P \to E$ **and** $B : P \to E$ **defined by**

$$A(u,v)(t) = \int_1^e G(t,s)f(s,u(s),v(s))\,\frac{ds}{s}, \qquad (Bu)(t) = \int_1^e G(t,s)g(s,u(s))\,\frac{ds}{s}.$$

**It is easy to see that** $u$ **is a solution to** $(1)-(2)$ **if and only if** $A(u,u)+Bu=u$**. Using** $(H1)$**, we establish that** $A:P\times P\to P$ **and** $B:P\to P$**. In view of** $(H2)$**, from one side, one may conclude that** $A$ **is mixed monotone and** $B$ **is increasing. In light of** $(H3)$**, from another side, for any** $\lambda\in(0,1)$**, and** $u,v\in P$**, we have**

$$\begin{aligned} A(\lambda u,\lambda^{-1}v)(t) &= \int_1^e G(t,s)f(s,\lambda u(s),\lambda^{-1}v(s))\,\frac{ds}{s}\\ &\geq \lambda^{\beta}\int_1^e G(t,s)f(s,u(s),v(s))\,\frac{ds}{s}\\ &= \lambda^{\beta}A(u,v)(t), \end{aligned}$$

**for all** $t\in[1,e]$**. Hence, for all** $\lambda\in(0,1)$**, and** $u,v\in P$**, we have**

$$A(\lambda u,\lambda^{-1}v)\succeq\lambda^{\beta}A(u,v).$$

**Thus, condition** $(4)$ **of Lemma**2.4 **is satisfied. From (H3), we conclude that for all** $\lambda\in(0,1)$**, and** $u\in P$**,**

$$\begin{aligned} B(\lambda u)(t) &= \int_1^e G(t,s)g(s,\lambda u(s))\,\frac{ds}{s}\\ &\geq \lambda\int_1^e G(t,s)g(s,u(s))\,\frac{ds}{s}\\ &= \lambda Bu(t), \end{aligned}$$

**for all** $t\in[1,e]$**. Then, for all** $\lambda\in(0,1)$ **and** $u\in P$**, it comes**

$$B(\lambda u)\succeq\lambda Bu.$$

**To this step, we conclude that** $B$ **is a sub-homogeneous operator. For the next, we prove that** $A(h,h)\in P_h$ **and** $Bh\in P_h$**. Indeed, using Lemma** 2.2 **and** $(H2)$**, we obtain**

$$A(h,h)(t)=\int_1^e G(t,s)f(s,h(s),h(s))\,\frac{ds}{s}\leq\frac{1}{\Gamma(\alpha)}h(t)\int_1^e(1-\log s)^{\alpha-2}f(s,h_{\max},0)\,\frac{ds}{s},$$

**where** $h_{\max}=\max\{h(t):\,t\in[1,e]\}$**. Again, Using Lemma** 2.2 **and** $(H2)$**, we have**

$$A(h,h)(t)=\int_1^e G(t,s)f(s,h(s),h(s))\,\frac{ds}{s}\geq\frac{1}{\Gamma(\alpha)}h(t)\int_1^e(1-\log s)^{\alpha-1}f(s,0,h_{\max})\,\frac{ds}{s}.$$

**Let us define** $\mu_1$**, and** $\mu_2$ **by**

$$\mu_1=\frac{1}{\Gamma(\alpha)}\int_1^e(1-\log s)^{\alpha-1}f(s,0,h_{\max})\,\frac{ds}{s}$$

**and**

$$\mu_2=\frac{1}{\Gamma(\alpha)}\int_1^e(1-\log s)^{\alpha-2}f(s,h_{\max},0)\,\frac{ds}{s}.$$

**We have**

$$\mu_1 h\preceq A(h,h)\preceq\mu_2 h.$$

**It yields from** $(H2)$ **and** $(H4)$**, that**

$$f(s,h_{\max},0)\geq f(s,0,h_{\max})\geq\delta_0 g(s,0)\geq 0.$$

**Since** $M(\{t\in[1,e]\,|\,g(t,0)\neq 0\})>0$**, we have**

$$\mu_2=\frac{1}{\Gamma(\alpha)}\int_1^e(1-\log s)^{\alpha-2}f(s,h_{\max},0)\,\frac{ds}{s}\geq\frac{\delta_0}{\Gamma(\alpha)}\int_1^e(1-\log s)^{\alpha-2}g(s,0)\,\frac{ds}{s}>0$$

and

$$\mu_1 = \frac{1}{\Gamma(\alpha)} \int_1^e (1-\log s)^{\alpha-1} f(s,0,h_{\max}) \frac{ds}{s} \geq \frac{(\alpha-1)\delta_0}{\Gamma(\alpha)} \int_1^e (1-\log s)^{\alpha-1} g(s,0) \frac{ds}{s} > 0.$$

Thus we proved that $A(h,h) \in P_h$. Similarly,

$$\frac{1}{\Gamma(h)} h(t) \int_1^e (1-\log s)^{\alpha-1} g(s,0) \frac{ds}{s} \leq Bh(t) \leq \frac{1}{\Gamma(\alpha)} h(t) \int_1^e (1-\log s)^{\alpha-2} g(s,h_{\max}) \frac{ds}{s}.$$

From $M(\{t \in [1,e] \,|\, g(t,0) \neq 0\}) > 0$, we have $Bh \in P_h$. Then condition $(i)$ of Lemma 2.4 is satisfied. In the following we show the condition $(ii)$ of Lemma 2.4 is satisfied. Let $u, v \in P$. From $(H4)$, we have

$$A(u,v)(t) = \int_1^e G(t,s) f(s,u(s),v(s)) \frac{ds}{s} \geq \delta_0 \int_1^e G(t,s) g(s,u(s)) \frac{ds}{s} = \delta_0 Bu(t).$$

Therefore for all $t \in [1,e]$, and $u, v \in P$, we have $A(u,v) \succeq \delta_0 Bu$. So the conclusion of Theorem 3.1 follows from Lemma 2.4.

Example

Let us consider the following fractional boundary value problem

$$D_{0^+}^{3/2} u(t) + 2\left(\sqrt{t} + u(t)\right) + \frac{1}{u(t)+4} = 0, \quad 0 < t < 1, \tag{8}$$

$$u(1) = u(e) = 0. \tag{9}$$

In this example, we have $\alpha = 3/2$. Consider the functions $f : [1,e] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [1,e] \times [0,+\infty) \to [0,+\infty)$ defined by

$$f(t,x,y) = \sqrt{t} + x + \frac{1}{y+4}$$

for all $t \in [1,e]$, $x, y \in [0,+\infty)$. Then (5) is equivalent to

$$D_{0^+}^{3/2} u(t) + f(t,u(t),u(t)) + g(t,u(t)) = 0, \quad 1 < t < e.$$

For all $\lambda \in (0,1)$, $t \in [1,e]$, $x \in [0,+\infty)$, we have

$$g(t,\lambda x) = \lambda x + \sqrt{t} \geq \lambda(\sqrt{t} + x) = \lambda g(t,x).$$

for all $\lambda \in (0,1)$, $t \in [1,e]$, $x, y \in [0,+\infty)$, we have

$$\begin{aligned} f(t,\lambda x,\lambda^{-1} y) &= \sqrt{t} + \lambda x + \frac{1}{\sqrt{\lambda^{-1} y + 4}} \\ &= \sqrt{t} + \lambda x + \frac{\sqrt{\lambda}}{\sqrt{y + 4\lambda}} \\ &\geq \sqrt{\lambda}\left(\sqrt{t} + x + \frac{1}{\sqrt{y+4}}\right) \\ &= \lambda f(t,x,y). \end{aligned}$$

for all $t \in [1,e]$, $x, y \in [0,+\infty)$, we have

$$f(t,x,y) = \sqrt{t} + x + \frac{1}{\sqrt{y+4}} \geq \sqrt{t} + x = g(t,x).$$

To this end, in view of Theorem 3.1, it follows that Problem $(5)-(6)$ has a unique positive solution $x^* \in P_h$, where $h(t) = (\log t)^{\alpha-1}(1-\log t)$, $t \in [1,e]$.

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