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On Hesitant Fuzzy Primary Ideal In Γ- ring

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Abstract

In this paper, we introduce the notions of hesitant fuzzy primary ideal and completely primary ideal, hesitant fuzzy semiprimary ideals of a Γ -ring, and discuss the relation between hesitant primary ideal, completely primary and semiprimary.

Keywords: Gamma ring, hesitant fuzzy primary ideal , hesitant fuzzy semiprimary ideal.

Introduction

N.Nobusawa [9] in troduces the notion of Γ-ring as more general that a ring .W.E.Barnes [1] we akened the conditis on of definition of Γ -ring in the sence of Nobusawa Barnes ,K.Yuno and Luh[6] studied the structure of Γ-ring and obtained various generalizations .

In 1991the notions of primary ideal by D.Malik and Mordeson.J.N [7] introduced the concept of fuzzy ring .

Fairooze .A, Abbasi . M.Y and sabahat Ali [3] introduced the notions of hesitant fuzzy ideal .

In 1982 S Kyuno ,[4] , study the concept of prime ideal in Γ-rings . Mohammad and other [8] in 2018 introduced the concepts hesitant fuzzy ideal , hesitant fuzzy primary ideal in ring and some other concepts .In this paper , we introduce the notions of hesitant fuzzy primary ideal and completely primary ideal , hesitant fuzzy semiprimary ideals of a Γ -ring , and discuss the relation between hesitant primary ideal ,completely primary and semiprimary.

2. HESITANT FUZZY PRIMARY IDEAL IN $\Gamma -$ RING.

Proposition 2.1

Let M be a Γ-ring and Let $h_1 \in$ HFI(M) , then for any X ∈ M , $\alpha \in \Gamma$

1- $h(x) \subseteq h(0)$

2- $h(-x) \subseteq h(x)$

Proof :

Assume that h∈HFI(M).

1. Since 0=x-x , hence $h(0)=h(x-x) =h(x)\cap h(x)=h(x)$.thus $h(x) = h(0)$.

2. $h(-x)=h(0-x) = h(0) \cap h(x)$, so by condition (1), we have $h(0) \cap h(x)=h(x)$.

Thus $h(-x) = h(x)$ .... (1) .

Also $h(x) =h(0-(-x)) = h(0) \cap h(-x)$,so by condition (1) , we have $h(0)\cap h(-x) =h(-x)$.

So that $h(x) =h(-x)$ ....(2).

From (1),(2) we get $h (-x)=(x)$.

Theorem 2.2

Let M be a Γ-ring and Let h be a hesitant fuzzy ideal of M, then

1- The t-lower bound $h_t^- \in$ HFI(M).

2-The t-upper bound $h_t^+ \in$ HFI(M).

Proof :

suppose that h∊HFI(M),let x,y ∊M,and α∊Γ.

We have h_t^-(x-y)={k∊h(x-y): k≤t}⊒{k∊h(x)∧k∊h(y):k≤t}

={k∊h(x)⁚k≤t∧k∊h(y):k≤t}

={k∊h(x)⁚k≤t}∩{k∊h(y):k≤t} =h_t^-(x)∩h_t^-(y).

Thus h_t^-(x-y)⊒ h_t^-(x)∩h_t^-(y)........(1) .

Now, h_t^-(xαy)={k∊h(xαy):k≤t} ⊒{k∊h(x)Uh(y):k≤t}

={k∊h(x)k∨∊h(y):k≤t}={k∊h(x):k≤t}U {k∊h(y):k≤t}

=h_t^-(x) U h_t^-(y).

Hence h_t^-(xαy) ⊒ h_t^-(x) U h_t^-(y).......(2).

Frome (1) and (2) , we get h_t^- ∊HFI(M).

by the same way,we can prove (2).

Definition 2.3

Let M be a Γ-ring and Let h_1 , h_2 are two hesitant fuzzy set of M , S > 0 if S is constant , then

1- $h^s(x) = \cup_{k\in h(x)}\{k^s\} \cong \{ k^s : k \in h(x) \}$

2- s.h (x) = $\cup_{k\in h(x)}\{1-(1-k)^s\} \cong \{ 1-(1-k)^s : k \in h(x) \}$

Proposition 2.4

Let M be a Γ-ring and Let h be a hesitant fuzzy ideal of M, then h^s is a hesitant fuzzy ideal ,where s > 0, s is constant.

Proof: Suppos that h is ahesitant fuzzy ideal of M and x,y∊ M and α∊Γ,then

h^s(x-y)={k^s: k∊h(x-y) }⊒{k^s⁚$k\epsilon h\epsilon x) \cap h(y)$ }

={k^s:k∊h(x)∧ k∊h(y)}

={k^s:k∊h(x)} ∩ {k^s:k∊h(y)}

=h^s(x) ∩ h^s(y).

Thus h^s(x-y)⊒h^s(x) ∩ h^s(y).........(1)

and h^s(xαy)={k^s : k∊h(xαy)} ⊒ {k^s⁚ k∊h(x) U h(y)}

={k^s⁚k∊h(x)∨k∊h(x)}

={k^s ⁚ k∊h(x)}∪{k^s⁚ k∊h(y)}

=h^s(x) ∪ $h^s(y)$.

Thus ,h^s(xαy)⊒h^s(x) ∪ h^s(y)........(2). .

From (1)and(2) we get h^s is a hesitant fuzzy ideal of M.

Definition 2.5`

Let M be a Γ-ring. A hesitant fuzzy ideal h of M is said to be a hesitant fuzzy Primary ideal (in Short , HFYI) if for any x,y ∈ M, $\alpha \in \Gamma$,

h(x α y) ⊆ h(x) ∪ h((y α)$^{n-1}$y) , for some n ∈ N .

We will denote the set of all hesitant fuzzy Primary ideals in M as HFYI (M).

Example 2.6

let (Z_2, $+_2$) , (Z , +) are additive abelion groups then (Z_2, $+_2$) is Z-ring where Z_2 = {$\underline{0}$, $\underline{1}$ } and h:Z_2 → P[0, 1], define as following : h = $\{[0.2\,,0.7]\ if\ x = 1\ [0.2\,,0.5]\ if\ x = 0$

then we easily wee that h ∈ HFI(Z_2) , for any m ∈ N h($\underline{0}$ $._2$ 0 $._2$ $\underline{0}$) ⊆ h($\underline{0}$) ∪ h(0^m)=h(0)U h(0)

h($\underline{0}$ $._2$ 1 $._2$ $\underline{0}$) ⊆ h($\underline{0}$) ∪ h(0^m)=h(0)U h(0)

h($\underline{1}$ $._2$ 2 $._2$ $\underline{1}$) = h($\underline{0}$) ⊆ h($\underline{1}$) ∪ h(1^m)=h(1)U h(1)

h($\underline{1}$ $._2$ 3 $._2$ $\underline{1}$) = h($\underline{1}$) ⊆ h($\underline{1}$) ∪ h(1^m)=h(1) U h(1)

Thus h is hesitant fuzzy semi primary ideal of Z-ring

Remark 2.7

Let M be a Γ-ring. If h ∈ HFYI (M), then h(x α y) = h(x) ∪ h((y α)$^{n-1}$y) for any x , y ∈ M and $\alpha \in \Gamma$

3- Main Results

Proposition 3.1

Let M be a Γ-ring, A hesitant fuzzy ideal of M is acalled hesitant fuzzy Primary ideal, if h(x α y) = h(0) implies h(x)=h(o) or h((y α)$^{n-1}$y) = h(0), for some m ∈ N.

Poof:

Assume that the condition is hold.

since h(x) ⊆ h(0) , for all x ∈ M by proposition 2.1

so h(x α y) = h(0) = h(0) ∪ h(0) = h(x) ∪ h(0) = h(0) ∪ h((y α)$^{n-1}$y)

= h(x) ∪ h((y α)$^{n-1}$y)

Thus h(x α y) = h(x) ∪ h((y α)$^{n-1}$y) , where h(x)=h(0) or h((y α)$^{n-1}$y) = h(0) for some n ∈ N.

Proposition 3.2

Let M be a Γ-ring. Every hesitant fuzzy Prime ideal is a hesitant Primary ideal of M.

The proof is directly from definition 2.5.

Definition3.3[2;10]

Let h_1 ,h_2 be two hesitant fuzzy sets on X and S>0 (costant),then

1. $(h_1 \oplus h_2)(x) = \cup_{k_1 \in h_1, k_2 \in h_2} \{k_1 + k_2 - k_1k_2\} \cong \{k_1 + k_2 - k_1k_2 \mid k_1 \in h_1, k_2 \in h_2\}$
2. $(h_1 \otimes h_2)(x) = \cup_{k1 \in h1, k2 \in h2} \{k_1k_2\} \cong \{k_1k_2 \dot{:} k_1 \in h_1, k_2 \in h_2\}$.
3. $.(h_1 \ominus h_2)(x) = \{t \dot{:}\ k_1 \in h_1(x), k_2 \in h_2 \in h_2(x)\}$

$$where\ t = \begin{cases} \frac{k_1 - k2}{1 - k2} & if\ k_1 \geq k_2,\ k_2 \neq 0 \\ 0 & otherwise \end{cases}$$

4. $(h_1 \oslash h_2)(x) = \{t \dot{:} k_1 \in h_1(x), k_2 \in h_2(x)\}\,.where\ t = \begin{cases} \frac{k_1}{k_2} & if\ k_1 \leq k2\ ,k2 \neq 0 \\ 1 & otherwise \end{cases}$

Proposition 3.4

Let M be a Γ-ring and let h be a hesitant fuzzy Primary ideal of M if s is constant. Then

1- h^s is a hesitant fuzzy Primary ideal of M, where S>0.

2- s.h is a hesitant fuzzy Primary ideal of M.

3- $(h^s)^s$ is a hesitant fuzzy Primary ideal of M.

Proof :

1.Assume that h is a hesitant fuzzy Primary ideal of M.

it is clear that h^s is a hesitant fuzzy ideal of M , By Definition 2.5 and proposition2.4

Thus, $h^s(x\,\alpha\,y) = \{k^s : k \in h(x\,\alpha\,y)\} = \{ k^s : k \in h(x) \cup h((y\,\alpha\,)^{n-1}y)\}$,for some $n \in N$

$= \{ k^s : k \in h(x) \vee h((y\,\alpha\,)^{n-1}y)\}$

$= \{ k^s : k \in h(x)\} \cup \{ k^s : k \in h((y\,\alpha\,)^{n-1}y)\}$

$= h^s(x) \cup h^s((y\,\alpha\,)^{n-1}y)$

This $h^s(x\,\alpha\,y) = h^s(x) \cup h^s((y\,\alpha\,)^{n-1}y)$ for some $n \in N$

Thus h^s is a hesitant fuzzy primary ideal of M.

2.Assume that h is a hesitant fuzzy Primary ideal of M

it is clear that s.h is a hesitant fuzzy ideal of M, By Definition 2.5 and Definition 3.3

Thus, $s.h(x\,\alpha\,y) = \{1-(1-k)^s : k \in h(x\,\alpha\,y)\} = \{1-(1-k)^s : k \in h(x) \cup h((y\,\alpha\,)^{n-1}y)\}$ for same $n \in N$

$= \{1-(1-k)^s : k \in h(x) \vee k \in h((y\,\alpha\,)^{n-1}y)\}$

$= \{1-(1-k)^s : k \in h(x)\} \cup \{1-(1-k)^s : k \in h((y\,\alpha\,)^{n-1}y)\}$

$= s.h(x) \cup s.h((y\,\alpha\,)^{n-1}y)$

This $s.h(x\,\alpha\,y) = s.h(x) \cup s.h((y\,\alpha\,)^{n-1}y)$ for some $n \in N$

Thus s.h is a hesitant fuzzy primary ideal of M.

4.Assume that h is a hesitant fuzzy Primary ideal of M

.By condition(1) , h^s is a hesitant fuzzy primary ideal of M

Thus, $(h^s)^s(x\,\alpha\,y) = \{(k^s)^s : k \in h^s(x\,\alpha\,y)\} = \{(k^s)^s : k \in h^s(x\,\alpha\,y)\}$ let $t = s.s$

$= \{ k^t : k \in h^s(x) \cup k \in h^s((y\,\alpha\,)^{n-1}y)\}$

$= \{ k^t : k \in h^s(x) \vee k \in h^s((y\,\alpha\,)^{n-1}y)\}$

$= \{ k^t : k \in h^s(x)\} \cup \{ k^t : k \in h^s((y\,\alpha\,)^{n-1}y)\}$

$= (h^s)^s(x) \cup (h^s)^s((y\,\alpha\,)^{n-1}y)$

This $(h^s)^s(x\,\alpha\,y) = (h^s)^s(x) \cup (h^s)^s((y\,\alpha\,)^{n-1}y)$

Thus $(h^s)^s$ is a hesitant fuzzy primary ideal of M.

Proposition 3.5

if hesitant fuzzy ideal $h:M \rightarrow P[0,l]$, such that M is Γ -ring is hesitant fuzzy Primary ideal of M where $t \in [0 , l]$. Then

1- h_t^- (lower bounded) is a hesitant fuzzy Primary ideal of M

2- h_t^+ (upper bounded) is a hesitant furzy Primary ideal of M

Proof:

1. Assume that h is a hesitant fuzzy Primary ideal of M

so h_t^- is a hesitant fuzzy ideal of M, from definition 2.5 and Proposition 2.2

since h_t^- $(x\,\alpha\, y)$ = { k ∈ h(x α y) : K ≤ t} = {k ∈ h(x) ∪ h($(y\,\alpha\,)^{n-1}y$) : K ≤ t }

= {k ∈ h(x) ∨ h($(y\,\alpha\,)^{n-1}y$) : K ≤ t }

= { k ∈ h(x) : K ≤ t } ∪ { k ∈ h ($(y\,\alpha\,)^{n-1}y$) : K ≤ t }

= h_t^- (x) ∪ h_t^- ($(y\,\alpha\,)^{n-1}y$)

Thus h_t^- $(x\,\alpha\, y)$ = h_t^- (x) ∪ h_t^- ($(y\,\alpha\,)^{n-1}y$)

Thus h_t^- is a hesitant fuzzy primary ideal of M,

By the same way, we can prove (2).

Definition 3.6

Let M be a Γ-ring. A hesitant fuzzy ideal h of M is said a hesitant fuzzy completely Primary ideal of M (in short, HFCYI) if for any two hesitant fuzzy paints X_t ,Y_q ∈ HFP(M), $x_t \circ y_q$ ∈ h and x_t ∉ h this implies that, there exists n ∈ N such that $((y_q\,\alpha\,)^{n-1}y_q)$ ∈ h , for some n ∈ N

Proposition 3.7

Let M be a Γ-ring every hesitant Fuzzy completely Primary ideal is a hesitant fuzzy Primary ideal of M.

Proof :

Assume that h ∈ HFCYI(M)

let h(x) ∪ h($(y\,\alpha\,)^{n-1}y$) ⊂ h(x α y) , for some x , y ∈ M and $\alpha \in \Gamma$, n ∈ N

Put t = h(x α y), then h(x) ∪ h($(y\,\alpha\,)^{n-1}y$) ⊂ t and $(x\,\alpha\, y)_t$ ∈ h.

so h(x) ⊂ t this implies $(x)_t$ ∉ h and h($(y\,\alpha\,)^{n-1}y$) ⊂ t

this implies $((y\,\alpha\,)^{n-1}y)_t$ ∉ h

This is a Contradiction.

Therefore for all x , y ∈ M , $\alpha \in \Gamma$, h(x) ∪ h($(y\,\alpha\,)^{n-1}y$) ⊇ h(x α y)

so h ∈ HFCYI(M)

Definition 3.8

Let M be a Γ-ring an ideal I of M is called semi primary if for $\alpha \in \Gamma$, n ∈ N x α y ∈ I implies that either a Power of x or a power of y belongs to I .

Definition 3.9

Let M be a Γ-ring A hesitant fuzzy ideal of M is said to be a hesitant fuzzy Semi Primary ideal of M if for all a , b ∈ M, $\alpha \in \Gamma$, either h(a α b) ⊆ h($(a\,\alpha\,)^{n-1}a$) or h(a α b) ⊆ h($(b\,\alpha\,)^{m-1}b$) for som n , m ∈ N.

Example 3.10

let (Z_2, $+_2$) and (Z , +) are additive abelian groups then (Z_2, $+_2$) is Z-ring where Z_2 = {$\underline{0}$, $\underline{1}$, $\underline{2}$ } and h :Z_2 → P[0, 1] , define as the following:

h = {$[0.1\,, 0.9]\ if\ x = 1\ [0.1\,, 0.7]\ if\ x = 0$

then we easily wee that h ∈ HFI(Z_2) , for any 0 , 1 ∈ Z_2

h($\underline{0}$ $\cdot_2$ 0 $\cdot_2$ $\underline{0}$) ⊆ h(0^n) or h(0^m)

h($\underline{0}$ $\cdot_2$ 1 $\cdot_2$ $\underline{0}$) ⊆ h(0^n) or h(0^m)

h($\underline{1}$ $\cdot_2$ 2 $\cdot_2$ $\underline{1}$) = h($\underline{0}$) or h(1^n) or h(1^m)

h($\underline{1}$ $\cdot_2$ 3 $\cdot_2$ $\underline{1}$) = h($\underline{1}$) or h(1^n) or h(1^m)

h(1 .2 2 .2 0) = h(0) or h(1^n) or h(0^m)
Thus h is hesitant t fuzzy semi primary

Proposition 3.11

Let M be Γ-ring. Every hesitant fuzzy Semi Primary ideal is a hesitant fuzzy completely Primary ideal of M .

Proof:

Assume that h is a hesitant fuzzy Semi Primary ideal of M and x_t, y_q ∈ HFP(M),

$x_t \circ y_q \in h$ then $(x\ \alpha\ y)_{t \cap q} \in h$,

put k = t ∩ q , so $(x\ \alpha\ y)_k \in h$ then k ⊆ h(x α y)

since h a hesitant fuzzy semi primary ideal of M.

So k ⊆ h(x α y) ⊆ h($(x\,\alpha)^{n-1}x$) hence k ⊆ h($(x\,\alpha)^{n-1}x$) ,then $((x_k\,\alpha)^{n-1}x) \in h$

or k ⊆ h(x α y) ⊆ h($(x\,\alpha)^{n-1}x$) hence k ⊆ h($(y\,\alpha)^{m-1}y$) ,then $((y_k\,\alpha)^{m-1}y) \in h$

so we get $(x\,\alpha\,y)_k \in h$ implies $((x_k\,\alpha)^{n-1}x) \in h$ or $((y_k\,\alpha)^{m-1}y) \in h$

Thus h is hesitant fuzzy completely primary ideal of M .

proposition 3.12

Let M be Γ-ring and let h_1,h_2 are two hesitant fuzzy semi primary ideal of M. then

1- $h_1 \oplus h_2$ is a hesitant fuzzy Semi primary ideal of M.
2- $h_1 \otimes h_2$ is a hesitant fuzzy Semi primary ideal of M.
3- $h_1 \oslash h_2$ is a hesitant fuzzy Semi primary ideal of M.
4- $h_1 \ominus h_2$ is a hesitant fuzzy Semi primary ideal of M.

Proof: we proof the point (3) and (4) the others are similarly

3. Assume h_1 , h_2 be hesitant fuzzy Semi primary ideal , imply h_1 , h_2 be hesitant fuzzy ideal. then $h_1 \oslash h_2$ is hesitant fuzzy ideal since $(h_1 \oslash h_2)(x) = \{t: k_1 \in h_1(x)\ ,\ k_2 \in h_2(x)\}\ \forall\ x \in X$

Where $t= \begin{cases} \frac{k_1}{k_2} & if\ \ k_1 \le k_2\ ,k_2 \ne 0 \\ 0 & other\ wise \end{cases}$

Now ,we must prove

$(h_1 \oslash h_2)\ (x\ \alpha\ y)\ = \{\ t : k_1 \in h_1(x\ \alpha\ y)\ ,\ k_2 \in h_2(x\ \alpha\ y)\ \}$

$\subseteq \{\ t : k_1 \in h_1((x\ \alpha)^{n-1}x)\ ,\ k_2 \in h_2((x\ \alpha)^{n-1}x)\}$

$= (h_1 \oslash h_2)((x\ \alpha)^{n-1}x)$

Thus $(h_1 \oslash h_2)\ (x\ \alpha\ y) \subseteq (h_1 \oslash h_2)((x\ \alpha)^{n-1}x)$

Similarly , we get $(h_1 \oslash h_2)\ (x\ \alpha\ y) \subseteq (h_1 \oslash h_2)((y\ \alpha)^{n-1}y)$

Thus $(h_1 \oslash h_2)$ is a hesitant fuzzy semi primary ideal of M.

4.Assume that h_1 , h_2 be hesitant fuzzy Semi Primary ideal imply h_1 , h_2 be hesitant fuzzy ideal imply $h_1 \ominus h_2$ is hesitant fuzzy ideal since $(h_1 \ominus h_2)(x) = \{t: k_1 \in h_1(x)\ ,\ k_2 \in h_2(x)\}\ \forall\ x \in M$

Where $t = \{\frac{k_1 - k_2}{1 - k_2}\ \ if\ \ k_1\ \le\ k_2\ , k_2\ \ne 1\ 0\ \ \ if\ \ \ \ \ \ other\ wise$

Now ,we must prove

$(h_1 \ominus h_2)\ (x\ \alpha\ y)\ = \{\ t : k_1 \in h_1(x\ \alpha\ y)\ ,\ k_2 \in h_2(x\ \alpha\ y)\ \}$

$\subseteq \{ t : k_1 \in h_1((x\,\alpha)^{n-1}x) , k_2 \in h_2((x\,\alpha)^{n}x)\}$

$= (h_1 \ominus h_2)((x\,\alpha)^{n-1}x)$

Thus $(h_1 \ominus h_2)(x\,\alpha\,y) \subseteq (h_1 \ominus h_2)((x\,\alpha)^{n-1}x)$

Similarly , we get $(h_1 \ominus h_2)(x\,\alpha\,y) \subseteq (h_1 \ominus h_2)((y\,\alpha)^{n-1}y)$

Thus $(h_1 \ominus h_2)$ is a hesitant fuzzy semi primary ideal of M.

We can prove the rest of the points by the same way

References

1. Barnes W.E : " on the Γ-rings of No pacific J. Math. , vol.18 , pp. 411-422: (1966)
2. Carios R.J .Aicantud a, Torra Vicenc, "Decomposition Theorems And Extension Principles For Hesitant Fuzzy Sets Fuzzy Sets" In Formation Fusion 41 (2018) 48-56.
3. FairoozeTaleel. A, Abbasi M.Y And Sabahat Ail Khan"Hesitant Fuzzy Ideal In Semigroup With Two Frontiers "Journal of Basic Applied Engineering Research , Volume 4,Issue 6, July-September, 2017,PP.385-388.
4. Kyuno S. ; " Prime Ideal In Gamm Ring" Paciic Journal Of Mathematics (VOL.98,NO.2 , 1982)
5. Liao H.C , Xu, Z.S. "Subtraction and division operations over hesitant fuzzy sets" ,Journal of Inteligent and fuzzy Systems (2013b), doi:10.3233/IFS-130978.
6. Luh J. ; " The stracture of primitive gamma rings" ; Osaka J. Math. ; vol. ; pp. 267-274 ; (1970)
7. Malik.D,Mordeson.J.N "Fuzzy Maximal,Radical, And Primary Ideal Of Aring " Information Sciences 5, 238-250(1991).
8. Mohammad Y.A , Aakif F. , Khan T.S. and Hilla K. ; " A Hesitant fuzzy set approach to ideal theory in Γ-semigroups"; Advances in Fuzzy systems; (2018)
9. Nobusawa N. ;" on a generalization of the ring theory" ; Osaka J. Math. : vol. 1 pp. 81-89 : (1964).
10. Zeshui Xu , "Hesitant Fuzzy Sets Theory "Springer Chan Heidelberg New York Pordrecht Hondoh Dol,10.10071978-3-319-04711-9.