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Rate of Growth of Triple Sequence Spaces Defined in Double Orlicz Function

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Abstract:

By this article we present the rate for growth of triple sequences space which defined by double Orlisz function and introduce universal properties of these spaces.

Key words: metric space, double Orlicz function, triple sequences spaces , solid.

1. Introduction

Some first work of double series is made known to in Apostol [1], and the initial works on double sequences are bring into being in Bromwich [2]. Later on, it was introduced by Hardy [3], M´oricz [4], and a lot of others . Later on studied from through Some initial work of the triple sequences spaces established in Sahiner , Esi [5],[6] and many others the work of Orlicz function is used to concept the spaces . Lindenstrauss and Tzafriri [7] explored to more parts for Orlicz sequence spaces. There are proved that to all Orlicz sequences space $l_Đ$ has a subspace isomorphic to $l_p(1 \leq p \leq \infty)$.

Orlicz function is a function $Đ: [0, \infty) \rightarrow [0, \infty)$ which is continuous , non- decreasing,

And convex with $Đ(0) = 0, Đ(Ƃ) > 0$, for $Ƃ > 0$ and $Đ(Ƃ) \rightarrow \infty$ as $Ƃ \rightarrow \infty$.[8] and [9]

2. Definition and Preliminaries

Definition 1.1.1. [10][11] : A triple sequence $\left(Ƃ_{h,d,b}\right)$ (real or complex)can be defined as a function $Ƃ: N\times N\times N\rightarrow R(C)$ where N, R and C denote the sets of natural,real and complex numbers respectively.

Definition 1.1.8.[12]: An Orlicz function is a function $Đ: [0, \infty) \rightarrow [0, \infty)$, which is continuous , non- decreasing and convex with $Đ(0) = 0$, $Đ(Ƃ) > 0$, for $Ƃ > 0$, and $Đ(Ƃ)\rightarrow\infty$, as $Ƃ\rightarrow\infty$.

Definition ..2.1[13] The double Orlicz function is a function

$Đ: [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times [0, \infty)$ such that

$Đ(Ƃ, ȡ) = \left(Đ_1(Ƃ), Đ_2(ȡ)\right)$,

$Đ_1: [0, \infty) \rightarrow [0, \infty)$ and $Đ_2: [0, \infty) \rightarrow [0, \infty)$,

such that $Đ_1, Đ_2$ are Orlicz functions which is continuous, non-decreasing,

even , convex and satisfies the following conditions

I. $Đ_1(0) = 0, Đ_2(0) = 0 \Rightarrow Đ(0, 0) = \left(Đ_1(0), Đ_2(0)\right) = (0, 0)$,
II. $Đ_1(Ƃ) > 0, Đ_2(ȡ) > 0 \Rightarrow Đ(Ƃ, ȡ) = \left(Đ_1(Ƃ), Đ_2(ȡ)\right) > (0, 0)$,

for $Ƃ > 0, ȡ > 0,$ we mean by $Đ(Ƃ, ȡ) > (0, 0)$, that $Đ_1(Ƃ) > 0, Đ_2(ȡ) > 0.$

III. $Đ_1(Ƃ)\to\infty, Đ_2(ȡ)\to\infty,$ as $Ƃ, ȡ\to\infty$, then

$Đ(Ƃ, ȡ) = \left(Đ_1(Ƃ), Đ_2(ȡ)\right) \to (\infty, \infty)$ as $(Ƃ, ȡ) \to (\infty, \infty)$, we mean by

$Đ(Ƃ, ȡ) \to (\infty, \infty)$, that $Đ_1(Ƃ)\to\infty, Đ_2(ȡ)\to\infty.$

Definition 1.1.19 .[14]: A triple sequence space V is said to be solid if

$(\alpha_{v,d,p} Ƃ_{v,d,p}) \in V$ whenever $\left(Ƃ_{v,d,p}\right) \in V$ and for all triple sequence$(\alpha_{v,d,p})$ of

scalars with $|\alpha_{v,d,p}|\leq 1$ for all $v, d, p \in N$.

3. Main Results

We denoted for the classes to entire and analytic scalar valued single sequences ($V$and Y), respectively.

Definition..3.1. A triple sequences (Ƃ, ȡ) is named a triple growth entire sequences for double Orlicz function , to a set X of sequences if

$$\left(Ƃ_{v,d,p}, ȡ_{v,d,p}\right) = (0,0)\ \left(z_{v,d,p}, ş_{v,d,p}\right) \Leftrightarrow Đ\left(\left|\frac{\left(Ƃ_{v,d,p}, ȡ_{v,d,p}\right)}{\left(z_{v,d,p}, ş_{v,d,p}\right)}\right|\right)^{\frac{1}{v+d+p}} \to (0, 0)\ as\ v, d, p\to\infty.$$

Definition..3.2:: $(z, ş)$is a sequence nemmed a triple growth analytic sequence of Orlicz function to a set X of sequences if:-

$$\left(\left(Ƃ_{v,d,p}\right) \vee \left(ȡ_{v,d,p}\right)\right) = (0,0)\ \left(\left(z_{v,d,p}\right) \vee \left(ş_{v,d,p}\right)\right) \Leftrightarrow Đ\left(\left|\frac{\left(\left(Ƃ_{v,d,p}\right)\vee\left(ȡ_{v,d,p}\right)\right)}{\left(\left(z_{v,d,p}\right)\vee\left(ş_{v,d,p}\right)\right)}\right|\right)^{\frac{1}{v+d+p}} < \left(\left(Đ_1\left|\frac{Ƃ_{v,d,p}}{z_{v,d,p}}\right|\right) \vee \left(Đ_2\left|\frac{Ƃ_{v,d,p}}{z_{v,d,p}}\right|\right)\right)^{\frac{1}{v+d+p}} < (\infty, \infty)\ \forall\ v, d, p.$$

New we let $\left(\left(Ƃ_{v,d,p}\right) \vee \left(ȡ_{v,d,p}\right)\right)$ become a triple sequence of real (R) or complex (C) numbers where the triple series defined by double Orlisz become $\sum_{v,d,p=1}^{\infty} \left(\left(Ƃ_{v,d,p}\right) \vee \left(ȡ_{v,d,p}\right)\right)$ give space is said become convergent $\Leftrightarrow$the triple sequence $X_{v,d,p}$ is convergent

$$X_{v,d,p} = \sum_{a,e,i}^{v,d,p} \left(\left(Ƃ_{a,e,i}\right) \vee \left(ȡ_{a,e,i}\right)\right)\ (v, d, p = 1,2,3,4,\ldots).$$

Hence the vector space for all triple analytic sequences ($sup_{v,d,p}\left|\left(\left(Ƃ_{v,d,p}\right) \vee \left(ȡ_{v,d,p}\right)\right)\right|^{\frac{1}{v+d+p}} < (\infty, \infty)$) are denoted by$Y^3$.
A sequence$(Ƃ, ȡ)$= $\left(\left(Ƃ_{v,d,p}\right) \vee \left(ȡ_{v,d,p}\right)\right)$ is named triple entire sequences if

$\left|\left(Ƃ_{v,d,p}\right) \vee \left(ȡ_{v,d,p}\right)\right|^{\frac{1}{v+d+p}} \to (0, 0)\ as\ v, d, p\to\infty.$

Now we get the vector space for all triple entire sequences defined through double Orlisz function denoted by $V^3_Đ$.

Suppose the set for sequences with the property become denoted by $V^3_Đ\ and Y^3_Đ$ are a metric space

$$d((Ƃ, ȡ), (\psi, ɏ)) = sup_{v,d,p}\left\{|(Ƃ, ȡ) - (\psi, ɏ)|^{\frac{1}{v+d+p}} : v, d, p: 1, 2, 3, 4, \ldots\ldots\right\},$$

For all (Ƃ, ȡ)= $\{(Ƃ_{v,d,p}) \vee (ȡ_{v,d,p})\}$ *and* (Ψ, Ɏ)= $\{(Ψ_{v,d,p}) \vee (Ɏ_{v,d,p})\}$ in $V^3_Đ$

Let $\emptyset = \{finite\ sequences\}$.
If Q are a triple sequences space ,we get on the following definitions:
1)Q^* = the continuous dual of Q;
2)$Q^{\sigma} = \left\{\alpha = (\alpha_{v,d,p}): \sum_{v,d,p=1}^{\infty} \left|\alpha_{v,d,p}(Ƃ_{v,d,p}, ȡ_{v,d,p})\right| < \infty, for\ each\ Ƃ, ȡ \in Q\right\}$
3) $Q^{\rho} = \left\{\alpha = (\alpha_{v,d,p}): \sum_{v,d,p=1}^{\infty} \alpha_{v,d,p}(Ƃ_{v,d,p}, ȡ_{v,d,p}) is\ convergent, for\ each\ Ƃ, ȡ \in Q\right\}$

Theorem 3-3. If V^3 has a growth sequence of double Orlicz then $V^3_{Đπ}$ has a growth sequence of double Orlicz function .
Proof ::

Suppose $V^3_{Đπ}$ be a growth sequence of double Orlicz Then $Đ\left(\left|\frac{(Ƃ_{v,d,p}, ȡ_{v,d,p})}{(ζ_{v,d,p}, ʂ_{v,d,p})}\right|\right)^{\frac{1}{v+d+p}} \to (0,0)$ *as* $v, d, p \to \infty$.
Let (Ƃ, ȡ) ∈ $V^3_{Đπ}$.

Then $\left\{\frac{Ƃ_{v,d,p}}{π_{v,d,p}}, \frac{ȡ_{v,d,p}}{π_{v,d,p}}\right\} \in V^3_Đ$.We have $Đ\left(\left|\frac{Ƃ_{v,d,p}}{π_{v,d,p}ζ_{v,d,p}}, \frac{ȡ_{v,d,p}}{π_{v,d,p}ʂ_{v,d,p,}}\right|\right)^{\frac{1}{v+d+p}} \leq$

$$\left\{\left(Đ_1\left|\frac{Ƃ_{v,d,p}}{π_{v,d,p}ζ_{v,d,p}}\right|\right)^{\frac{1}{v+d+p}}, \left(Đ_2\left|\frac{ȡ_{v,d,p}}{π_{v,d,p}ʂ_{v,d,p}}\right|\right)^{\frac{1}{v+d+p}}\right\} \leq$$

$$Đ\left(\left|Ƃ_{v,d,p}, ȡ_{v,d,p}\right|\right)^{\frac{1}{v+d+p}} \leq \left\{\left(Đ_1\left|Ƃ_{v,d,p}\right|\right)^{\frac{1}{v+d+p}}, \left(Đ_2\left|ȡ_{v,d,p}\right|\right)^{\frac{1}{v+d+p}}\right\} \leq$$

$\left|π_{v,d,p}ζ_{v,d,p}, π_{v,d,p}ʂ_{v,d,p}\right| \to 0$ as $v, d, p \to \infty$, which means that
$Đ\left(\left|Ƃ_{v,d,p}, ȡ_{v,d,p}\right|\right)^{\frac{1}{v+d+p}} \leq \left|π_{v,d,p}ζ_{v,d,p}, π_{v,d,p}ʂ_{v,d,p}\right| \to 0$ as $v, d, p \to \infty$.

The growth $\left\{π_{v,d,p}ζ_{v,d,p}, π_{v,d,p}ʂ_{v,d,p}\right\}$ is a growth for triple sequence ($V^3_{Đπ}$).

Theorem 3-4 .
Let $V^3_Đ$ be a matric space *then*the rate space $V^3_{Đπ}$ have a growth sequences for double Orlicz function .
Proof **::**
(Ƃ, ȡ) ∈ $V^3_{Đπ}$. Then $\left\{\frac{Ƃ_{v,d,p}}{π_{v,d,p}}, \frac{ȡ_{v,d,p}}{π_{v,d,p}}\right\} \in V^3_Đ$.
Put B_{vdp}(Ƃ, ȡ) =($\frac{Ƃ_{v,d,p}}{π_{v,d,p}}, \frac{ȡ_{v,d,p}}{π_{v,d,p}}$)∀ (Ƃ, ȡ) ∈ $V^3_{Đπ}$. Then B_{vdp} is a continuous functional on $V^3_{Đπ}$. Where $\left|B_{vdp}\right| \to 0$ as $v, d, p \to \infty$.

Also to the every positive integer as $v, ძ, p$ we have $\left(Đ\left(\left|\frac{Ƃ_{v,ძ,p}}{π_{v,ძ,p}}, \frac{ȡ_{v,ძ,p}}{π_{v,ძ,p}}\right|\right)^{\frac{1}{v+ძ+p}} \le \left\{\left(Đ_1\left|\frac{Ƃ_{v,ძ,p}}{π_{v,ძ,p}}\right|\right)^{\frac{1}{v+ძ+p}}, \left(Đ_2\left|\frac{ȡ_{v,ძ,p}}{π_{v,ძ,p}}\right|\right)^{\frac{1}{v+ძ+p}}\right\}\right) = \left|B_{vძp}(Ƃ, ȡ)\right| \le \left|B_{vძp}(Ƃ, ȡ)(π_{v,ძ,p}, π_{v,ძ,p})\right| Đ\left(\left|Ƃ_{v,ძ,p}, ȡ_{v,ძ,p}\right|\right)^{\frac{1}{v+ძ+p}} \le \left|B_{vძp}(Ƃ, ȡ\right.$

as $v, ძ, p \to \infty$.

Hance $(Ƃ_{v,ძ,p}, ȡ_{v,ძ,p}) = (0, 0)\left(B_{vძp}(Ƃ, ȡ)(π_{v,ძ,p}, π_{v,ძ,p})\right)$

Thus $\left\{B_{vძp}(Ƃ, ȡ)(π_{v,ძ,p}, π_{v,ძ,p})\right\}$ is a growth sequence for $V^3_{Đπ}$.

Theorem 3-5

$$\left(V^3_π\right)^σ = Y^3_{\frac{1}{π}}$$

Proof: let $(Ƃ, ȡ) \in Y^3_{\frac{1}{π}}$. then there exists $Đ > 0$ where $Đ(Ƃ, ȡ) = \left(Đ_1(Ƃ), Đ_2(ȡ)\right)$

With $\left|π_{v,ძ,p}\left(Ƃ_{v,ძ,p}, ȡ_{v,ძ,p}\right)\right| \le Đ^{v+n+c}$

$\to \left|π_{v,ძ,p}\left(Ƃ_{v,ძ,p}, ȡ_{v,ძ,p}\right)\right| \le \left(Đ_1(Ƃ), Đ_2(ȡ)\right)^{v+ძ+p} \quad \forall v, ძ, p \ge 1.$

Choose $\epsilon > 0$ such that $\epsilon\,(Đ(Ƃ, ȡ)) < 1$.

If $(ɰ, ɏ) \in V^3_π$, we have $\left(\left|\frac{ɰ_{v,ძ,p}}{π_{v,ძ,p}}, \frac{ɏ_{v,ძ,p}}{π_{v,ძ,p}}\right|\right) \le \epsilon^{v+ძ+p} \; \forall v, n, c \ge v_0, ძ_0, p_0$ depending on ϵ.

Therefore $\sum\left|\left(Ƃ_{v,ძ,p}, ȡ_{v,ძ,p}\right)\left(ɰ_{v,ძ,p}, ɏ_{v,ძ,p}\right)\right| \le \sum (Đ\epsilon)^{v+ძ+p} < \infty$, hence

$Y^3_{\frac{1}{π}} \subset \left(V^3_π\right)^σ$ …………….. (1)

And the other hand , let $(Ƃ, ȡ) \in \left(V^3_π\right)^σ$.

Suppose that $(Ƃ, ȡ) \notin Y^3_{\frac{1}{π}}$. than there exists an increasing sequence $\left\{Ƃ_{v,n,c}q_{v,n,c}, ɓ_{v,n,c}ƀ_{v,n,c}\right\}$

Of positive integers such that

$$\left|π_{\left(Ƃ_{v,ძ,p}q_{v,ძ,p}, ɓ_{v,ძ,p}ƀ_{v,ძ,p}\right)}\left(Ƃ_{\left(Ƃ_{v,ძ,p}q_{v,ძ,p}, ɓ_{v,ძ,p}ƀ_{v,ძ,p}\right)}, ȡ_{\left(Ƃ_{v,ძ,p}q_{v,ძ,p}, ɓ_{v,ძ,p}ƀ_{v,ძ,p}\right)}\right)\right| > (v + ძ + p)^{3\left(Ƃ_{v,ძ,p}+q_{v,ძ,p}, ɓ_{v,ძ,p}+ƀ_{v,ძ,p}\right)}$$

$\forall v, ძ, p > v_0, ძ_0, p_0$

Take $(ɰ, ɏ) = \left\{ɰ_{v,ძ,p}, ɏ_{v,ძ,p}\right\}$ by

$\left(ɰ_{v,ძ,p}, ɏ_{v,ძ,p}\right) = \{\frac{π_{v,ძ,p}}{(v+n+c)^{3\left(Ƃ_{v,ძ,p}+q_{v,ძ,p}, ɓ_{v,ძ,p}+ƀ_{v,ძ,p}\right)}}$, $for\ (Ƃq, ɓƀ, ꞥɨ) = (Ƃ_v q_v, ɓ_ძ ƀ_ძ, ꞥ_p ɨ_p)\ 0$, $for\ (Ƃq, ɓƀ, ꞥɨ) \ne (Ƃ_v q_v, ɓ_ძ ƀ_ძ, ꞥ_p$

………..(2)

Then $\left\{ɰ_{v,ძ,p}, ɏ_{v,ძ,p}\right\} \in V^3_π$, but $\sum\left|\left(Ƃ_{v,ძ,p}, ȡ_{v,ძ,p}\right)\left(ɰ_{v,ძ,p}, ɏ_{v,ძ,p}\right)\right| = \infty$, *a contradiction.*

This contradiction shows that

$\left(V^3_{\pi}\right)^{\sigma} \subset Y^3_{\frac{1}{\pi}}$ ………………..(3)

From (1) and (2) we get the $\left(V^3_{\pi}\right)^{\sigma} = Y^3_{\frac{1}{\pi}}$

Proposition 3-6 . $V^3_{Đ} \subset V^3_{Đ}$

Proof: let $(Ƃ, ȡ) \in V^3_{Đ\pi}$

Then we have $Đ\left(\left|\frac{Ƃ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}, \frac{ȡ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}} \leq$

$$\left\{\left(Đ_1\left|\frac{Ƃ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}}, \left(Đ_2\left|\frac{ȡ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}}\right\} \to 0 \quad as \quad \upsilon, d, p \to \infty$$

Here, we get $Đ\left(\left|\frac{Ƃ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}, \frac{ȡ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}} \to 0$ as $\upsilon, d, p \to \infty$.

Thus we have $(Ƃ, ȡ) \in V^3_{Đ\pi}$ and so $V^3_{Đ} \subset V^3_{Đ}$.

Proposition 3-7 . The rate of growth of the triple sequence spaces defined by double orlizc $(V^3_{Đ\pi})$ is solid

Proof:

Let $\left|Ƃ_{\upsilon,d,p}, ȡ_{\upsilon,d,p}\right| \leq \left|ɰ_{\upsilon,d,p}, ƕ_{\upsilon,d,p}\right|$ and let $(ɰ, ƕ) = \left(ɰ_{\upsilon,d,p}, ƕ_{\upsilon,d,p}\right) \in V^3_{Đ\pi}$.

We have

$$\left(\left|\frac{Ƃ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}, \frac{ȡ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}} \leq \left(\left|\frac{ɰ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}, \frac{ƕ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}}.$$

But

$$\left(\left|\frac{ɰ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}, \frac{ƕ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}} \in V^3_{Đ\pi}.$$

Because $(ɰ, ƕ) \in V^3_{Đ\pi}$. That is $\left(\left|\frac{ɰ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}, \frac{ƕ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}} \to 0 \Longrightarrow \left(\left|\frac{Ƃ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ʐ_{\upsilon,d,p}}, \frac{ȡ_{\upsilon,d,p}}{\pi_{\upsilon,d,p}ȿ_{\upsilon,d,p}}\right|\right)^{\frac{1}{\upsilon+d+p}} \to 0$ as $\upsilon, d, p \to \infty$

Therefore $(Ƃ, ȡ) = (Ƃ_{\upsilon,d,p}, ȡ_{\upsilon,d,p}) \in V^3_{Đ\pi}$

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