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New Class of Holomorphic Univalent Functions Defined by Linear Operator

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Abstract:

In the present work, we submit and study a new class $\mathcal{AN}(\lambda,\tau,n,\rho)$ containing holomorphic univalent functions defined by linear operator in the open unit disk $\Delta=\{s\in\mathbb{C}:|s|<1\}$. We get some geometric properties, such as, coefficient inequality, growth and distortion bounds, convolution properties, convex set, neighborhood property, radii of starlikeness and convexity, weighted mean and arithmetic mean for functions belonging to the class $\mathcal{AN}(\lambda,\tau,n,\rho)$.

Keywords: Holomorphic function, Univalent function, Coefficient inequality, Distortion bound, Convex set, convolution property and radii of starlikeness and convexity.

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1. Introduction

Let $\mathcal{A}$ be symbolize the function class of the form:

$$k(s)=s+\sum_{n=2}^{\infty}d_n s^n, \tag{1}$$

which are holomorphic and univalent in the open unit disk $\Delta=\{s\in\mathbb{C}:|s|<1\}$.

Let $\mathcal{N}$ be symbolize the function subclass of $\mathcal{A}$ containing of functions of the form:

$$k(s)=s-\sum_{n=2}^{\infty}d_n s^n, \qquad (d_n\geq 0). \tag{2}$$

For function $k(s)\in\mathcal{N}$, given by (2), and $h(s)\in\mathcal{N}$ given by

$$h(s)=s-\sum_{n=2}^{\infty}c_n s^n \qquad (s\in\Delta; c_n\geq 0), \tag{3}$$

the convolution (or Hadamard product) of $k(s)$ and $h(s)$ is defined by

$$(k*h)(s)=s-\sum_{n=2}^{\infty}d_n c_n s^n=(h*k)(s). \tag{4}$$

A function $k(s)\in\mathcal{A}$ is called univalent starlike of order γ $(0\leq\gamma<1)$, if $k(s)$ satisfies the condition:

$$Re\left(\frac{sk'(s)}{k(s)}\right)>\gamma \qquad (s\in\Delta). \tag{5}$$

Also, a function $k(s)\in\mathcal{A}$ is called univalent convex of order γ $(0\leq\gamma<1)$, if $k(s)$ satisfies the condition:

$$Re\left(1+\frac{sk''(s)}{k'(s)}\right)>\gamma \qquad (s\in\Delta). \tag{6}$$

Symbolized by $S^*(\gamma)$ and $C^*(\gamma)$ the classes of univalent starlike and univalent convex functions of order γ, respectively.

The following infinite series is named after Émile Leonard Mathieu (1835–1890) who investigated it in his 1890 monograph [7] on elasticity of solid bodies:

$$\mathcal{S}(\rho)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+\rho^2)^2},\qquad (\rho>0).$$

The series $\mathcal{S}(\rho)$ has a closed integral form provided by (see[3])

$$\mathcal{S}(\rho)=\frac{1}{\rho}\int_0^{\infty}\frac{t\sin(\rho t)}{e^t-1}dt.$$

The Mathieu-type power series is described as follows (see[10])

$$\mathcal{S}(\rho;s)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+\rho^2)^2}s^n,\qquad (\rho>0,|s|<1).$$

This series was defined specifically for functions of real variables, however it was redefined for complex variables by Bansal and Sokól [2]. Since $\mathcal{S}(\rho;s)\notin\mathcal{A}$ so using following normalization, we have

$$\mathcal{S}(\rho;s)=\frac{(\rho^2+1)^2}{2}\sum_{n=1}^{\infty}\frac{2n}{(n^2+\rho^2)^2}s^n,$$

$$=s+\sum_{n=2}^{\infty}\frac{n(\rho^2+1)^2}{(n^2+\rho^2)^2}s^n, \tag{7}$$

By using the Hadamard product in conjunction with (1) and (7), Liu at el. [6] introduced a new linear operator $Q(n,\rho):\mathcal{A}\to\mathcal{A}$ as follows

$$Q(n,\rho)k(s)=k(s)*\mathcal{S}(\rho;s)=s+\sum_{n=2}^{\infty}\frac{n(\rho^2+1)^2}{(n^2+\rho^2)^2}d_ns^n. \tag{8}$$

Now, by using the linear operator $Q(n,\rho)$, we define the following:

Definition 1 A function $k(s)\in\mathcal{N}$ is said to be in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$ if it satisfies the following condition:

$$\left|\frac{\lambda s^2(Q(n,\rho)k(s))'''+s(Q(n,\rho)k(s))''}{\lambda s(Q(n,\rho)k(s))''+(1+\lambda)(1-\tau)(Q(n,\rho)k(s))'}\right|<1,$$

where $0\le\lambda<1, 0\le\tau<1, \rho>0, n\in\mathbb{N}$ *and* $s\in\Delta$.

Some of the following properties studied for other classes in [5,1,11,8].

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for a function $k(s)$ to be in class $\mathcal{AN}(\lambda,\tau,n,\rho)$.

Theorem 1 A function $k(s)\in\mathcal{N}$ is in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$ if and only if

$$\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}d_n\le(1+\lambda)(1-\tau), \tag{9}$$

where $0\le\lambda<1, 0\le\tau<1, \rho>0, n\in\mathbb{N}$ *and* $s\in\Delta$.

The result is sharp for the function $k(s)$ given by

$$k(s)=s-\frac{(1+\lambda)(1-\tau)(n^2+\rho^2)^2}{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}s^n,\quad (n\ge2). \tag{10}$$

Proof. Assume that inequality (9) holds true and $|s| = 1$. Then we have

$|\lambda s^2(Q(n,\rho)k(s))''' + s(Q(n,\rho)k(s))''| - |\lambda s(Q(n,\rho)k(s))'' + (1+\lambda)(1-\tau)(Q(n,\rho)k(s))'|$

$$= \left|\sum_{n=2}^{\infty} n^2(n-1)(1+\lambda n-2\lambda)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, s^{n-1}\right| - \left|(1+\lambda)(1-\tau) - \sum_{n=2}^{\infty} n^2(\lambda(n-\tau)+1-\tau)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, s^{n-1}\right|$$

$$\le \sum_{n=2}^{\infty} n^2(n-1)(1+\lambda n-2\lambda)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, |s|^{n-1} - (1+\lambda)(1-\tau) + \sum_{n=2}^{\infty} n^2(\lambda(n-\tau)+1-\tau)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, |s|^{n-1}$$

$$= \sum_{n=2}^{\infty} n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, |s|^{n-1} - (1+\lambda)(1-\tau) \le 0,$$

by hypothesis. Hence by maximum modulus principle, we get $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$.

Conversely, let $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$. Then

$$\left|\frac{\lambda s^2(Q(n,\rho)k(s))''' + s(Q(n,\rho)k(s))''}{\lambda s(Q(n,\rho)k(s))'' + (1+\lambda)(1-\tau)(Q(n,\rho)k(s))'}\right|$$

$$= \left|\frac{\sum_{n=2}^{\infty} n^2(n-1)(1+\lambda n-2\lambda)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, s^{n-1}}{(1+\lambda)(1-\tau) - \sum_{n=2}^{\infty} n^2(\lambda(n-\tau)+1-\tau)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, s^{n-1}}\right| < 1.$$

Since $R(s) \le |s|$ for all s ($s \in \Delta$), then we obtain

$$Re\left(\frac{\sum_{n=2}^{\infty} n^2(n-1)(1+\lambda n-2\lambda)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, s^{n-1}}{(1+\lambda)(1-\tau) - \sum_{n=2}^{\infty} n^2(\lambda(n-\tau)+1-\tau)\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\, d_n\, s^{n-1}}\right) < 1. \tag{11}$$

Now, choosing the values of s on the real axis and allowing $s \to 1$ from the left through real values, the inequality (11) immediately yields the desired condition in (9). Finally, it is observed that the result is sharp for the function is given by (10).

Corollary 1 If $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$, then

$$d_n \le \frac{(1+\lambda)(1-\tau)(n^2+\rho^2)^2}{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}, \qquad (n \ge 2). \tag{12}$$

3. Growth and Distortion Bounds

Next, we prove the growth and distortion bounds for the linear operator $Q(n,\rho)$.

Theorem 2 If $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$, then

$$r - \frac{(1-\tau)}{2(2-\tau)}r^2 \le |Q(n,\rho)k(s)| \le r + \frac{(1-\tau)}{2(2-\tau)}r^2, \quad (0 < |s| = r < 1). \tag{13}$$

The result is sharp for the function $k(s)$ is given by

$$k(s) = s - \frac{(1-\tau)(4+\rho^2)^2}{4(2-\tau)(\rho^2+1)^2}s^2\,. \tag{14}$$

Proof. Let $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$. Then by Theorem 1

$$4(2-\tau)(1+\lambda)\frac{(\rho^2+1)^2}{(4+\rho^2)^2}\sum_{n=2}^{\infty} d_n \le \sum_{n=2}^{\infty} n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}d_n \le (1+\lambda)(1-\tau)$$

$$\sum_{n=2}^{\infty} d_n \le \frac{(1-\tau)(4+\rho^2)^2}{4(2-\tau)(\rho^2+1)^2}. \tag{15}$$

Hence,

$$|Q(n,\rho)k(s)| \le |s| + \sum_{n=2}^{\infty} \frac{n(\rho^2+1)^2}{(n^2+\rho^2)^2} d_n\, |s|^n \le r + \frac{2(\rho^2+1)^2}{(4+\rho^2)^2} r^2 \sum_{n=2}^{\infty} d_n \ \le r + \frac{(1-\tau)}{2(2-\tau)} r^2. \tag{16}$$

Similarly

$$|Q(n,\rho)k(s)| \ge |s| - \sum_{n=2}^{\infty} \frac{n(\rho^2+1)^2}{(n^2+\rho^2)^2} d_n\, |s|^n \ge r - \frac{2(\rho^2+1)^2}{(4+\rho^2)^2} r^2 \sum_{n=2}^{\infty} d_n \ \ge r - \frac{(1-\tau)}{2(2-\tau)} r^2. \tag{17}$$

From (16) and (17), we get (13). Thus, the proof is complete.

Theorem 3 If $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$, then

$$1 - \frac{(1-\tau)}{(2-\tau)} r \le \left|\left(Q(n,\rho)k(s)\right)'\right| \le 1 + \frac{(1-\tau)}{(2-\tau)} r, \quad (0 < |s| = r < 1). \tag{18}$$

The result is sharp for the function $k(s)$ is given by (14).

Proof. Let $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$. Then by Theorem 1

$$2(2-\tau)(1+\lambda)\frac{(\rho^2+1)^2}{(4+\rho^2)^2}\sum_{n=2}^{\infty} n d_n \le \sum_{n=2}^{\infty} n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2} d_n \le (1+\lambda)(1-\tau)$$

$$\sum_{n=2}^{\infty} n d_n \ \le \ \frac{(1-\tau)(4+\rho^2)^2}{2(2-\tau)(\rho^2+1)^2}. \tag{19}$$

Thus

$$|(Q(n,\rho)k(s))'| \le 1 + \sum_{n=2}^{\infty} \frac{n(\rho^2+1)^2}{(n^2+\rho^2)^2} n d_n |s|^{n-1} \le 1 + \frac{2(\rho^2+1)^2}{(4+\rho^2)^2} r \sum_{n=2}^{\infty} n d_n \le \ 1 + \frac{(1-\tau)}{(2-\tau)} r. \tag{20}$$

Similarly

$$|(Q(n,\rho)k(s))'| \ge 1 - \sum_{n=2}^{\infty} \frac{n(\rho^2+1)^2}{(n^2+\rho^2)^2} n d_n\, |s|^{n-1} \ge 1 - \frac{2(\rho^2+1)^2}{(4+\rho^2)^2} r \sum_{n=2}^{\infty} n d_n \ \ge 1 - \frac{(1-\tau)}{(2-\tau)} r. \tag{21}$$

From (20) and (21), we get (18). Thus, the proof is complete.

4. Convolution Properties

Theorem 4 Let the function k_j $(j = 1,2)$ defind by

$$k_j(s) = s - \sum_{n=2}^{\infty} d_{n,j}\, s^n, \qquad \left(d_{n,j} \ \ge 0, \qquad j = 1,2\right), \tag{22}$$

be in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$. Then $k_1 * k_2 \in \mathcal{AN}(\mu,\tau,n,\rho)$, where

$$\mu \le \frac{n^2[\,\lambda(n^2-2n+2-\tau)+n-\tau\,]^2(\rho^2+1)^2 - (n-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2}{(n^2-2n+2-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2 - n^2[\,\lambda(n^2-2n+2-\tau)+n-\tau\,]^2(\rho^2+1)^2}.$$

Proof. We must find the largest μ such that

$$\sum_{n=2}^{\infty} \frac{n^2[\,\mu(n^2-2n+2-\tau)+n-\tau\,](\rho^2+1)^2}{(1-\tau)(1+\mu)(n^2+\rho^2)^2} d_{n,1} d_{n,2} \le 1.$$

Since $k_j \in \mathcal{AN}(\lambda,\tau,n,\rho)$, $(j = 1,2)$, then

$$\sum_{n=2}^{\infty} \frac{n^2[\,\lambda(n^2-2n+2-\tau)+n-\tau\,](\rho^2+1)^2}{(1-\tau)(1+\lambda)\,(n^2+\rho^2)^2} d_{n,j} \le 1, \qquad (j = 1,2). \tag{23}$$

By Cauchy – Schwarz inequality, we get

$$\sum_{n=2}^{\infty}\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}\sqrt{d_{n,1}d_{n,2}}\leq 1. \tag{24}$$

We want only to show that

$$\frac{n^2[\mu(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\mu)(n^2+\rho^2)^2}d_{n,1}d_{n,2}\leq\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}\sqrt{d_{n,1}d_{n,2}}.$$

This equivalently to

$$\sqrt{d_{n,1}d_{n,2}}\leq\frac{(1+\mu)[\lambda(n^2-2n+2-\tau)+n-\tau]}{(1+\lambda)[\mu(n^2-2n+2-\tau)+n-\tau]}. \tag{25}$$

From (24), we get

$$\sqrt{d_{n,1}d_{n,2}}\leq\frac{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}.$$

Thus, it is sufficient to show that

$$\frac{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}\leq\frac{(1+\mu)[\lambda(n^2-2n+2-\tau)+n-\tau]}{(1+\lambda)[\mu(n^2-2n+2-\tau)+n-\tau]},$$

which implies to

$$\mu\leq\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau]^2(\rho^2+1)^2-(n-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2}{(n^2-2n+2-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2-n^2[\lambda(n^2-2n+2-\tau)+n-\tau]^2(\rho^2+1)^2}.$$

Theorem 5 Let the function k_j $(j=1,2)$ defined by (22) be in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$.

Then the function k defined by

$$k(s)=s-\sum_{n=2}^{\infty}(d_{n,1}^2+d_{n,2}^2)s^n, \tag{26}$$

belong to the class $\mathcal{AN}(\epsilon,\tau,n,\rho)$, where

$$\epsilon\leq\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau]^2(\rho^2+1)^2-2(n-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2}{2(n^2-2n+2-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2-n^2[\lambda(n^2-2n+2-\tau)+n-\tau]^2(\rho^2+1)^2}.$$

Proof. We must find the largest ϵ such that

$$\sum_{n=2}^{\infty}\frac{n^2[\epsilon(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\epsilon)(n^2+\rho^2)^2}(d_{n,1}^2+d_{n,2}^2)\leq 1.$$

Since $k_j\in\mathcal{AN}(\lambda,\tau,n,\rho)$, $(j=1,2)$, we get

$$\sum_{n=2}^{\infty}\left(\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}\right)^2 d_{n,1}^2$$

$$\leq\sum_{n=2}^{\infty}\left(\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}d_{n,1}\right)^2\leq 1 \tag{27}$$

and

$$\sum_{n=2}^{\infty}\left(\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}\right)^2 d_{n,2}^2$$

$$\leq\sum_{n=2}^{\infty}\left(\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}d_{n,2}\right)^2\leq 1. \tag{28}$$

Combining the inequalities (27) and (28), gives

$$\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\lambda)(n^2+\rho^2)^2}\right)^2\left(d_{n,1}^2+d_{n,2}^2\right)\le 1. \tag{29}$$

But, $k\in\mathcal{AN}(\epsilon,\tau,n,\rho)$ if and only if

$$\sum_{n=2}^{\infty}\frac{n^2[\epsilon(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\epsilon)(n^2+\rho^2)^2}\left(d_{n,1}^2+d_{n,2}^2\right)\le 1. \tag{30}$$

The inequality (30) will be satisfied if

$$\frac{n^2[\epsilon(n^2-2n+2-\tau)+n-\tau](\rho^2+1)^2}{(1-\tau)(1+\epsilon)(n^2+\rho^2)^2}\le\frac{n^4[\lambda(n^2-2n+2-\tau)+n-\tau]^2(\rho^2+1)^4}{2(1-\tau)^2(1+\lambda)^2(n^2+\rho^2)^4}.$$

So that

$$\epsilon\le\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau]^2(\rho^2+1)^2-2(n-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2}{2(n^2-2n+2-\tau)(1-\tau)(1+\lambda)^2(n^2+\rho^2)^2-n^2[\lambda(n^2-2n+2-\tau)+n-\tau]^2(\rho^2+1)^2}.$$

5. Convex Set

Theorem 6 The class $\mathcal{AN}(\lambda,\tau,n,\rho)$ is convex set.

Proof. Let functions k and h be in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$. Then for every $0\le m\le 1$, we must show that

$$(1-m)k(s)+m\,h(s)\in\mathcal{AN}(\lambda,\tau,n,\rho). \tag{31}$$

We have

$$(1-m)k(s)+mh(s)=s-\sum_{n=2}^{\infty}[(1-m)d_n+mc_n]s^n.$$

So, by Theorem 1, we get

$$\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\,[(1-m)d_n+mc_n]$$

$$=(1-m)\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}d_n+m\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}c_n$$

$$\le(1-m)(1+\lambda)(1-\tau)+m(1+\lambda)(1-\tau)=(1+\lambda)(1-\tau).$$

6. Neighborhood Property

Following the earlier investigations of Goodman [4] and Ruscheweyh [9], we define the $\sigma-$neighborhood of function $k(s)\in\mathcal{N}$ by

$$N_\sigma(k)=\left\{h\in\mathcal{N}:h(s)=s-\sum_{n=2}^{\infty}c_ns^n\ \ and\ \ \sum_{n=2}^{\infty}n|d_n-c_n|\le\sigma\right\} \tag{32}$$

In particular, for the identity function $e(s)=s$, we immediately have

$$N_\sigma(e)=\left\{h\in\mathcal{N}:h(s)=s-\sum_{n=2}^{\infty}c_ns^n\ \ and\ \ \sum_{n=2}^{\infty}n|c_n|\le\sigma\right\} \tag{33}$$

Definition 2 A function $k(s)\in\mathcal{N}$ is said to be in the class $\mathcal{AN}_y(\lambda,\tau,n,\rho)$ if there exists a function $h\in\mathcal{AN}(\lambda,\tau,n,\rho)$, such that

$$\left|\frac{k(s)}{h(s)}-1\right|<1-y,\qquad(s\in\Delta,0\le y<1).$$

Theorem 7 If $h(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$ and

$$y = 1 - \frac{2\sigma(2-\tau)(\rho^2+1)^2}{4(2-\tau)(\rho^2+1)^2-(1-\tau)(4+\rho^2)^2}, \tag{34}$$

then $N_\sigma(h) \subset \mathcal{AN}_y(\lambda,\tau,n,\rho)$.

Proof. Let $k(s) \in N_\sigma(h)$. Then we get from (32)

$$\sum_{n=2}^{\infty} n|d_n - c_n| \le \sigma,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |d_n - c_n| \le \frac{\sigma}{2}.$$

Also, since $h(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$, we get from Theorem 1

$$\sum_{n=2}^{\infty} c_n \le \frac{(1-\tau)(4+\rho^2)^2}{4(2-\tau)(\rho^2+1)^2}.$$

So that

$$\left|\frac{k(s)}{h(s)} - 1\right| \le \frac{\sum_{n=2}^{\infty}|d_n - c_n|}{1-\sum_{n=2}^{\infty} c_n} \le \frac{2\sigma(2-\tau)(\rho^2+1)^2}{4(2-\tau)(\rho^2+1)^2-(1-\tau)(4+\rho^2)^2} = 1-y.$$

Thus by Definition 2, $k(s) \in \mathcal{AN}_y(\lambda,\tau,n,\rho)$ for y given by (34).

7. Radii of Starlikeness and Convexity

In the next theorems, we will find the radii of starlikeness and convexity for the functions in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$.

Theorem 8 Let $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$. Then the function $k(s)$ is univalent starlike of order γ $(0 \le \gamma < 1)$ in the disk $|s| < R_1$, where

$$R_1 = \inf_n \left[\frac{n^2[\lambda(n^2-2n+2-\tau)+n-\tau](1-\gamma)(\rho^2+1)^2}{(n-\gamma)(1+\lambda)(1-\tau)(n^2+\rho^2)^2}\right]^{\frac{1}{n-1}}, \qquad (n \ge 2).$$

The result is sharp for the function $k(s)$ given by (10).

Proof. It is sufficient to prove

$$\left|\frac{sk'(s)}{k(s)} - 1\right| \le 1-\gamma \quad (0 \le \gamma < 1),$$

for $|s| < R_1$, we get

$$\left|\frac{sk'(s)}{k(s)} - 1\right| \le \frac{\sum_{n=2}^{\infty}(n-1)d_n\,|s|^{n-1}}{1-\sum_{n=2}^{\infty} d_n\,|s|^{n-1}}.$$

Thus

$$\left|\frac{sk'(s)}{k(s)} - 1\right| \le 1-\gamma,$$

if

$$\sum_{n=2}^{\infty} \frac{n-\gamma}{1-\gamma} d_n\,|s|^{n-1} \le 1. \tag{35}$$

Therefore, by using Theorem 1, (35) will be true if

$$\frac{n-\gamma}{1-\gamma}|s|^{n-1} \leq \frac{n^2[\,\lambda(n^2-2n+2-\tau)+n-\tau\,](\rho^2+1)^2}{(1+\lambda)(1-\tau)(n^2+\rho^2)^2}.$$

Hence

$$|s| \leq \left[\frac{n^2[\,\lambda(n^2-2n+2-\tau)+n-\tau\,](1-\gamma)(\rho^2+1)^2}{(n-\gamma)(1+\lambda)(1-\tau)(n^2+\rho^2)^2}\right]^{\frac{1}{n-1}}, \qquad (n \geq 2).$$

Setting $|s| = R_1$, we obtain the desired result.

Theorem 9 Let $k(s) \in \mathcal{AN}(\lambda,\tau,n,\rho)$. Then the function $k(s)$ is univalent convex of order γ $(0 \leq \gamma < 1)$ in the disk $|s| < R_2$, where

$$R_2 = \inf_n \left[\frac{n[\,\lambda(n^2-2n+2-\tau)+n-\tau\,](1-\gamma)(\rho^2+1)^2}{(n-\gamma)(1+\lambda)(1-\tau)(n^2+\rho^2)^2}\right]^{\frac{1}{n-1}}, \qquad (n \geq 2\,).$$

The result is sharp for the function $k(s)$ given by (10).

Proof. It is sufficient to show that

$$\left|\frac{sk''(s)}{k'(s)}\right| \leq 1-\gamma \quad (0 \leq \gamma < 1),$$

for $|s| < R_2$, we get

$$\left|\frac{sk''(s)}{k'(s)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)d_n\,|s|^{n-1}}{1-\sum_{n=2}^{\infty} nd_n\,|s|^{n-1}}.$$

Thus

$$\left|\frac{sk''(s)}{k'(s)}\right| \leq 1-\gamma,$$

if

$$\sum_{n=2}^{\infty} \frac{n(n-\gamma)}{1-\gamma} d_n\,|s|^{n-1} \leq 1. \qquad (36)$$

Therefore, by using Theorem 1, (36) will be true if

$$\frac{n(n-\gamma)}{1-\gamma}|s|^{n-1} \leq \frac{n^2[\,\lambda(n^2-2n+2-\tau)+n-\tau\,](\rho^2+1)^2}{(1+\lambda)(1-\tau)(n^2+\rho^2)^2},$$

and hence

$$|s| \leq \left[\frac{n[\,\lambda(n^2-2n+2-\tau)+n-\tau\,](1-\gamma)(\rho^2+1)^2}{(n-\gamma)(1+\lambda)(1-\tau)(n^2+\rho^2)^2}\right]^{\frac{1}{n-1}}, (n \geq 2\,).$$

Setting $|s| = R_2$, we get the desired result.

8. Weighted Mean and Arithmetic Mean.

Definition 3 Let $k(s)$ and $h(s)$ belong to $\mathcal{N}$. Then the weighted mean $E_q(s)$ of $k(s)$ and $h(s)$ is presented by

$$E_q(s) = \tfrac{1}{2}[(1-q)k(s)+(1+q)h(s)], \qquad (0 < q < 1).$$

The following theorem shows the weighted mean for the functions belong to the class $\mathcal{AN}(\lambda,\tau,n,\rho)$.

Theorem 10 Let $k(s)$ and $h(s)$ be in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$. Then the weighted mean of $k(s)$ and $h(s)$ is also in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$.

Proof. By Definition 3, we have

$$E_q(s) = \frac{1}{2}[(1-q)k(s)+(1+q)h(s)]$$

$$=\frac{1}{2}\left[(1-q)\left(s-\sum_{n=2}^{\infty}d_n s^n\right)+(1+q)\left(s-\sum_{n=2}^{\infty}c_n s^n\right)\right]$$

$$=s-\sum_{n=2}^{\infty}\frac{1}{2}[(1-q)d_n+(1+q)c_n]s^n.$$

Since $k(s)$ and $h(s)$ are in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$, so by Theorem 1 we get

$$\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\,d_n\ \le (1+\lambda)(1-\tau)$$

and

$$\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\,c_n\ \le (1+\lambda)(1-\tau).$$

Hence,

$$\sum_{n=2}^{\infty}\ n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\left[\frac{1}{2}(1-q)d_n+\frac{1}{2}(1+q)c_n\right]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}$$

$$=\frac{1}{2}(1-q)\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\,d_n$$

$$+\frac{1}{2}(1+q)\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\,c_n$$

$$\le\frac{1}{2}(1-q)(1+\lambda)(1-\tau)+\frac{1}{2}(1+q)(1+\lambda)(1-\tau)=(1+\lambda)(1-\tau).$$

This shows that $E_q\in\mathcal{AN}(\lambda,\tau,n,\rho)$.

In the following theorem, we shall demonstrate that the class $\mathcal{AN}(\lambda,\tau,n,\rho)$ is closed under arithmetic mean.

Theorem 11 Let $k_1(s),k_2(s),k_3(s),\dots,k_v(s)$ that defined by

$$k_m(s)=s-\textstyle\sum_{n=2}^{\infty}d_{n,m}\,s^n,\qquad \left(d_{n,m}\ \ge 0,\ m=1,2,\dots v,\ n\ge 2\right), \tag{37}$$

are in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$. Then the arithmetic mean of $k_m(s)$ $(m=1,2,\dots,v)$ that defined by

$$g(s)=\frac{1}{v}\sum_{m=1}^{v}k_m(s), \tag{38}$$

is also in the class $\mathcal{AN}(\lambda,\tau,n,\rho)$.

Proof. By (37) and (38), we can write

$$g(s)=\frac{1}{v}\sum_{m=1}^{v}\left(s-\sum_{n=2}^{\infty}d_{n,m}\,s^n\right)=s-\sum_{n=2}^{\infty}\left(\frac{1}{v}\sum_{m=1}^{v}\ d_{n,m}\right)s^n.$$

Since $k_m(s)\in\mathcal{AN}(\lambda,\tau,n,\rho)$ for every $m=1,2,\dots,v$, so by Theorem 1, we have

$$\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\left(\frac{1}{v}\sum_{m=1}^{v}d_{n,m}\right)$$

$$=\frac{1}{v}\sum_{m=1}^{v}\left(\sum_{n=2}^{\infty}n^2[\lambda(n^2-2n+2-\tau)+n-\tau]\frac{(\rho^2+1)^2}{(n^2+\rho^2)^2}\ d_{n,m}\right)$$

$$\le\frac{1}{v}\sum_{m=1}^{v}\ (1+\lambda)(1-\tau)=(1+\lambda)(1-\tau).$$

This ends the proof.

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