DOI https://doi.org/10.24297/jam.v21i.9254

On the existence of a bounded variation solution of a fractional integral equation in $L_1[0, T]$ due to the spread of COVID 19

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Abstract

In this article, we will investigate the existence and uniqueness of a bounded variation solution for a fractional integral equation in the space $L_1[0, T]$ of Lebesgue integrable functions.

Keywords: Nemytskii operator, Fractional calculus, Hausdorff measure of noncompactness, Functions of bounded variation, Darbo fixed point theorem.

1 Introduction

In investigating the problem of the spread of covid-19, some scientists such as Sabri T.M. Thabet, Mohammed S. Abdo, Kamal Shah, Thabet Abdeljawad [19] reached to an integral equation

$$x(t) = g(t) + \frac{1-\alpha}{N(\alpha)} f(t, x(t)) + \lambda \int_0^t (t-s)^{(\alpha-1)} f(s, x(s)) ds, \quad 0 < \alpha \leq 1, \quad t \in [0, T] \tag{1}$$

that reduced from a system of differential equations in the operation of dynamical mathematical modeling.
This paper studies the existence of at least one solution of this fractional integral equation in the space $L_1[0, T]$ of functions of bounded variation.

2 Preliminaries

This section is devoted to recall some notations and results that will be needed in the sequel. Denote by $L_1 = L_1[0, T]$ the space of Lebesgue integrable functions on the interval $[0, T]$, with the standard norm

$$\|x\| = \int_0^T |x(t)| dt, \quad x \in L_1.$$

The most important operator in nonlinear functional analysis is the so-called Nemytskii operator ([2], [8], [9]).

Definition 2.1 *If $f(t, x) = f : I \times R \to R$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in [0, T]$. Then to every function $x(t)$ being measurable on $[0, T]$ we may assign the function*

$$(Fx)(t) = f(t, x(t)) \qquad t \in I$$

The operator F is called the Nemytskii (or superposition) operator generated by f.

Furthermore, we propose a theorem which gives necessary and sufficient condition for the Nemytskii operator to map the space L_1 into itself continuously.

Theorem 2.1 *([2], [16]) If f satisfies Carathéodory conditions, then the Nemytskii operator F generated by the function f maps continuously the space L_1 into itself if and only if*

$$|f(t,x)| \leq a(t) + b|x|,$$

for every $t \in [0,T]$ and $x \in R$, where $a(t) \in L_1$ and $b \geq 0$ is a constant.

Definition 2.2 *([4], [11], [17])*

The Hausdorff measure of noncompactness $\chi(X)$ (see also [10], [12]) is defined as

$$\chi(X) = \inf\{r > 0 : \text{there exists a finite subset } Y \text{of } E \text{ such that } x \subset Y + B_r\}.$$

It is worthwhile to mention that the first important example of measure of weak noncompactness has been defined by De Blasi [7] by:

$$\beta(X) = \inf\{r > 0 : \text{there exists a weakly compact subset } W \text{ of } E \text{ such that } x \subset W + B_r\}.$$

Let us recall that there exists a formula allowing us to express De Blasi measure of weak noncompactness in the space L_1. This formula has been recently given by Appell and De Pascale [3]:

$$\beta(X) = \lim_{\varepsilon \to 0}\{\sup_{x \in X}\{\sup[\int_D |x(t)|\, dt : \ D \subset I, meas(D) \leq \varepsilon]\}\}. \tag{2}$$

The Hausdorff measure of noncompactness χ and De Blasi measure of weak noncompactness β are related by the following theorem:

Theorem 2.2 *[3]*
Let X be an arbitrary nonempty and bounded subset of $L_1[0,T]$. If X is compact in measure then $\beta(X) = \chi(X)$.

Now, we give Darbo fixed point theorem (cf.[6], [14], [15]).

Theorem 2.3 *Let $Q \subset E$ be nonempty, bounded, closed and convex and assume that $A : Q \to Q$ is a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. $\exists$ a constant $k \in [0,1)$ where*

$$\mu(AX) \leq k\mu(X),$$

for any nonempty subset X of Q. Then A has at least one fixed point in the set Q.

In the sequel, we give a short note about the fractional calculus.

Definition 2.3 *(Riemman–Liouville) ([1], [13])*
Let $f \in L_1[a,b]$, $\alpha \in R^+$. The Riemman-Liouville (R-L) fractional integral of the function f of order α is defined as

$$I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \ \alpha > 0, \ a \leq t \leq b$$

Definition 2.4 *[18] Let g be an absolutely continuous function on $[a,b]$. Then the fractional derivative of order $\alpha \in (0,1]$ of g is defined as*

$$D_a^\alpha g(t) = I_a^{1-\alpha} Dg(t),\ D = \frac{d}{dt}.$$

Definition 2.5 *(Functions of bounded variation) ([5], [17])*
Let $x : [a,b] \to R$ be a function. For each partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of the interval $[a,b]$, we define

$$Var(x, [a,b]) = \sup \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|,$$

where the supremum is taken over the interval $[a,b]$. If $Var(x) < \infty$, we say that x has bounded variation and we write $x \in BV$.
We denote by $BV = BV[a,b]$ the space of all functions of bounded variation on $[a,b]$.

Theorem 2.4 *[3] Assume that $X \subset L_1(I)$ is of locally generalized bounded variation, then Conv X (convex hull of X) and $\bar{X}$ are of the same type.*

Corollary 2.1 *[3] Let $X \subset L_1(I)$ is of locally generalized bounded variation, then Conv X is also such.*

Next, we will have the following theorem, which we will further use (cf. [3]).

Theorem 2.5 *Let X be a bounded subset of $L_1[0,T]$ of locally generalized bounded variation. If, in addition, for some $t_0 \in [0,T]$ the set $X(t_0) = \{x(t_0) : x \in X\}$ is bounded, then every sequence $\{x_n\} \subset X$ contains a subsequence which converges on $[0,T]$ to a function of locally bounded variation.*

3 Main result

We can write (1) in operator form as

$$Ax = g(t) + \frac{1-\alpha}{N(\alpha)} Fx + \lambda I^\alpha (Fx),$$

where $(Fx) = f(t,x)$ and $I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds.$

We will treat equation (1) under the following assumptions listed below:

(i) $g \in L_1[0,T]$ and is of locally generalized bounded variation on $[0,T]$.

(ii) $f : [0,T] \times R \to R$ satisfies Carathéodory conditions and there exist a function $a \in L_1[0,T]$ and a constant $b \geq 0$ such that

$$|f(t,x)| \leq a(t) + b|x|, \quad \text{for all } t \in [0,T] \text{ and } x \in R.$$

(iii) there exists a constant $k > 0$ such that

$$|f(t,x) - f(t,y)| \leq k|x - y|.$$

Moreover, there exists a constant $M > 0$ such that for every $n \in N$, every partition $0 = t_0 < t_1 < \ldots < t_n = T$, the following inequality holds:

$$\sum_{i=1}^{n} |f(t_i, x_{i-1}) - f(t_{i-1}, x_{i-1})| \leq M.$$

(iv) $b(\frac{1-\alpha}{N(\alpha)} + \frac{\lambda T^{\alpha}}{\alpha}) < 1$.

Theorem 3.1 *Let the hypotheses (i)–(iv) be satisfied, then equation (1) has at least one solution $x \in L_1[0,T]$ which is a function of locally bounded variation on $[0,T]$.*

Proof. Taking an arbitrary $x \in L_1[0,T]$, depending on assumption (ii) and Theorem 2.1 it is easy to see that $Ax \in L_1[0,T]$. Now, for $x \in B_r$ and by our assumptions, we have

$$\begin{aligned}
\|Ax\| &= \int_0^T |g(t) + \frac{1-\alpha}{N(\alpha)} f(t,x(t)) + \lambda \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds|dt \\
&\leq \int_0^T |g(t)|dt + \frac{1-\alpha}{N(\alpha)} \int_0^T |f(t,x(t))|dt + \int_0^T \int_0^t (t-s)^{\alpha-1}|f(s,x(s))|dsdt \\
&\leq \|g\| + \frac{1-\alpha}{N(\alpha)} \int_0^T [a(t) + b|x(t)|]dt + \lambda \int_0^T \int_s^T (t-s)^{\alpha-1}[a(s) + b|x(s)|]dtds \\
&\leq \|g\| + \frac{1-\alpha}{N(\alpha)}[\|a\| + b\|x\|] + \lambda \frac{(t-s)^{\alpha}}{\alpha}|_s^T . \int_0^T [a(s) + b|x(s)|]ds \\
&\leq \|g\| + \frac{1-\alpha}{N(\alpha)}[\|a\| + b\|x\|] + \frac{\lambda (T-s)^{\alpha}}{\alpha}[\|a\| + b\|x\|] \\
&\leq \|g\| + \|a\|[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda T^{\alpha}}{\alpha}] + b[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda T^{\alpha}}{\alpha}]\|x\| \\
&\leq \|g\| + \|a\|[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda T^{\alpha}}{\alpha}] + b[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda T^{\alpha}}{\alpha}]r \\
&\leq r.
\end{aligned}$$

From the above estimate, the operator $A : B_r \to B_r$, where

$$r = \frac{\|g\| + \|a\|[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda T^{\alpha}}{\alpha}]}{1 - b[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda T^{\alpha}}{\alpha}]} > 0.$$

Next, let us choose an $x \in B_r$. Observe that

$$\begin{aligned}
|(Ax)(0)| &= |g(0) + \frac{1-\alpha}{N(\alpha)} f(0,x(0))| \\
&\leq |g(0)| + \frac{1-\alpha}{N(\alpha)} |f(0,x(0))| \\
&< \infty. \qquad (3)
\end{aligned}$$

So we get that all functions belonging to AB_r are bounded at $t = 0$.
Moreover, fix $T > 0$ and assume that the sequence t_i such that $0 = t_0 < t_1 < t_2 \ldots < t_n = T$. Then, using the above

assumptions leads us to

$$\begin{aligned}
\sum_{i=1}^{n} |(Ax)(t_i) - (Ax)(t_{i-1})| &\leq \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| \\
&+ \frac{1-\alpha}{N(\alpha)} \sum_{i=1}^{n} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))| \\
&+ \lambda \sum_{i=1}^{n} |\int_0^{t_i} (t_i - s)^{\alpha-1} f(s, x(s))ds - \int_0^{t_{i-1}} (t_{i-1} - s)^{\alpha-1} f(s, x(s))ds| \\
&\leq V(g, T) + \frac{1-\alpha}{N(\alpha)} \sum_{i=1}^{n} |f(t_i, x(t_i)) - f(t_i, x(t_{i-1}))| \\
&+ \frac{1-\alpha}{N(\alpha)} \sum_{i=1}^{n} |f(t_i, x(t_{i-1})) - f(t_{i-1}, x(t_{i-1}))| \\
&+ \lambda \sum_{i=1}^{n} |\int_0^{t_i} (t_i - s)^{\alpha-1} f(s, x(s))ds - \int_0^{t_i} (t_{i-1} - s)^{\alpha-1} f(s, x(s))ds| \\
&+ \lambda \sum_{i=1}^{n} |\int_0^{t_i} (t_{i-1} - s)^{\alpha-1} f(s, x(s))ds - \int_0^{t_{i-1}} (t_{i-1} - s)^{\alpha-1} f(s, x(s))ds| \\
&\leq V(g, T) + \frac{1-\alpha}{N(\alpha)} [k \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| + M] \\
&+ \lambda \sum_{i=1}^{n} \int_0^{t_i} |(t_i - s)^{\alpha-1} - (t_{i-1} - s)^{\alpha-1}||f(s, x(s))|ds \\
&+ \lambda \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |(t_{i-1} - s)^{\alpha-1}||f(s, x(s))|ds \\
&\leq V(g, T) + \frac{1-\alpha}{N(\alpha)} [kV(x, T) + M] \\
&+ \frac{\lambda}{\alpha} \sum_{i=1}^{n} [| - (t_i - s)^{\alpha} + (t_{i-1} - s)^{\alpha}]_0^{t_i} [\|a\| + b\|x\|] \\
&+ \frac{\lambda}{\alpha} \sum_{i=1}^{n} [|(t_{i-1} - s)^{\alpha}]_{t_{i-1}}^{t_i} [\|a\| + b\|x\|] \\
&\leq V(g, T) + \frac{1-\alpha}{N(\alpha)} [kV(x, T) + M] \\
&+ \frac{\lambda}{\alpha} \sum_{i=1}^{n} [|(t_{i-1} - t_i)^{\alpha} + ({t_i}^{\alpha} - {t_{i-1}}^{\alpha})][\|a\| + b\|x\|] \\
&+ \frac{\lambda}{\alpha} \sum_{i=1}^{n} [| - (t_{i-1} - t_i)^{\alpha}|][\|a\| + b\|x\|] \qquad (4)
\end{aligned}$$

By mean value theorem there is $z,\ \ t_{i-1} < z < t_i$ such that

$$t_i^{\alpha} - t_{i-1}^{\alpha} = (t_i - t_{i-1})\alpha z^{\alpha-1} \leq (t_i - t_{i-1})\alpha T^{\alpha-1}$$

Then

$$\begin{aligned}
V(Ax,T) &\leq V(g,T)+\frac{1-\alpha}{N(\alpha)}[kV(x,T)+M] \\
&+ \frac{\lambda}{\alpha}\sum_{i=1}^{n}[|(t_{i-1}-t_i)^{\alpha}|+\alpha(t_i-t_{i-1})T^{\alpha-1}+|-(t_{i-1}-t_i)^{\alpha}|][\|a\|+b\|x\|] \\
&\leq V(g,T)+\frac{1-\alpha}{N(\alpha)}[kV(x,T)+M] \\
&+ \frac{\lambda}{\alpha}\sum_{i=1}^{n}[2|(t_i-t_{i-1})^{\alpha}|+\alpha(t_i-t_{i-1})T^{\alpha-1}][\|a\|+b\|x\|] \\
&\leq V(g,T)+\frac{1-\alpha}{N(\alpha)}[kV(x,T)+M] \\
&+ \sup\frac{\lambda}{\alpha}\sum_{i=1}^{n}[2(t_i-t_{i-1})+\alpha(t_i-t_{i-1})T^{\alpha-1}][\|a\|+b\|x\|] \\
&\leq V(g,T)+\frac{1-\alpha}{N(\alpha)}[kV(x,T)+M] \\
&+ \frac{\lambda}{\alpha}[2(t_n-t_0)+\alpha(t_n-t_0)T^{\alpha-1}][\|a\|+b\|x\|] \\
&\leq V(g,T)+\frac{1-\alpha}{N(\alpha)}[kV(x,T)+M]+\frac{\lambda}{\alpha}(2T+\alpha T^{\alpha})[\|a\|+b\|x\|] \\
&< \infty, \qquad (5)
\end{aligned}$$

from the previous estimate, all functions belonging to AB_r have variation majorized by the same constant on every closed subinterval of $[0,T]$.

In the following, let the set Q_r=Conv AB_r, it is clear that $Q_r \subset B_r$. We will show that Q_r is nonempty, bounded convex, closed and compact in measure.

To prove Q_r is nonempty, let $x(t)=\frac{r}{2}$, we get

$$\|x\| \leq \int_0^1 |\frac{r}{2}|dt=\frac{r}{2}\leq r.$$

Since, $Q_r \subset B_r$ then it is bounded.

To prove the convexity of Q_r, take $x_1,x_2 \in Q_r$ which gives $\|x_i\|\leq r, \quad i=1,2$. Let

$$z(t)=\lambda x_1(t)+(1-\lambda)x_2(t), \quad t\in[0,T],\ \lambda\in[0,T].$$

Then

$$\begin{aligned}
\|z\| &\leq \lambda\|x_1\|+(1-\lambda)\|x_2\| \\
&\leq \lambda r+(1-\lambda)r=r.
\end{aligned}$$

So, we get Q_r is convex.

Now, we prove that the closeness of Q_r . To do this, suppose $\{x_n\}$ is the sequence of elements in Q_r that converges to x in $L_1[0,T]$, then this sequence is convergent in measure and as a result of the Vitali convergence theorem and the characterization of convergence in measure (the Riesz theorem) this leads to the existence of $\{x_{n_k}\}\subset\{x_n\}$ that converges to x almost uniformly on $[0,T]$ that means $x\in Q_r$ and thus the set Q_r is closed.

Further, by (3) we conclude that the functions from Q_r are equibounded at the point t_0. Moreover, by (5) and Corollary 2.1 we deduce that Q_r is the set of locally bounded variation. Combining the above mentioned properties of Q_r and by Theorem 2.5 we get Q_r is compact in measure.

Finally, we prove that the operator G is a contraction with respect to the measure of noncompactness χ. Take a subset $X \subset Q_r$ and $\varepsilon > 0$ is fixed, then $\forall x \in X$ and for a set $D \subset [0,T]$, meas$D \leq \varepsilon$, we get

$$\begin{aligned}\int_D |(Ax)(t)|dt &\leq \int_D |g(t|dt + \frac{1-\alpha}{N(\alpha)}\int_D |f(t,x(t))|dt + \lambda\int_D |\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds|dt \\ &\leq \int_D g(t)dt + \frac{1-\alpha}{N(\alpha)}\int_D [a(s)+b|x(s)|]ds + \lambda \int_D \int_s^T (t-s)^{\alpha-1}|f(s,x(s))dt|ds \\ &\leq \int_D g(t)dt + \frac{1-\alpha}{N(\alpha)}[\int_D a(s)ds + b\int_D |x(s)|ds] + \frac{\lambda}{\alpha}(t-s)^{\alpha-1}|_s^T \int_D [a(s)+b|x(s)|]ds \\ &\leq \int_D g(t)dt + (\frac{1-\alpha}{N(\alpha)} + \frac{\lambda}{\alpha}T^{\alpha-1})\int_D a(s)ds + b(\frac{1-\alpha}{N(\alpha)} + \frac{\lambda}{\alpha}T^{\alpha-1})\int_D |x(s)|ds.\end{aligned}$$

Therefore, using the fact that

$$\lim_{\varepsilon\to 0} \sup\{\int_D g(t)dt : \ D \subset [0,T], \ \text{meas}D \leq \varepsilon\} = 0,$$

and

$$\lim_{\varepsilon\to 0} \sup\{\int_D a(t)dt : \ D \subset [0,T], \ \text{meas}D \leq \varepsilon\} = 0,$$

Then using (2), we get

$$\beta(AX) \leq b[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda}{\alpha}T^{\alpha-1}]\beta(X).$$

Since X is a subset of Q_r and Q_r is compact in measure, we have

$$\chi(AX) \leq b[\frac{1-\alpha}{N(\alpha)} + \frac{\lambda}{\alpha}T^{\alpha-1}]\chi(X).$$

Therefore, by using assumption (iv) we can apply Darbo fixed point theorem. This completes the proof. ■

4 Uniqueness of the solution

Now, we can prove the existence of our unique solution.

Theorem 4.1 *If the assumptions of Theorem 3.1 is satisfied but instead of assuming (iv), let* $k(\frac{\alpha(1-\alpha)+\lambda N(\alpha)T^\alpha}{\alpha N(\alpha)}) < 1$. *Then, equation (1) has a unique solution on* $[0,T]$.

Proof. To prove that equation (1) has a unique solution, let $x(t), y(t)$ be any two solutions of equation (1) in B_r, we have

$$\begin{aligned}\|x-y\| &\leq \frac{1-\alpha}{N(\alpha)}\int_0^T |f(t,x(t)) - f(t,y(t))|dt + \lambda\int_0^T\int_0^t (t-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|dsdt \\ &\leq \frac{k(1-\alpha)}{N(\alpha)}\int_0^T |x(t)-y(t)|dt + \lambda k \int_0^T\int_s^T (t-s)^{\alpha-1}|x(s)-y(s)|dtds \\ &\leq \frac{k(1-\alpha)}{N(\alpha)}\|x-y\| + \lambda k \frac{(t-s)^\alpha}{\alpha}|_s^T.\|x-y\| \\ &\leq \frac{k(1-\alpha)}{N(\alpha)}\|x-y\| + \frac{\lambda k T^\alpha}{\alpha}\|x-y\|.\end{aligned}$$

Therefore,

$$[1 - k(\frac{\alpha(1-\alpha)+\lambda N(\alpha)T^\alpha}{\alpha N(\alpha)})]\|x-y\|_{L_1} \leq 0,$$

This yields $\|x - y\| = 0, \Rightarrow x = y$, which completes the proof.

Data Availability (excluding Review articles)

Applicable.

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Supplementary Materials

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Funding Statement

The research was self-sposored by the author.

Acknowledgments

The authors like to thank the anonymous referees for their valuable comments and suggestions, which greatly improved the presentation of this paper.